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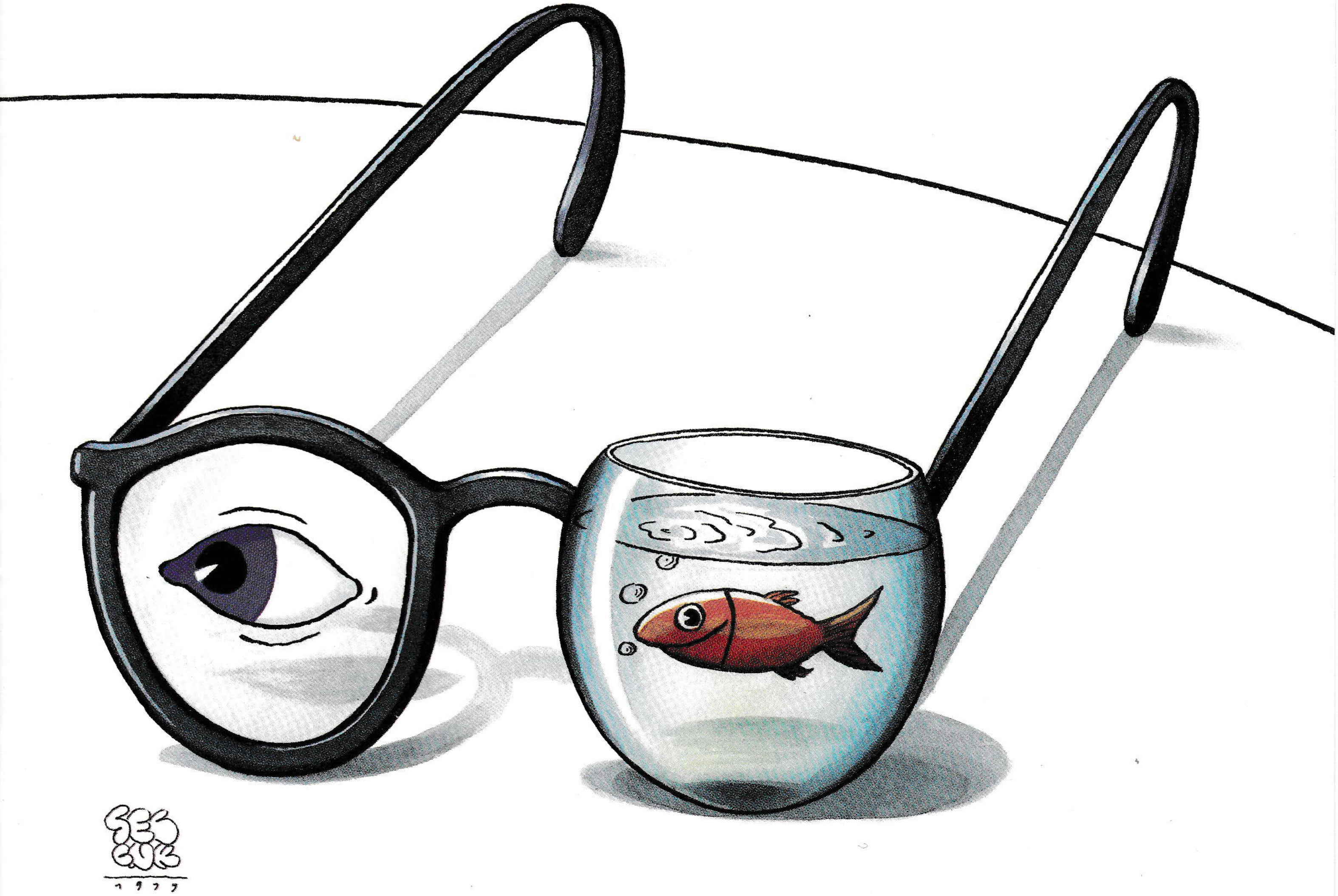

