# Stop Faking It!

Finally Understanding Science So You Can Teach It

## MATH

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08 07 06 4 3 2 1

**Library of Congress Cataloging-in-Publication Data**
Robertson, William C.
Math/by William C. Robertson.
p. cm.—(Stop faking it! finally understanding science so you can teach it)
Includes index.
ISBN 0-87355-240-7
1. Mathematics—Study and teaching—Popular works. I. Title. II. Series: Robertson, William C. Stop faking it!
Q11.2.R63 2006
510.071—dc22
2005033676

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# Contents

# Preface

The book you have in your hands is the seventh in the *Stop Faking It!* series. The previous six books have been on various science topics, so obviously a book on math is a slight departure from what I have done so far. The focus on understanding rather than memorization, though, remains. When I speak of understanding, I'm not talking about what rules and formulas to apply when, but rather knowing the meaning behind all the rules, formulas, and procedures. I *know* that it is possible for science and math to make sense at a *deep level*—deep enough that you can teach it to others with confidence and comfort.

Why do science and math have such a bad reputation as being so difficult? What makes them so difficult to understand? Well, my contention is that science and math are *not* difficult to understand. It's just that from kindergarten through graduate school, we present the material *way* too fast and at too abstract a level. To truly understand science and math, you need *time* to wrap your mind around the concepts. However, very little science and math instruction allows that necessary time. Unless you have the knack for understanding abstract ideas in a quick presentation, you can quickly fall behind as the material flies over your head. Unfortunately, the solution many people use to keep from falling behind is to *memorize* the material. Memorizing your way through the material is a surefire way to feel uncomfortable when it comes time to teach the material to others. You have a difficult time answering questions that aren't stated explicitly in the textbook, you feel inadequate, and let's face it—it just isn't any fun!

So, how do you go about *understanding* science and math? You could pick up a high school or college science textbook and do your best to plow through the ideas, but that can get discouraging quickly. You could plunk down a few bucks and take an introductory college course, but you might be smack in the middle of a too-much-material-too-fast situation. Chances are, also, that the undergraduate credit you would earn wouldn't do the tiniest thing to help you on your teaching pay scale. Elementary and middle school textbooks generally include brief explanations of the concepts, but the emphasis is definitely on the word *brief*. Finally, you can pick up one or fifty "resource" books that contain many cool classroom activities but also include too brief, sometimes incorrect, and vocabulary-laden explanations.

There is an established method for helping people learn concepts, and that method is known as the Learning Cycle. Basically, it consists of having someone do a hands-on activity or two, or even just think about various questions or situations, followed by explanations based on those activities. By connecting new concepts to existing ideas, activities, or experiences, people tend to develop understanding rather than memorization. Each chapter in this book, then, is broken up into two kinds of sections.

One kind of section is titled, "Things to do before you read the explanation," and the other is titled, "The explanation." If you actually do the things I ask prior to reading the explanations, I guarantee you'll have a more satisfying experience and a better chance of grasping the material.

I do suggest that you read the section titled "About This Book" before you start plowing through the material.

## Dedication

I dedicate this book to the two best math teachers I ever had—Drexel Pope in seventh and eighth grade and Harmon Unkrich in high school.

## About the Author

As the author of NSTA Press's *Stop Faking It!* series, Bill Robertson believes science can be both accessible and fun—if it's presented so that people can readily understand it. Robertson is a science education writer, reviews and edits science materials, and frequently conducts inservice teacher workshops as well as seminars at NSTA conventions. He has also taught physics and developed K–12 science curricula, teacher materials, and award-winning science kits. He earned a master's degree in physics from the University of Illinois and a PhD in science education from the University of Colorado.

## About the Illustrator

The recently-out-of-debt, soon-to-be-famous, humorous illustrator Brian Diskin grew up outside of Chicago. He graduated from Northern Illinois University with a degree in commercial illustration, after which he taught himself cartooning. His art has appeared in many books, including *The Beerbellie Diet* and *How a Real Locomotive Works*. You can also find his art in newspapers, on greeting cards, on T-shirts, and on refrigerators. At any given time he can be found teaching watercolors and cartooning, and hopefully working on his ever-expanding series of *Stop Faking It!* books. You can view his work at *www.briandiskin.com.*

# About This Book

I'm thinking that many of you are sitting back, looking at this book, and wondering why in the world I'm writing it. Sure, it's difficult to understand lots of science concepts, but everyone understands mathematics, right? You add, subtract, multiply, divide, mess around with fractions, solve equations, and all that, and just about everyone who has to teach math knows all the rules. Well yes, just about everyone who teaches math *does* understand the rules, but there's more to math than the rules. Here's one of my favorite examples:

When you add two fractions, you have to get a common denominator. Then in order to add these fractions, you add the numerators of the fractions but don't add the denominators. The denominator of the result is the same as the denominator of the two fractions you're adding. When you multiply two fractions together, however, you don't need to get a common denominator. You simply multiply the numerators together, and then you multiply the denominators together.

Yep, those are the rules for adding and multiplying fractions. Now. . .do those rules make sense? Why do you need a common denominator when adding fractions, but not when multiplying them? Why is it that when adding fractions you add the numerators but not the denominators, yet when multiplying fractions you multiply both the numerators and the denominators?

If the questions above bother you a bit, or if you have never even thought about them, then perhaps this book is for you. There are great reasons behind the rules for adding and multiplying fractions, just as there are great reasons for just about everything you do in math. Many people learn math, however, without ever learning the reasoning behind the rules.

So what's wrong with just knowing the rules? Well, without understanding the reasoning behind the rules of math, chances are you are simply memorizing procedures. If you're memorizing procedures, then chances are you are teaching your students to memorize procedures. There's a big difference between memorizing math and really understanding it. In my humble opinion, when you really understand what's going on in math (or science, or anything else), then you are more comfortable teaching it and might just do a better job of teaching it.

Which brings us to the next point—the broad scope of this book. I begin with adding numbers in Base 10, and end up with calculus. Why does a kindergarten or second grade teacher need to know how to solve algebraic equations, figure out geometric formulas, and be able to do calculus? The short answer is that you don't need to know how to *do* these math calculations, but that you just might benefit from *understanding* these math calculations. When you know more than your students are likely to ask about, you feel more comfortable. On the occasion that your students *do* ask complicated questions, wouldn't it be nice to at least have a basic understanding from which to answer the questions? Answer: yes.

Which brings us to another point. The purpose of this book is to help you gain a deep understanding of the meaning behind the rules and operations of math. The purpose is *not* to ensure that you will be able to *do* the various calculations with proficiency. You won't become an expert at solving algebraic equations, solving geometry problems, or solving calculus problems by going through this book. To become proficient, you need more than understanding; you need lots and lots of practice. Any math course or textbook in the appropriate subject area can guide you through that practice. What I'm trying to do is make that process less painful, should you decide to pursue it.

And another point. Because this math book is part of a science book series, I will usually take the scientist's point of view rather than the mathematician's point of view whenever there is a conflict. As a rule, scientists tend to be just a wee bit less formal in their use of math than are mathematicians. Makes sense, because for the most part math is a tool for scientists. Scientists bend the strict rules of math when it makes sense to do so. For example, division by zero is absolutely undefined in the rules of math, but scientists tend to think in terms of dividing by things so close to zero that those things are essentially equal to zero. So, when they see division by zero, they don't always think that's a terrible thing. That said, many scientific theories have grown from pure mathematics. I don't want to give the impression that math is somehow secondary to science; it isn't. It's just that scientists and mathematicians sometimes view math differently.

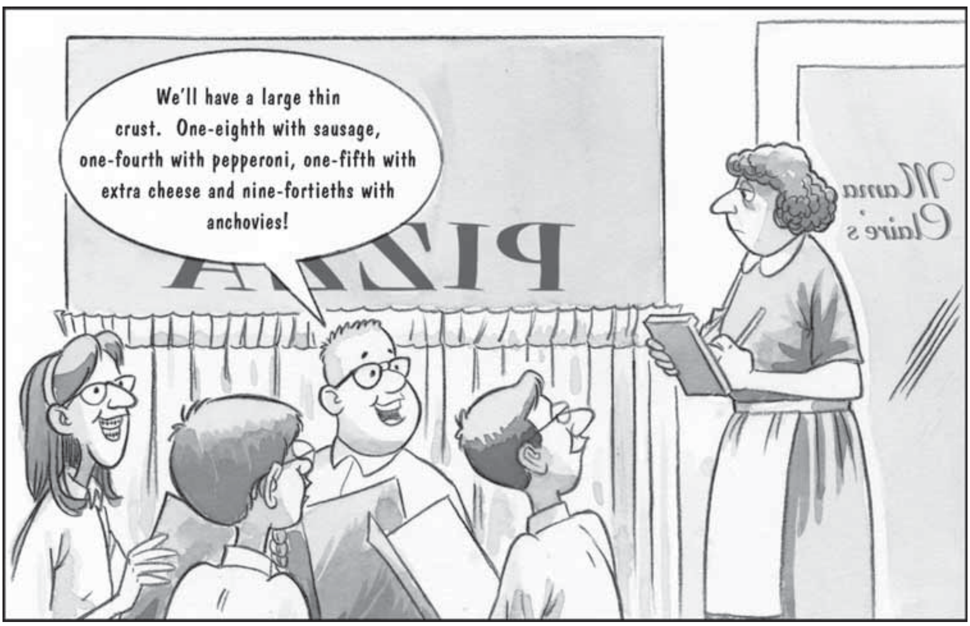
I hope that whatever level of science or math you teach, you find this book useful. If some of the early chapters are second nature to you, by all means ignore them. If some of the later chapters seem like something you'll never use, by all means put them away and maybe revisit them at another time. All I'm trying to do is provide a perspective on math that, all too often, doesn't find its way to our students.

Finally, as you go through this book, you will notice a number of text boxes that are labeled *Guidepost*. My most valuable reviewer, my wife, complained that as she went through the text she sometimes got lost as to the purpose of a given section. To make things clearer, I added these Guideposts. They are there to remind you of the purpose of an activity or what exactly I'm explaining at a given point. Think of them as a tool for helping you stay on the same road I'm traveling.

Chapter 2

# Fractions and More Rules

This chapter deals with fractions and decimals. This is one place where people who are otherwise fine with math jump off the boat and decide it's just too weird to continue. As with everything else, though, it's only a matter of understanding what's going on. Try to memorize your way through, and you're in trouble. Insist that everything makes sense, though, and you're fine.