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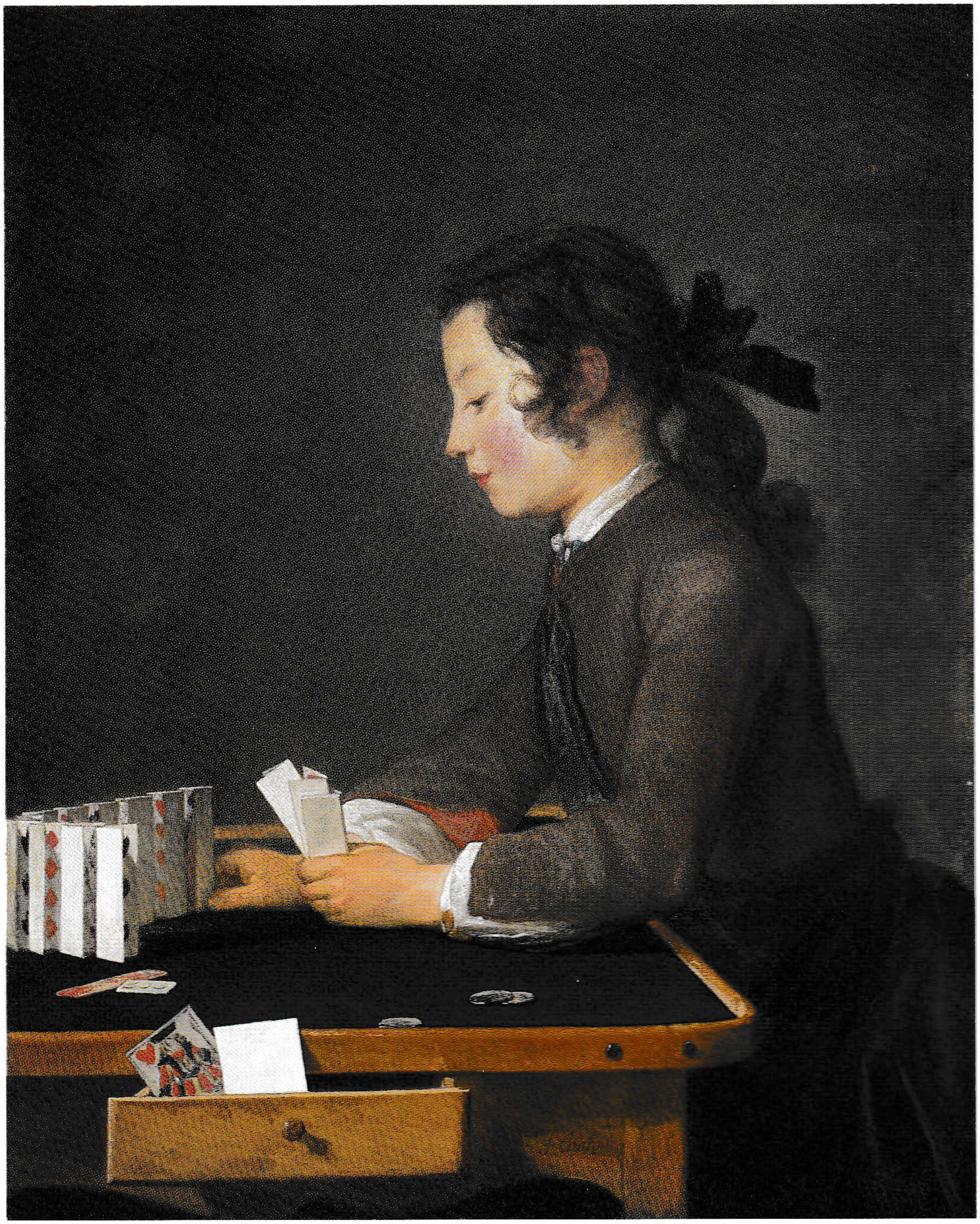
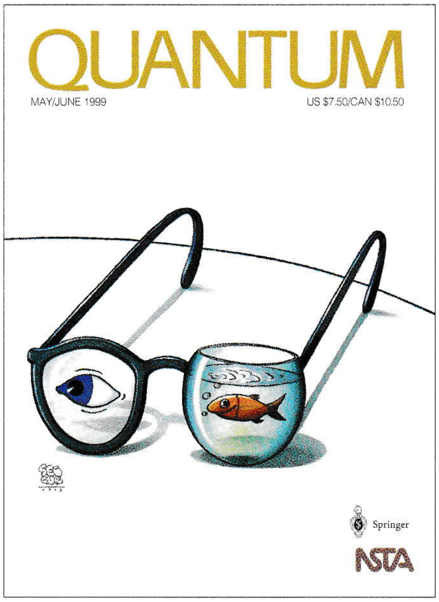


The house of cards (c. 1735) by Jean Siméon Chardin

CLEANING UP AFTER A PARTY, A YOUNG SERvant tests his fine motor skills. According to the National Gallery of Art, the painting "points to idleness and the vanity of worldly constructions." Indeed, we know from experience that after a period of time, things tend to fall apart. For an examination of various physical systems for which we can quantify the stability of equilibrium, turn to page 4.

QUANTUM

MAY/JUNE 1999 VOLUME 9, NUMBER 5



Cover art by Selçuk Demirel

If human eyes operated like fish eyes, having to move the lenses back and forth to focus an image, how bug-eyed or socket-faced would people look when attempting to sharply focus the images of objects at various distances? Eyeball this and other challenges in this month's Physics Contest on page 30.

The eyes may not have it in this month's Kaleidoscope, but the laws of optics rule. Turn to page 28 to test your ray-tracing savvy.

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FEATURES

DEPARTMENTS

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QUANTUM

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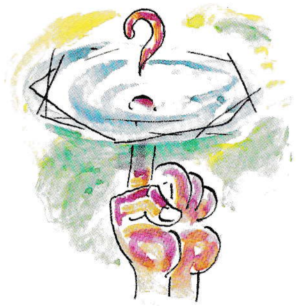
Just for the fun of it!

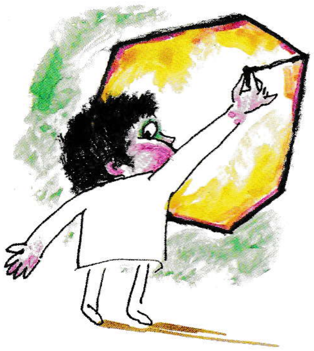


B261

Possible polygons. Does a convex polygon exist that has neither line symmetry nor point symmetry, but such that when it is rotated around some point by an angle less than 180°, it returns to its original position?

B262

Taking the prize. In a mathematical Olympiad, 100 students were offered four problems to solve. The first problem was solved by exactly 90 students, the second by exactly 80 students, the third by exactly 70, and the fourth by exactly 60. No participant solved all four problems. Students who solved both the third and fourth problems were awarded a prize. How many students received a prize?



B263

Nontriangulation. What is the maximum number of diagonals that can be drawn in a convex heptagon (convex 7-gon) so that no triangle is created by these diagonals, whose vertices are also vertices of the heptagon?

B264

Game show gambit. A participant in a game show is offered a choice of one of three boxes, only one of which contains a prize. After the participant chooses a box, the emcee opens one of the other two boxes and shows that it is empty. Then the emcee asks, "Do you want to stick with your choice, or would you rather pick the other box?" Naturally one wants to maximize the odds, but will switching do it? What would you do? Make your decision—and justify your answer!

B265

Pressure in a bottle. An unsealed bottle sank in the ocean to a depth of several kilometers. Does the holding capacity (storage) of the bottle increase or decrease due to the huge hydrostatic pressure at the ocean floor?

ANSWERS, HINTS & SOLUTIONS ON PAGE 52

Art by Pavel Chernusky

When things fall apart

A small tribute to Humpty-Dumpty, Bryan Berg, and many others

by Boris Korsunsky

HOW DO YOU COMBINE PLAYing cards and playing physics? Well, how about building a house of cards, say, the tallest house of cards in the world? The competition would be tough. In 1995 an 83-story "monster" was created in Boston. That world-record house of cards was almost 5 meters tall!