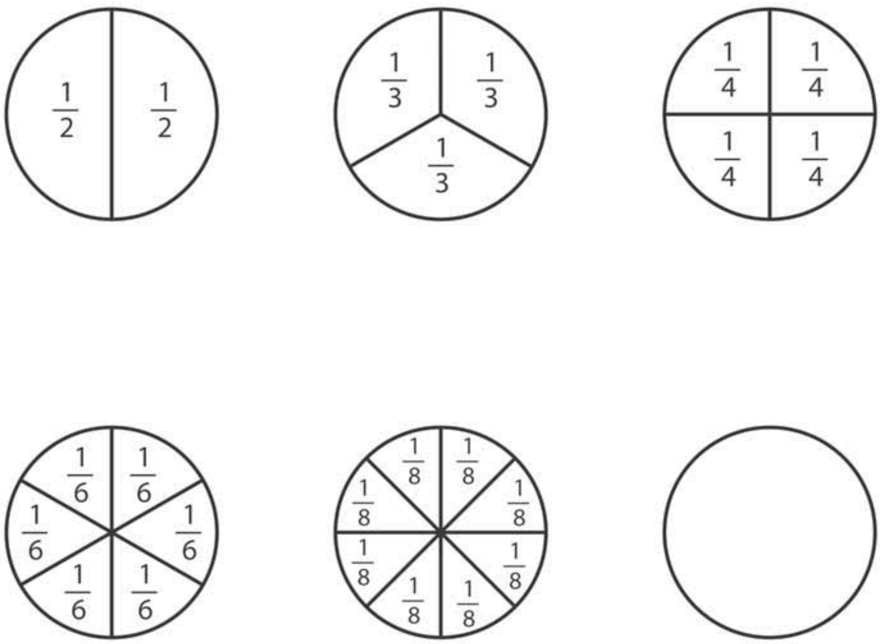

Rose dreaded the Math Club's "Pizza Night."

## Things to do coupled with explanations

I usually have "things to do" sections followed by "explanation" sections. This chapter doesn't lend itself to that format, so rather than separate the things to do from the explanation, it will be easier to understand if we explain as we go along.

For starters, take a look at the circles below in Figure 2.1. In fact, you should make a copy of this page, because you're going to be cutting the circles out and you probably don't want to ruin the book.

**Figure 2.1**




Each circle, except for the blank one, is divided into fractional pieces. For example, the second one is divided into three equal-sized pieces we call thirds. Each section is one third of the whole circle, and we write that fraction as $\frac{1}{3}$. This is a physical representation of a fraction, and this representation is the one most people think of when they think of fractions. It's pretty easy to picture one-half of an apple, one-fourth of a glass of water, or two-fifths of a candy bar. Fractions also represent the procedure of division, though. The fraction $\frac{1}{3}$ is the number one divided by the number three, so $\frac{1}{3}$ is the same as $1 \div 3$.

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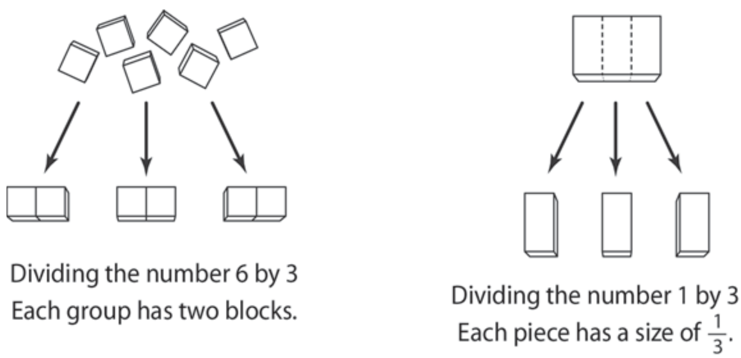
| **Guidepost** Different ways to picture fractions. |

Previously we pictured division as separating a large collection of blocks into smaller groups, but here we have the number 1, which would be 1 block, divided into three groups. In order to do that, we have to break the block into smaller pieces. That's essentially what the second circle in Figure 2.1 shows. You can't divide one circle into three groups of circles, but you can divide it into three separate pieces. Figure 2.2 compares everyday division with the division represented by a fraction.

**Figure 2.2**

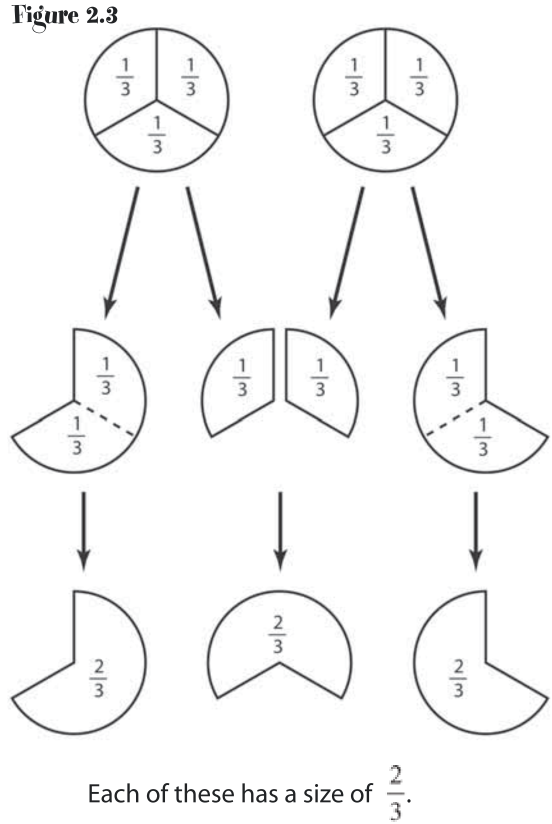



OK, what about a fraction like $\frac{2}{3}$? How do you divide two whole things into three "groups?" Pretty much the same way you divide one thing into three groups, or rather pieces. Figure 2.3 shows two circles divided into three equal groups. Each group contains two pieces of size $\frac{1}{3}$, and represents two divided by three, or $\frac{2}{3}$.

Figure 2.3 shows that a piece of a circle that has a size of $\frac{2}{3}$ is equivalent to two pieces with size $\frac{1}{3}$. This again points out the different ways we can think of fractions. One is as a simple division (the number 2 divided by the number 3) and one is as a physical entity (a section whose size is equivalent to that of two pieces, each one-third in size).

Before moving on, I should explain the terminology used for fractions. Whatever is on the top of the fraction is

**Figure 2.3**

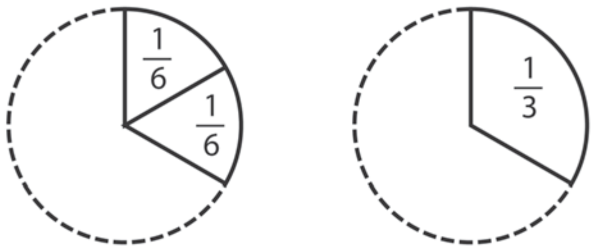


called the **numerator** and whatever is on the bottom of the fraction is called the **denominator**.

Time to cut out all the circle pieces. Find a piece that is exactly half the size of a $\frac{1}{3}$ piece.

That would be a $\frac{1}{6}$ piece, as shown in Figure 2.4.

**Figure 2.4**




A $\frac{1}{6}$ piece is exactly one half size of a $\frac{1}{3}$ piece.

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| **Guidepost** The rule for multiplying fractions—making sense of it. |

This illustrates that "half of $\frac{1}{3}$ is $\frac{1}{6}$." This also illustrates what it means to multiply fractions. Another way to represent "half of" something is to multiply the something by $\frac{1}{2}$. In other words, $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. This also tells us the rule for multiplying fractions. You multiply the numerators to get the numerator of the answer ($1 \cdot 1 = 1$) and you multiply the denominators to get the denominator of the answer ($2 \cdot 3 = 6$). And this also fits with our earlier representation of multiplication as a series of additions. $4 \cdot 5$ means add 5 four times. $\frac{1}{2} \cdot \frac{1}{3}$ means add $\frac{1}{3}$ one half of a time, or simply take one half of $\frac{1}{3}$.

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| **Guidepost** Understanding equivalent fractions and how to convert from a fraction to its equivalent. |

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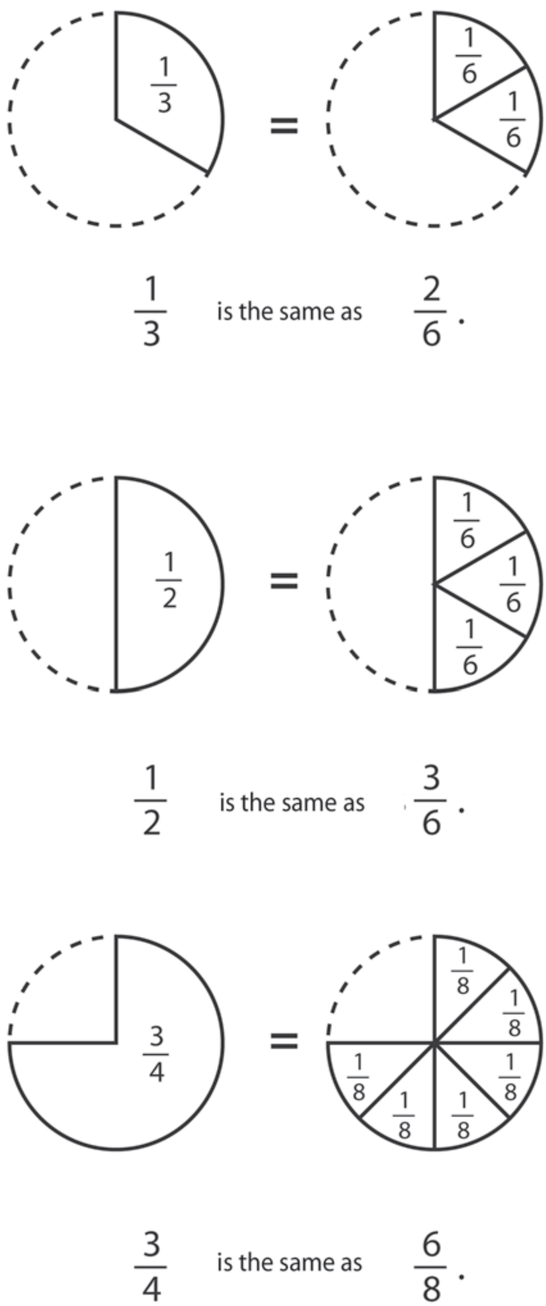
| When you multiply fractions in math problems, I'm not expecting that you'll break the process down like this. You will simply multiply the numerators and denominators, following the rule. You should always, however, be able to stop and explain *why* that's the proper rule. When teaching, you should stop every once in a while and ask the students *why* the rule makes sense. If they can't give an answer, then I suggest it would be worthwhile to stop and talk about the reasoning behind the rule. I further suggest that this would be appropriate all the way through the college level. Too often we think that illustrating fractions with pie slices and reasoning through rules is "baby stuff" that isn't appropriate once kids get to, say, the fifth grade and beyond. My position is that we don't do nearly enough of that baby stuff at all levels. |