Humans have always been fascinated by the art of fine balancing. Bryan Berg, who built the "ceiling-scraper" card structure, became famous. So did John Evans, a Brit who balanced 66 bricks on his head. So did Ashley Brophy of Australia, who managed to walk almost 12 kilometers in just over three hours. Oh, did I mention that she was walking on a tightrope?[1]

What do these accomplishments have in common? Among other things, they all deal with the stability of equilibrium. And this is exactly the subject of this article. Whether a particular equilibrium is stable makes all the difference in the world: You would not want to live in Berg's house, and you would think more than twice before accompanying Brophy on her morning

Art by Sergey Ivanov

[1] All of these facts are taken from *The Guinness Book of Records*, Guinness Publishing, 1996.

walk, right? And we all know what happens if an unstable equilibrium is violated: All the king's horses and all the king's men may not be able to help . . .

In most high school physics courses, however, the concept of stability is barely touched. Usually, you solve a few problems dealing with the general conditions for equilibrium, and as far as *stability* goes, you take a quick look at the picture of a little ball on a hill and that same ball in a pit, and that is it. This is unfortunate. There are many interesting problems dealing with this concept not only in mechanics but also in gas laws, electricity, hydrostatics—everywhere forces are involved.

I offer a few little-known problems and exercises that focus on the stability of equilibrium in very different systems, rather than equilibrium itself. In the first problem we analyze unstable equilibrium, and in the others we apply the conditions for stable equilibrium.

A Batman trick

Suppose you want to balance a baseball bat on the tip of your finger. Why is it easier to do with the thick end up?

To answer, we have to consider what it takes to retain balance. Imagine that your hand shakes a bit, and the bat tilts from the vertical. Naturally, you have to move your finger to bring the bat back into equilibrium. Why does it help to have the thick end up?

Consider a model of this situation. If a light rod with a heavy ball attached as shown in figure 1 is pivoted at the bottom, we can calculate its angular acceleration at any given moment. Using the common symbols in Newton's second law for rotation, we have

$$\alpha = \frac{\tau}{I} = \frac{mgl\sin\theta}{ml^2} = \frac{g}{l}\sin\theta.$$

As we can see, the acceleration is less if the ball is attached closer to the top of the rod (in other words, if the center of mass is farther from the pivot point). This explains why the bat falls

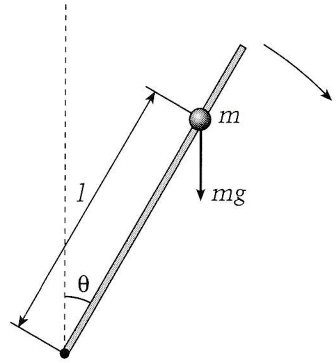


Figure 1

more slowly with the heavy end up and you have more time to adjust the position of the supporting finger. The bat is more stable (or, rather, less unstable) in this position.

Now let's take a closer look at some "stable" systems. As we know, equilibrium is stable if small deviations produce a net force (or a net torque) toward the equilibrium position. The following examples illustrate this point.

A swing dance

In figure 2, a rod of mass m *is held vertically by an unknown mass* M *hanging over a pulley as shown. Assuming* h *and* H *are known, find the minimum value of* M *that makes the equilibrium stable.*

To find the answer, consider the net torque on the rod corresponding to a small deviation, as shown in figure 3. The rod will return to the vertical if the torque due to the tension is greater than the torque due to

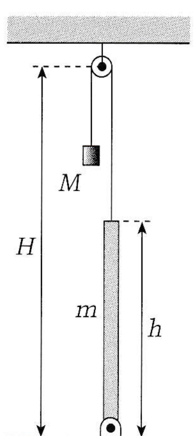


Figure 2

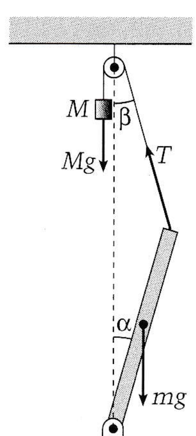


Figure 3

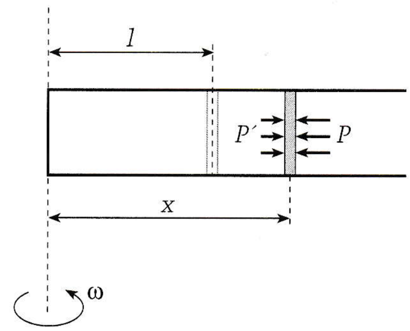


Figure 4

gravity. In the "critical" case, these torques must be equal:

$$MgH\sin\beta = mg\frac{h}{2}\sin\alpha.$$

From the law of sines,

$$\frac{\sin\beta}{h} = \frac{\sin(\alpha+\beta)}{H}.$$

Or, noticing that both α and β are small,

$$\beta = \frac{\alpha h}{H-h},$$

and

$$M = m\frac{H-h}{2H}.$$

A swirled smoothie

A test tube is filled with air. A movable piston of mass m *and cross-section* A *is inserted in the test tube a distance* l *from the sealed end. Initially, the air pressure on both sides of the piston is* P. *Then the tube is rotated at an angular speed* ω *about the vertical axis as shown in figure 4. Find the new equilibrium distance between the piston and the sealed end of the test tube. Assume constant temperature and no friction.*

If the new pressure between the piston and the sealed end is P', then Newton's second law for the piston can be written

$$m\omega^2 x = (P - P')A.$$

Also, since PV = constant for the air inside the tube, $P'x = Pl$. Combining these equations and solving the resulting quadratic equation

$$\frac{m\omega^2}{PA}x^2 - x + l = 0,$$

we get two answers (to make them look better let $k = 2m\omega^2$):

$$x_{1,2} = \frac{PA}{k}\left(1 \pm \sqrt{1 - \frac{2kl}{PA}}\right).$$

Analysis shows that, if solutions exist, the positive root corresponds to unstable equilibrium and the negative root corresponds to stable equilibrium. (Hint: consider graphs of

$$y = x$$

and

$$y = \frac{m\omega^2 x^2}{PA} + l;$$

their intersection points correspond to the roots.) In other words, both answers are legitimate, but only the smaller one can be observed "in real life."[2]

The charge in charge

A small particle of charge q *and mass* m *is placed at the top point of the inner surface of a smooth sphere of diameter* d. *What minimum charge* Q *must be placed at the bottom point of the sphere to keep charge* q *(a) in equilibrium? (b) in stable equilibrium?*

This problem is a wonderful illustration of the difference between stable and unstable equilibrium. Part (a) is fairly easy: the electrostatic force acting on charge q must be greater than or equal to the force of gravity, so, for the minimum value of Q,

$$\frac{kqQ}{d^2} = mg$$

or

$$Q = \frac{mgd^2}{kq},$$

[2] A detailed discussion of quadratic equations in physics problems can be found in my article "Fair and Squared!" *Quantum*, May/June 1997. Problem 5 in that article especially resembles this one.

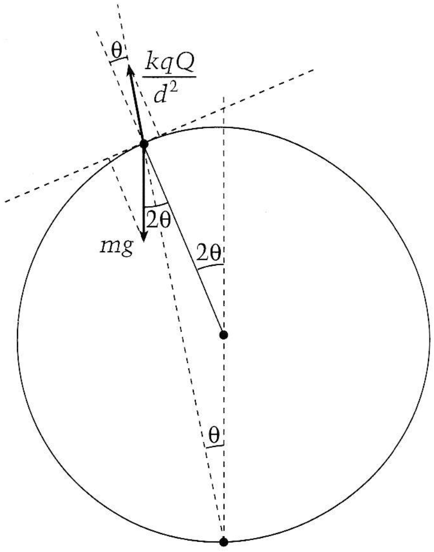


Figure 5

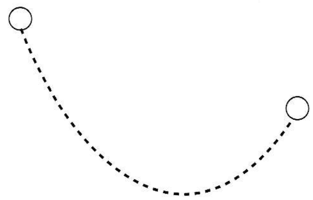


Figure 6

where k is the coefficient in Coulomb's law.

The second part of the problem is more interesting: We have to consider the net force on charge q corresponding to a small deviation from equilibrium. It is convenient to consider the components of all forces along the tangent to the sphere (fig. 5):

$$\frac{kqQ}{d^2}\sin\theta = mg\sin 2\theta.$$

(Again, the net force is zero for the minimum value of Q). Since q is small, one can replace $\sin\theta$ by θ, and the answer is

$$Q = 2\frac{mgd^2}{kq},$$

twice the value obtained in part (a)!

Examples 1 and 2 demonstrated the use of torques; examples 3 and 4 utilized forces. However, there is another way to determine whether an equilibrium situation is stable: energy considerations. Every system in the Universe "wants" to lower its potential energy. One of the conclusions of this ubiquitous law of nature is that a system's equilibrium is stable if it corresponds to a local minimum of its potential energy. Our last problem nicely demonstrates this idea.

A chain reaction

A flexible chain is attached to two nails driven into the wall as shown in figure 6. Suppose you pull the chain "downward" somewhere near the midpoint. Does the center of mass of the chain go up or down? Hint: Note the word somewhere.

Of course, it would not be wise to try to locate the center of mass analytically. In fact, the conditions of the problem are so vague—it is just impossible. So it may come as a surprise that a definite answer can be given. Furthermore, it may not be the answer that you would expect: The center of mass goes up even though you pull the chain down. Since the chain is in stable equilibrium before it is pulled, its center of mass must be in the lowermost position to provide the lowest potential energy, so up is the only way for it to move! (Of course, some sideways displacement may also occur).

I have picked problems that illustrate the richness and beauty of the stability of equilibrium. If you enjoyed them, maybe you will like thinking about the following conceptual exercises:

Exercise 1. Place a table tennis ball in a vertical air stream over a nozzle. Will it stay there?

Exercise 2. A system of electric charges placed in empty space is in equilibrium. Is it stable?

Exercise 3. A rubber balloon filled with gas is placed deep under water at a point where it is in equilibrium. Is it stable?

Exercise 4. A loop of current is placed in a uniform magnetic field. How many equilibrium positions does it have? What will happen if equilibrium is slightly disturbed?

Boris Korsunsky is a physics teacher at Thayer Academy in Braintree, Massachusetts, and has been a coach of the U.S. Physics Team.

ANSWERS, HINTS & SOLUTIONS ON PAGE 57

HOW DO YOU FIGURE?

Challenges

Math

M261

Functionally continuous? The function $f(x) = \sqrt{2} - 1 + x\sqrt{2}$ satisfies the condition $f(f(x)) = 1 + 2x$. Show that there does not exist a function defined for all real numbers such that $f(f(x)) = 1 - 2x$ for all real numbers x.

M262

Subset selection. How many ways can we choose a subset of the set $\{1, 2, 3, \ldots, 11\}$ that does not contain three consecutive integers?

M263

Packing problem. Using one straight cut, divide a 10 cm × 20 cm rectangle into two parts so that they can be placed in a circle of diameter 19.5 cm without overlapping.

M264

Draw the line. Given two points in a plane and a straightedge whose length is less than the distance between them (but no compass!), construct the line passing through the two points.

You may want to use a special case of *Desargues' theorem*: Suppose we have two triangles, ABC and $A_1B_1C_1$ positioned in such a way that AA_1, BB_1, and CC_1 intersect in a point. Let lines AB and A_1B_1 meet at point K, lines BC and B_1C_1 meet at P, and CA and C_1A_1 at M. Then, points K, P, and M are collinear (fig. 1).