Take two of your $\frac{1}{6}$ pieces and place them on the second circle in Figure 2.1. Convince yourself that $\frac{2}{6}$, which is two pieces of size $\frac{1}{6}$, is equivalent to a piece of size $\frac{1}{3}$. Similarly, convince yourself that $\frac{1}{2}$ is equivalent to $\frac{3}{6}$, and that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$. See Figure 2.5.

Figure 2.5




We can show that these fractions are equivalent just using math symbols. First, you have to recall that anything multiplied by 1 is just the thing you started with. So, you can multiply by 1 all day long and not change the value of a fraction. Second, you have to recall that anything divided by itself is equal to 1, as in $\frac{2}{2} = 1$, $\frac{5}{5} = 1$, and $\frac{\text{a squid}}{\text{a squid}} = 1$. Having recalled that, let's multiply the fraction $\frac{1}{3}$ by the number 1.

$$\frac{1}{3} \cdot 1 = \frac{1}{3}\text{, right?}$$

Now let's rewrite the number 1 as $\frac{2}{2}$, which we can do because 2 divided by 2 is equal to 1. Now we have

$$\frac{1}{3} \cdot 1 = \frac{1}{3} \cdot \frac{2}{2}.$$

Using our rule for multiplying fractions, we have

$$\frac{1}{3} \cdot \frac{2}{2} = \frac{2}{6}.$$

This confirms what you discovered with the circle pieces, namely that $\frac{1}{3}$ and $\frac{2}{6}$ represent the same fraction. This also illustrates how we can change the denominator in a fraction while making sure the fraction has the same value. Just multiply by 1, disguised as a number divided by itself. For example, you saw with the circle pieces that $\frac{1}{2}$ is the same as $\frac{3}{6}$. We can do the same thing by multiplying $\frac{1}{2}$ by $\frac{3}{3}$, as in

$$\frac{1}{2} \cdot \frac{2}{2} = \frac{3}{6}.$$

You also saw with the pieces that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$, something we can do with math symbols as

$$\frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}.$$

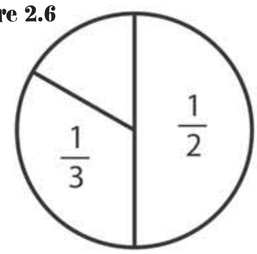
OK, we've covered what fractions are, how to multiply fractions, and how to change the denominator of a fraction by multiplying the entire fraction by 1, disguised as a number divided by itself.

Let's move on to adding fractions, starting with the rule for doing so, which you can find in any math textbook.* The rule is that you need to get a common denominator for the fractions, followed by adding the numerators and keeping the denominators the same. Do you really need to get a common denominator, though?

> **Guidepost** Understanding why you're supposed to get a common denominator when adding fractions.

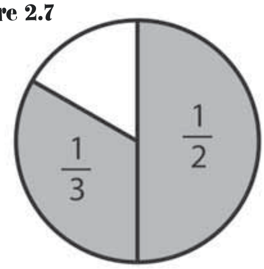
Check it out using the circle pieces. On a blank circle, place a $\frac{1}{2}$ piece and a $\frac{1}{3}$ piece, as in Figure 2.6.

**Figure 2.6**




This is a physical representation of $\frac{1}{2} + \frac{1}{3}$, so clearly we *can* add fractions that have different denominators. There's a problem, though. Unless you have carefully drawn circles and can measure exactly the fraction of the circle taken up by these two pieces, you don't have an exact answer for $\frac{1}{2} + \frac{1}{3}$. See Figure 2.7.

**Figure 2.7**




It's not obvious how much of the circle is shaded.

Our problem, then, is that we can add fractions that have different denominators, but we don't necessarily arrive at an exact answer. We can, however, get an exact answer if the fractions have the same denominator. For example, $\frac{1}{6} + \frac{2}{6} = \frac{3}{6}$, which you can verify using your cut-out circles from Figure 2.1. This kind of fraction addition also makes common sense. If you have one of something (a sixth, in this case) and add two of those somethings

* In much of this book, I try to use activities to *develop* the rules of math. I'm starting with the rule this time for a change of pace, so you can see how we can justify the rule while making sure it makes sense. Of course, I'm also assuming you already know this rule, and many other rules, of math.

(two-sixths) to it, you end up with three of the somethings (or three-sixths).

The solution to the problem of adding fractions with unlike denominators, therefore, is to transform the fractions so they have a common denominator. We do that by multiplying each fraction by 1, disguised as one number divided by itself. Let's use that idea to tackle $\frac{1}{2} + \frac{1}{3}$.

If we multiply $\frac{1}{2}$ by $\frac{3}{3}$, we get $\frac{3}{6}$.