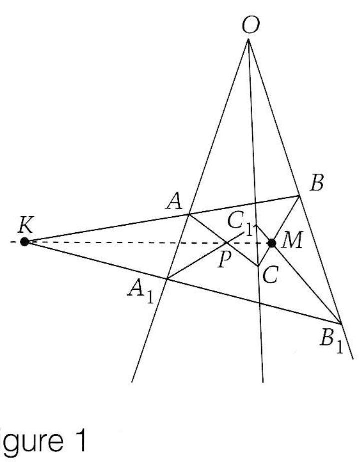


Figure 1

M265

Angling for an answer. Let AA_1, BB_1, and CC_1 be the bisectors of the interior angles of triangle ABC (where A_1, B_1, and C_1 are on the sides of the triangle). If $\angle AA_1C = \angle AC_1B_1$, find $\angle BCA$.

Physics

P261

A moving sidewalk. The following design for a moving sidewalk was proposed: A person steps from the ground to the first moving band, then goes to the next band, which runs faster, and so on. Let the first band move at a constant $v_1 = 2$ m/s. A person steps onto it perpendicular to its motion. Taking a firm position on this band (that is, without sliding), the person steps onto the second band, again perpendicular to its motion. The maximum projected load for such a moving band (the number of people coming to it) is $N = 10$ people/s, and the mass of an average person is assumed to be $M = 80$ kg. What is the minimum force necessary to pull the band horizontally at a constant speed? What force must be applied to the second band to move it at a constant $v_2 = 3$ m/sec? Assume the mean number of people on each band to be the same. (A. Zilberman)

P262

Underwater heat capacity. A student submerged a heater in a beaker of water. Every 3 minutes the student recorded the temperature in °C (see table, top row). Then the student cooled the water, sank a small piece of metal into the jar, and repeated the measurements (see table, lower row). The power line had a voltage of 35 V, the current through the heater was $I = 0.2$ A, and the room temperature was $T_0 = 20$°C. Find the heat capacity of the sample. (L. Bakanina)

25.2	26.4	27.6	28.7	29.7	30.6	31.5	32.3	33.1
22.6	23.8	25.0	26.0	27.0	28.0	28.9	29.8	30.6

P263

Tester and solar cell. A multirange voltmeter composed of a sensitive microammeter and a set of series resistances is used to study a solar cell. When it is connected to the cell using the 1-volt scale, it reads $V_1 = 0.7$ V, and us-

CONTINUED ON PAGE 33

Generating functions

A method for every student's bag of tricks

by S. M. Voronin and A. G. Kulagin

THE METHOD OF GENERATING FUNCTIONS IS a useful mathematical device that allows us to solve various problems in such fields as number theory, probability theory, combinatorics, and the calculus. It often turns out that the analytical reformulation obtained using this method soon yields a solution while all other approaches to the problem yield nothing. However, we should say at once that the method of generating functions is no magic wand that solves all problems (see below, for instance, Fermat's problem, for which it proves useless).

We first give, without many theoretical details, four classic examples of problems that will show how the method of generating functions works. Next, we give a brief explanation of this method, and we conclude with a set of problems that can be solved using the method. Some of the details in the solutions call on techniques from the calculus. However, the reader who does not follow these details can still read most of the exposition.

Weighing problem

In the middle of the eighteenth century, Leonhard Euler (1701–1783) solved the following problem: what loads can be weighed using $1, 2, 2^2, 2^3, \ldots, 2^m, \ldots$ gram weights, and in how many different ways can we do this? We don't know how, or how quickly, Euler found the solution, but his final reasoning is stunningly unusual. Judge for yourself.

Euler considered the product

$$\alpha(z) = (1+z)(1+z^2)(1+z^4)(1+z^8)\cdots \tag{1}$$

Removing all the parentheses and collecting similar terms, he obtained "an infinite polynomial in z":

$$\alpha(z) = 1 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + \cdots \tag{2}$$

And what are the numbers A_k ($k = 1, 2, \ldots$)? Each A_k is the coefficient of z^k, and z^k appears as the product of certain monomials z^{2m} (no more than one monomial from each term in (1)). Therefore, A_k is the number of different ways of representing k as a sum of different numbers from the set $1, 2, 4, 8, \ldots, 2^m, \ldots$ In other words, A_k is the number of ways to weigh a k-gram load using the given set of weights!

So, the original problem will be solved if we calculate all the numbers A_k. Of course, we can try to calculate some of them directly, especially when k is small. You will undoubtedly guess the correct answer: all the numbers A_k are equal to 1. But it is not very easy to prove this statement for all k by computation. Instead, Euler found another trick. He wrote down the following identities:

$$(1-z)(1+z) = 1-z^2,$$
$$(1-z^2)(1+z^2) = 1-z^4,$$
$$(1-z^4)(1+z^4) = 1-z^8, \ldots$$

Then he multiplied these equalities, eliminated common factors on both sides and obtained

$$(1-z)(1+z)(1+z^2)(1+z^4)(1+z^8)\cdot \cdots = 1,$$

which can be written as $(1-z)\cdot\alpha(z) = 1$. Finally:

$$\alpha(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots \tag{3}$$

(we've used the formula for the sum of a geometric progression[1]). Comparing equation (3) with equation (2), we

[1]This formula is valid when $|z| < 1$. We return to this question later in the article.

Art by Pavel Chernusky

conclude that $A_k = 1$ for all k. In other words, it is possible to weigh every load whose weight is an integer number of grams, using weights of 1, 2, 4, . . ., 2^m, . . . grams, and in a unique way.

The problem of the number of decompositions

This was the title Euler gave to the following problem: how many different integer solutions are there to the equation

$$x_1 + x_2 + \cdots + x_m = k? \tag{4}$$

Here k and m are fixed natural numbers. This time the solution comes from the consideration of another expression:

$$\beta(z) = (1 + z + z^2 + z^3 + z^4 + \cdots)^m. \tag{5}$$

Removing the parentheses, we can write

$$\beta(z) = 1 + B_1z + B_2z^2 + B_3z^3 + B_4z^4 + \cdots \tag{6}$$

What can we say about the B_ks? Each B_k is the coefficient of z^k. We can see that B_k is equal to the number of different ways to choose one monomial from each factor in the right part of equation (5) so that the sum of their powers is k. That is, B_k is equal to the number of solutions of equation (4)! Euler's problem thus reduces to the calculation of B_k.

It is difficult to do this calculation directly. But, if we remember the formula for the sum of an infinite geometric progression, we can write

$$1 + z + z^2 + z^3 \cdots = \frac{1}{1-z}. \tag{7}$$

or $\beta(z) = (1 - z)^{-m}$. Hence we can write

$$(1 - z)^{-m} = 1 + B_1z + B_2z^2 + B_3z^2 + B_4z^4 + \ldots$$

Let's take the derivative on both sides of this equality:

$$m(1 - z)^{-m-1} = B_1 + 2B_2z + 3B_3z^2 + 4B_4z^3 + \cdots$$

Letting $z = 0$, we obtain $B_1 = m$. Now if we take the second derivative and again let $z = 0$, we obtain $B_2 = m(m + 1)/2$. Following in Euler's footsteps, we take the k^{th} derivative of equation (7) and let $z = 0$. We can now see that the number of solutions of equation (4) is

$$B_k = \frac{m(m+1)\cdot \cdots \cdot(m+k-1)}{n!}.$$

The change problem

How many different ways are there to make change for one dollar using 1¢, 5¢, 10¢ and 25¢ coins? This question can be reformulated in a somewhat different way: How many solutions are there for the equation

$$x_1 + 5x_2 + 10x_3 + 25x_4 = 100?$$

A more general form of this problem is as follows: how many nonnegative integer solutions has the equation

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = n, \tag{8}$$

where a_i are positive integers and n is a fixed natural number?

Consider the expression

$$\begin{aligned}\gamma(z) &= \left(1 + z^{a_1} + z^{2a_1} + z^{3a_1} + \ldots\right)\left(1 + z^{a_2} + z^{2a_2} + z^{3a_2} + \ldots\right)\\ &\quad\times\left(1 + z^{a_m} + z^{2a_m} + z^{3a_m} + \ldots\right)\\ &= 1 + C_1z + C_2z^2 + C_3z^3 + C_4z^4 + \ldots\end{aligned} \tag{9}$$

Reasoning as before, we can say that for each n, C_n is equal to the number of solutions of equation (8). But how can we compute C_n? Again, we can use the formula for the sum of an infinite geometric progression:

$$1 + z^{a_1} + z^{2a_1} + z^{3a_1} + \cdots = \frac{1}{1 - z^{a_1}},$$

we can represent $\gamma(z)$ in the form

$$\gamma(z) = \frac{1}{(1 - z^{a_1})(1 - z^{a_2})\cdots(1 - z^{a_m})},$$

but it is still unclear how to write a general formula for C_n. However, equation (9) is already enough to find the answer for small n.

The problem of four squares

Is it true that every natural number can be represented in the form of the sum of the squares of four integers? If it is, then how many ways are there to do it? In other words, how many solutions are there for the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = m, \tag{10}$$

where m is a given natural number?

This problem is more difficult than the previous problems we have discussed, though it reduces, as before, to the computation of the coefficients of an "infinite polynomial." First we write down the equality

$$\begin{aligned}\delta(z) &= (1 + 2z + 2z^4 + 2z^9 + 2z^{16} + \cdots)^4\\ &= 1 + D_1z + D_2z^2 + D_3z^3 + D_4z^4 + \cdots\end{aligned} \tag{11}$$

It is not difficult to see that the number of solutions of equation (10) is equal to D_m. Taking the m^{th} derivative and letting $z = 0$, we get the following formula for D_m:

$$D_m = \frac{1}{m!}\frac{d^m}{dz^m}(\delta(z))\Big|_{z=0}.$$

Here the symbol

$$\frac{d^m}{dz^m}$$

means that the expression should be evaluated for $z = 0$. When m is small, we can use this formula to compute D_m, but it is not easy to do this in the general case. The original problem of number theory is reduced to an analytic one, but the latter is still to be solved!

The great German mathematician C. G. Jacobi (1804–1851) proved a wonderful formula for $\delta(z)$. This formula is

$$\delta(z) = 1 + 8 \sum_{m \neq 4k} \frac{mz^m}{1 - z^m}.$$

Later this formula was rediscovered by the famous Indian mathematician Srinivasa Ramanujan (1887–1920). One can use it to show that equation (10) always has solutions, and one can even use it to find the number of solutions it has.

Explanations

Let's try to understand the main idea of the method that worked so amazingly in various problems. In each case we considered an "infinite expression" with the letter z (an infinite product of binomials in (1), the finite powers of an infinite sum in equations (5) and (11), and the product of infinite sums in equation (9)). The next move was to rewrite this expression in the form of an "infinite polynomial in z"

$$\alpha(z) = 1 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + \cdots + A_nz^n + \cdots,$$

and it turned out that the coefficients A_k gave a clue to the solution of this problem. In some cases we managed to calculate the numbers A_k through various manipulations of infinite expressions. We performed arithmetic operations with them, took derivatives, transformed them using the formula for a geometric progression, and so on.

It is natural to ask what the infinite expressions that we considered are, and whether we are justified in manipulating them as we did above. The author of this method, Leonhard Euler, didn't much trouble himself with these questions. He worked with infinite expressions as if they had been finite, and worried only about the result. We know that many errors can arise from the thoughtless handling of infinity.