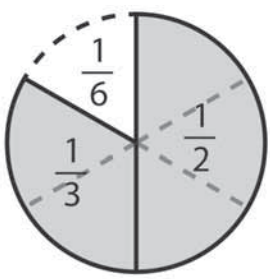
If we multiply $\frac{1}{3}$ by $\frac{2}{2}$, we get $\frac{2}{6}$.

Recalling that $\frac{3}{3}$ and $\frac{2}{2}$ are both equal to 1, and that multiplying anything by 1 doesn't change its value, we get

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} \cdot \frac{2}{2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

which means that our exact answer is $\frac{5}{6}$. You can verify this answer using pieces of a circle by placing the $\frac{1}{2}$ piece and the $\frac{1}{3}$ piece on the fourth circle in Figure 2.1 (the one divided into sixths). See Figure 2.8.

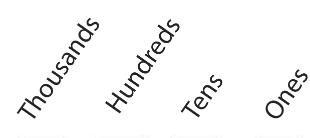
Figure 2.8




Just as with multiplication, I don't expect that you will go through this reasoning process each time you add fractions or ask your students to add fractions. I would suggest, however, asking students at any level of math, all the way up to college courses, *why* they have to get a common denominator when adding and subtracting fractions. I'm betting most won't know. You might get answers such as "because you can't add apples and oranges" or "because they have to be the same," but that's just restating the rule. Of course, you already know that you *can* add fractions without a common denominator (Figure 2.7). The only reason you need a common denominator is so you can get an exact answer.

The last thing I'm going to deal with in this section and in this chapter is the idea of using decimals rather than fractions. As you'll hopefully discover, decimals and fractions are just different notations for the same thing. Decimals really are fractions. To get started on this, let's expand our previous idea of place value in Base 10. Earlier, Base 10 place value was shown as in Figure 2.9.

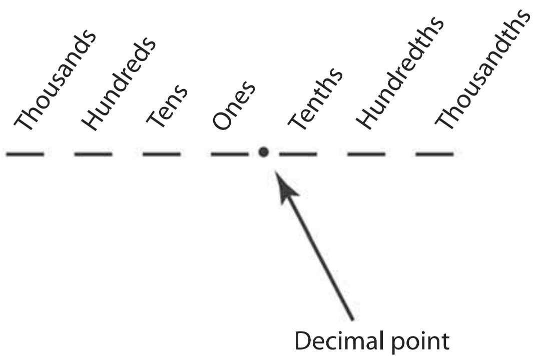
Figure 2.9




**Guidepost** Understanding decimals and their relationship to fractions.

This figure isn't complete, though, because as we all know, there is a decimal point and there are place values to the right of the decimal point. Figure 2.10 shows a more complete picture of Base 10 place value.

**Figure 2.10**




The places to the left of the decimal point represent ones, tens, hundreds, etc. The places to the right of the decimal point represent the fractions $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc. So, the number 456.291 means you have 4 thousands, 5 hundreds, 6 ones, 2 tenths, 9 hundredths, and 1 thousandth. If we were to represent this number using fractions rather than decimals, it would be $456 + \frac{2}{10} + \frac{9}{100} + \frac{1}{1000}$. So, the only thing special about decimals is that they are shorthand for fractions, using Base 10.

OK, how do we go about converting a fraction like $\frac{3}{4}$ to a decimal? Well, we use the calculator and divide 3 by 4, ending up with 0.75. What might be more instructive, however, is to see what happens when we use long division. You remember long division, don't you? It's that process all of us used before calculators. Anyway, let's divide 3 by 4, using long division. We start with the following familiar look:

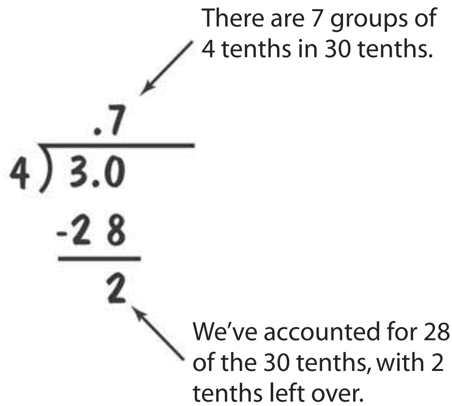
$$4\overline{)\,3\qquad}$$

Those symbols are basically asking you how many ones you get when you divide 3 by 4. The answer is that you don't get any ones, because 3 divided by 4 is less than 1. The next step is to convert those 3 ones to 30 tenths, which means adding a decimal point and a zero to keep track of the fact that we're now dealing with tenths.

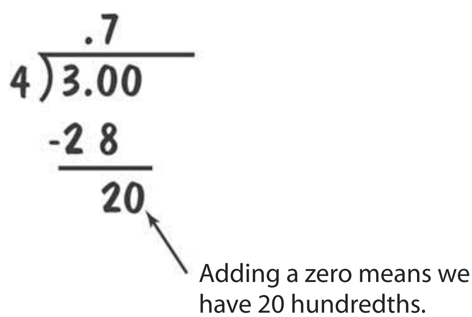
$$4\overline{)\,3.0\qquad}$$

Adding a zero means we have 30 tenths.