A very unpleasant example

Consider two "infinite polynomials"

$$\alpha(z) = 1 - z + z^2 - z^3 + z^4 - \cdots,$$
$$\beta(z) = -1 + z - z^2 + z^3 - z^4 + \cdots$$

Let's try to calculate the infinite sum

$$S(z) = \alpha(z) + \beta(z) + \alpha(z) + \beta(z) + \alpha(z) + \cdots$$

If we arrange the terms in $S(z)$ into pairs, starting with the first pair, we get

$$S(z) = [\alpha(z) + \beta(z)] + [\alpha(z) + \beta(z)] + \cdots = 0 + 0 + \cdots = 0.$$

But if we arrange them in a slightly different way, we get

$$S(z) = \alpha(z) + [\beta(z) + \alpha(z)] + [\beta(z) + \alpha(z)] + \cdots$$
$$= \alpha(z) + 0 + 0 + \cdots = \alpha(z)!$$

And if we change the order of terms inside $\alpha(z)$ and $\beta(z)$, we can arrive at still more surprising results—for instance, $S(z) = 17z^{17}$ (we invite the reader to think about how to do this).

It is now high time that we give strict definitions for infinite expressions and find out which manipulations with them are justified.

Formal power series

A *formal power series* is an expression such as

$$\alpha(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots, \qquad (12)$$

where a_i are real numbers (the *coefficients* of series (12)), and z is just a letter (the *formal variable* of the series). Mathematicians also say that $\alpha(z)$ is a *generating function* for the sequence of coefficients $a_0, a_1, \ldots, a_n, \ldots$ One shouldn't think, however, in the general case (which we consider here) that $\alpha(z)$ can be regarded as an ordinary function of the variable z. When we say that $\alpha(z)$ is a generating function of the sequence $a_0, a_1, \ldots, a_n, \ldots$, it means only that $\alpha(z)$ is shorthand for the formal expression in the right side of (12).

For those readers who are acquainted with the notion of an ordinary power series, we stress that the *formal* series (12) may diverge for some values of z, or even all nonzero values of z. Even in these cases, we can say that equation (12) is correct, because $\alpha(z)$ denotes the series itself, and not its sum for a given value of z.

The outstanding German mathematician C. Weierstrass (1815–1897), who developed the theory of power series and proposed strict logical foundations for it, considered only *converging* power series and criticized Euler for his dubious tricks with "infinite polynomials." The modern algebraic theory of formal power series, elements of which we discuss here, is quite strict, but it is essentially closer to Euler's "naïve" approach, than to the strict analytic theory of Weierstrass.

One can add and multiply formal power series. Suppose that in addition to the series $\alpha(z)$ (see equation (12)) another formal power series is given:

$$\beta(z) = b_0 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots \qquad (13)$$

Then we define the sum and the product of $\alpha(z)$ and $\beta(z)$ by the following natural formulas:

$$\begin{aligned} \alpha(z) + \beta(z) &= (a_0 + b_0) + (a_1 + b_1)z + (a_2 + b_2)z^2 + \cdots \\ &\quad + (a_n + b_n)z^n + \cdots, \\ \alpha(z) \cdot \beta(z) &= a_0b_0 + (a_1b_0 + a_0b_1)z \\ &\quad + (a_0b_2 + a_1b_1 + a_2b_0)z^2 + \cdots \end{aligned} \qquad (14)$$

(these operations are similar to the addition and multiplication of finite polynomials). Addition and multiplication of formal power series, defined in this way, possess all the standard properties of the addition and multiplication of numbers or polynomials: the associative, commutative, and distributive rules hold. Note, by the way, that one can consider the usual polynomials as particular cases of formal power series—that is, as a series whose coefficients, except for several of the first ones, vanish. Of course, this identification maps the usual sum and product of polynomials into their sum and product as formal power series.

Further, one can differentiate and integrate formal power series. By definition:

$$\frac{d}{d(z)}[\alpha(z)] = [\alpha(z)]' = a_1 + 2a_2 z + 3a_3 z^2 + \cdots na_n z^{n-1} + \cdots,$$

$$\int \alpha(z) = \int_0^z \alpha(x)dx = a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \cdots + \frac{a_n}{n+1}z^{n+1} + \cdots$$

Just like in the case of integers, one can subtract any formal power series from any other series. The result will be another formal power series. But it is not always possible to divide formal power series, just as it is not always possible to divide two integers. For instance, it is prohibited to divide by "zero"—that is, by the following formal series

$$0 = 0 + 0 \cdot z + 0 \cdot z^2 + \cdots + 0 \cdot z^n + \cdots$$

One can't also divide the "unit" series

$$1 = 1 + 0 \cdot z + 0 \cdot z^2 + \ldots + 0 \cdot z^n + \cdots$$

by the series

$$z = 0 + 1 \cdot z + 0 \cdot z^2 + \cdots + 0 \cdot z^n + \cdots$$

In fact, let's suppose that there exists a series

$$\alpha(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

such that $\alpha(z) \cdot z = 1$. That is,

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(0 + 1 \cdot z + 0 \cdot z^2 + \cdots) = 1 + 0 \cdot z + 0 \cdot z^2 + \cdots$$

According to formula (14), this means that $a_0 \cdot 0 = 1$, which certainly is impossible. Similarly, one can prove that $\alpha(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is divisible by z^n if and only if $a_0 = a_1 = \cdots = a_{n-1} = 0$.

It is easy to check that 1 is divisible by $\alpha(z)$ if and only if $a_0 \neq 0$, and in the latter case the coeffiecients of the resulting series are determined in a unique way.

As an example, let's see what happens when we divide the unit series by the series $1 - z$. We write

$$(1 - z)(a_0 + a_1 z + a_2 z^2 + \cdots) = 1 + 0 \cdot z + 0 \cdot z^2 + \cdots$$

Now, removing the parentheses and comparing the coefficients of equal powers of z, we get the system of equalities:

$$1 \cdot a_0 = 1,\ a_1 - a_0 = 0,\ a_2 - a_1 = 0, \cdots, a_n - a_{n-1} = 0, \cdots,$$

which shows that

$$a_0 = a_1 = \cdots = a_n = \cdots = 1,$$

which means that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

We've already used this formula above (see equation (3) and footnote 1), although there it meant something different than from what it means now. Here it is correct as an equality of formal power series (and has nothing to do with the condition $|z| < 1$).