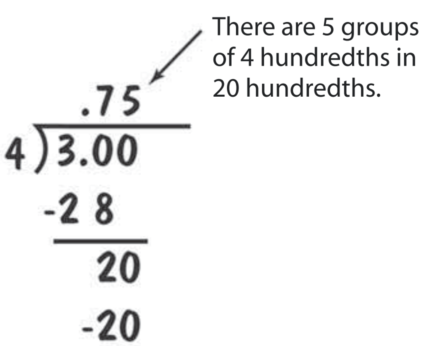
Now we're dividing 30 tenths by 4, asking how many groups of 4 tenths there are in 30 tenths. Well, there are 7 groups of 4 tenths in 30 tenths, with a couple of tenths left over. In our long division, it looks like

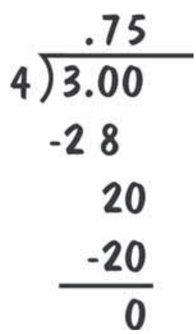



Now, since there are only 2 tenths left over, we can no longer divide them into groups of 4 tenths. Therefore, we rewrite the 2 tenths as 20 hundredths.



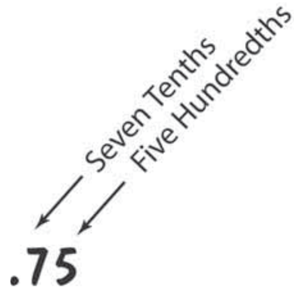

Now we're dividing 20 hundredths by 4, asking how many groups of 4 hundredths there are in 20 hundredths. The answer is 5, so we get



We're done, because there are no hundredths left over. We've converted the fraction three-fourths to seven-tenths plus 5 hundredths. See Figure 2.11.

**Figure 2.11**




And that's all there really is to converting fractions to decimals. Converting decimals to fractions is also pretty easy. 0.75 is 7 tenths and 5 hundredths, or $\frac{75}{100}$. Most math books cover pretty well what comes next, which is the process of "reducing fractions." We can rewrite $\frac{75}{100}$ as $\frac{3 \cdot 25}{4 \cdot 25}$ which, by the reverse of our rule for multiplying fractions, is equal to $\frac{3}{4} \cdot \frac{25}{25}$. Because $\frac{25}{25}$ is equal to 1, this expression is equal to $\frac{3}{4}$. By the way, this is the process behind the procedure known as "canceling." Because we can remove the $\frac{25}{25}$ from the above expression by using the reverse of fraction multiplication, we can take a shortcut and just cancel the 25s, as in $\frac{3 \cdot \cancel{25}}{4 \cdot \cancel{25}}$.

## Chapter Summary

- You can picture fractions as portions of a whole or as simple representations of division.
- You do not change the value of a fraction, or any other term for that matter, when you multiply it by the number 1.
- Anything divided by itself is equal to 1.
- When you multiply fractions, you multiply the numerators and multiply the denominators. There's a reason for that procedure.
- You *can* add fractions that have unlike denominators, but to get an exact answer you must first find a common denominator for the fractions.
- Decimals are a way of representing fractions and mixed numbers with Base 10 notation.

## Applications

1. There's a rule for converting mixed numbers, such as $4\frac{2}{3}$, to what is called an "improper fraction." What you do is, in this case, multiply the 3 and 4, add the 2, and put the result as a numerator over the denominator of 3. Thus, $4\frac{2}{3}$ is equal to $\frac{3 \cdot 4+2}{3}$, which is equal to $\frac{14}{3}$. OK, fine, but why is that the rule? Does it make sense? Yes, it does, but it takes a bit of thinking to figure it out. First, $4\frac{2}{3}$ is read "4 *and* $\frac{2}{3}$" or "$4 + \frac{2}{3}$." In order to add these together, you have to rewrite the 4 as a fraction with a denominator of 3. Since there are three 3s in the number 1, there are twelve 3s in the number 4. Therefore, you can write the number 4 as $\frac{12}{3}$. Now we add $\frac{2}{3}$ to the 4, which is $\frac{12}{3} + \frac{2}{3}$. We have a common denominator of 3, so we can write this as $\frac{12+2}{3}$, or $\frac{14}{3}$.

2. We covered the rule for multiplying fractions in this chapter, making sense of it, so this should be a simple application. Let's multiply $\frac{6}{1}$ by $\frac{1}{6}$. We multiply the numerators and multiply the denominators, so $\frac{6}{1} \cdot \frac{1}{6} = \frac{6 \cdot 1}{1 \cdot 6} = \frac{6}{6}$. Because we know that anything divided by itself is equal to 1, we know that $\frac{6}{6}$ is equal to 1. So, $\frac{6}{1} \cdot \frac{1}{6}$ is equal to 1. There is a special name for two numbers that, when multiplied together, equal 1. Those numbers are **reciprocals** of each other. Finding the reciprocal of a fraction is pretty easy. Just invert the fraction so the numerator and denominator are reversed, and you have the reciprocal. The reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$. The reciprocal of $\frac{15}{27}$ is $\frac{27}{15}$. Reciprocals are a nice thing to know about when you start solving math equations.

3. In various science applications there's a procedure known as the "factor label method." It's a fancy name that describes how one changes units associated with quantities. For example, consider something like 62 hours. The "hours" represent the units associated with the number 62. Suppose we want to know how many seconds are equivalent to 62 hours. Well, first we can convert to minutes by multiplying 62 hours by the number 1, disguised as $\frac{60 \text{ minutes}}{1 \text{ hour}}$. This fraction is equal to 1 because 60 minutes and 1 hour are the same thing. So, we have 62 hours $\cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$, or $\frac{(62 \text{ hours}) \cdot (60 \text{ minutes})}{1 \text{ hour}}$.

   Now we can rewrite the 62 hours as $62 \cdot (1 \text{ hour})$ and end up with $\frac{62 \cdot (1 \text{ hour}) \cdot (60 \text{ minutes})}{1 \text{ hour}}$.

   The next thing to do is cancel the "1 hour" terms, because $\frac{1 \text{ hour}}{1 \text{ hour}}$ is equal to 1. We get $\frac{62 \cdot (\cancel{1 \text{ hour}}) \cdot (60 \text{ minutes})}{\cancel{1 \text{ hour}}}$ or just $62 \cdot (60 \text{ minutes})$, which equals 3720 minutes. To convert this further to seconds, we multiply 3720 minutes by $\frac{60 \text{ seconds}}{1 \text{ minute}}$. Because there are 60 seconds in a minute, this again is the same as multiplying by 1. Just as with the hours units earlier, the minutes units now cancel, and we're left with $3720 \cdot (60 \text{ seconds})$, which equals 227,200 seconds.

Now, in using the "factor-label method," there's a shortcut way to do a conversion like this. You simply write 62 hours $\cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}}$. The hours units and minutes units all cancel, you multiply all the numbers together, and you end up with units of seconds. Many students view this factor-label method as some kind of magic trick, but all you're doing is multiplying by the number 1, disguised as one quantity divided by an equivalent quantity, over and over. Let's convert 1000 miles to meters using the full-blown shortcut, and relying on the fact that there are 0.6 kilometers in a mile and 1000 meters in a kilometer.

$$1000 \text{ miles} \cdot \frac{0.6 \text{ kilometers}}{1 \text{ mile}} \cdot \frac{1000 \text{ meters}}{1 \text{ kilometer}}$$

equals $$1000 \cancel{\text{miles}} \cdot \frac{0.6 \cancel{\text{kilometers}}}{1 \cancel{\text{mile}}} \cdot \frac{1000 \text{ meters}}{1 \cancel{\text{kilometer}}}$$

equals 600,000 meters.

4. I dealt with the multiplication of fractions in this chapter, but not the division of fractions. Check out any math book and you'll find that the rule for dividing by a fraction is to "invert and multiply." This means that if you have $\frac{4}{5} \div \frac{2}{3}$, you should invert the $\frac{2}{3}$ to form $\frac{3}{2}$, and then do the multiplication $\frac{4}{5} \cdot \frac{3}{2}$, resulting in $\frac{12}{10}$, or $\frac{6}{5}$, or $1\frac{1}{5}$. That's the correct answer, but why does this rule work? To understand, let's use a simpler example and use our "definition" of division involving separating things into piles of blocks or pieces of blocks.

Consider $\frac{13}{2} \div \frac{1}{2}$.

The rule says to invert the $\frac{1}{2}$ and multiply, meaning you end up with $\frac{13}{2} \cdot \frac{1}{2}$, which equals 13. Let's see if that makes sense in terms of our earlier definition of division. When you divide $\frac{13}{2}$ by $\frac{1}{2}$, you are in essence asking how many $\frac{1}{2}$s there are in $\frac{13}{2}$. Well, there are 13 halves in $\frac{13}{2}$, so we end up with the answer we get by following the rule. When you're doing something more complicated like $\frac{4}{5} \div \frac{2}{3}$, ending up with $1\frac{1}{5}$, the process is harder to follow. The answer does make sense, though. Because $\frac{2}{3}$ is smaller than $\frac{4}{5}$, it makes sense that there should be more than one $\frac{2}{3}$ in the fraction $\frac{4}{5}$. As with many other math rules, "invert and multiply" is a useful shortcut. If you don't understand the reasoning behind this rule, though, then you really don't understand what you're doing. You are simply following rules for the sake of following rules.

5. Here's another rule. When you multiply a number by 10, you simply move the decimal point to the right one place. When you divide by 10, you move the decimal point to the left one place. Multiplying or dividing by 100

moves the decimal point two places, multiplying or dividing by 1000 moves the decimal point three places, and so on. Convenient rule, but why does it work? Because we use the Base 10 number system. Let's look at the number 23.74. Knowing our place values, we know that this number represents 2 tens + 3 ones + 7 tenths + 4 hundredths. If you multiply this number by 10, then you multiply the 2 tens (20) by 10, multiply the 3 ones (3) by 10, multiply the 7 tenths (0.7) by 10, and multiply the 4 hundredths (0.04) by 10. $20 \cdot 10$ is equal to 200 (add up 10 groups of 20 each), 3 times 10 is 30, 0.7 (which is the same as the fraction $\frac{7}{10}$) times 10 is equal to 7, and 0.04 (which is the same as the fraction $\frac{4}{100}$) times 10 is equal to $\frac{4}{10}$, or 0.4. We end up with the number represented by 2 hundreds, 3 tens, 7 ones, and 4 tenths. We write that as 237.4, which corresponds to moving the decimal point one place to the right in the number 23.74. I'll leave it up to you to figure out why dividing by 10 simply moves the decimal point one place to the right. And once again, we have a shortcut accompanied by the reasoning behind the shortcut. In case you can't tell, that's basically the theme of this book.

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