Infinite sums and products

In the beginning of this article we solved a problem using an infinite product. Infinite sums are also very handy (one can regard a formal power series as an infinite sum of monomials). Here we're going to give the corresponding definitions to avoid possible misunderstandings.

Let $\alpha_1(z), \alpha_2(z), \ldots, \alpha_k(z), \ldots$ be a sequence of formal power series (in particular, polynomials). We want to define their infinite sum and infinite product

$$\alpha_1(z) + \alpha_2(z) + \cdots + \alpha_k(z) + \cdots, \tag{15}$$

$$(\alpha_1(z)) \cdot (\alpha_2(z)) \cdot \ \ldots \ \cdot (\alpha_k(z)) \cdot \ \ldots \tag{16}$$

in the way we did it above, by removing brackets and collecting similar terms. However, such a definition wouldn't apply in general. In fact, let's start multiplying out and collecting similar terms in the "infinite product"

$$(1 + z) \cdot (1 + z) \cdot \ \cdots \ \cdot (1 + z) \cdot \ \cdots$$

Take, for instance, the coefficient at z^2. We at once see that it keeps growing as we remove parentheses. There is an essential difference between this example and the one we considered above

$$(1 + z)(1 + z^2)(1 + z^4) \cdot \ \cdots \ \cdot (1 + z^{2m}) \cdot \ \cdots,$$

in which all factors gradually *stabilize*, because the "distant" parentheses do not contain monomials with "small" powers of the formal variable.

So the *infinite sum* (15) and *infinite product* (16) are defined only if the coefficients stabilize when one removes the brackets and collects like terms. They are defined to be equal to the formal power series whose coefficients are obtained in this way.

If you prefer to deal with strict statements, here's a precise definition of "stabilization": For all natural d one can find a natural number N such that for all $n \geq N$ series $\alpha_n(z)$ does not contain powers of z less than d.

Now it's time to conclude our brief excursion to the theory of formal series. We suggest that that the reader return to the examples in the beginning of the article to verify that no dubious "tricks" were performed with infinite polynomials and, in fact, only wholly legal manipulations of formal power series were undertaken.

Generating functions

A thoughtful reader who has perused the examples in the beginning of the article probably has noticed that the success of our method came from the possibility of writing the generating function of a power series $\alpha(z)$ in convenient form. For instance, one can write the generating function of the sequence 1, 1, 1, . . . as $1/(1-z)$, since

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \tag{17}$$

Manipulating this equality, we can obtain many formulas for other generating functions. For instance, if we multiply both sides in equation (17) by z and take the derivative, we get

$$\frac{d}{dz}\left(\frac{z}{1-z}\right) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots \tag{18}$$

This means that the function $(z/(1-z))' = 1/(1-z)^2$ is the generating function for the progression 1, 2, 3, 4, . . . If we once again multiply this equality by z and differentiate it, we obtain

$$\frac{1+z}{(1-z)^3} = 1 + 2^2 z + 3^2 z^2 + \cdots + (n+1)^2 z^n + \cdots \tag{19}$$

We have found an explicit formula for the generating function of the progression $1^2, 2^2, 3^2, 4^2, \ldots$

The reader who knows how to divide polynomials can try to obtain formula (19) in a somewhat different way, namely by dividing the polynomial $1 + z$ by the polynomial $(1 + z)^3 = 1 + 3z + 3z^2 + z^3$.

Let's start with the same formula (17) again, but now we will integrate it. We get

$$\int_0^z \frac{dx}{1-x} = \int_0^z 1 \cdot dx + \int_0^z x\,dx + \cdots + \int_0^z x^n dx + \cdots$$

and

$$-\ln(1-z) = z + \frac{z}{2} + \frac{z^2}{3} + \cdots$$

We can see that the generating function of the progression 1, 1/2, 1/3, . . . is $-\ln(1-z)$.

Problems

1. Prove that every positive integer has a unique binary representation. *Hint:* Compare this question with the weighing problem.

2. Prove that every positive integer number has a unique decimal representation

Hint. Use the following relation (prove it)

$$\begin{aligned}&(1 + z + z^2 + \cdots + z^9) \cdot (1 + z^{10} + z^{20} + \cdots + z^{90})\\ &\quad\times (1 + z^{100} + z^{200} + \cdots + z^{900}) \cdot \cdots\\ &= 1 + z + z^2 + z^3 + \cdots = 1/(1-z).\end{aligned}$$

3. Prove that there are as many ways to represent any positive integer in the form of a sum of *different* positive integers as there are to represent it in the form of a sum of (not necessarilly different) positive odd numbers. (For instance, for 6 there are four such representations:

$$6 = 1 + 5 = 2 + 4 = 1 + 2 + 3$$

and

$$1 + 5 = 3 + 3 = 1 + 1 + 1 + 3 = 1 + 1 + 1 + 1 + 1 + 1.)$$

Hint. Prove the formula

$$\begin{aligned}&\frac{1}{1-z}\cdot\frac{1}{1-z^3}\cdot\frac{1}{1-z^5}\cdot \cdots \cdot\\ &= \left(1 + z + z^2 + z^3 + \cdots\right)\left(1 + z^3 + z^6 + z^9 + \cdots\right)\\ &\quad\times\left(1 + z^5 + z^{10} + z^{15} + \cdots\right)\cdots\\ &= (1+z)\left(1+z^2\right)\left(1+z^3\right)\left(1+z^4\right)\cdots\\ &= \left(1-z^m\right)\left(1+z^m\right)\left(1+z^{2m}\right)\left(1+z^{4m}\right)\left(1+z^{8m}\right)\cdots = 1.\end{aligned}$$

and use it to solve the problem.

4. Find an explicit formula for the general term of the progression $(u_n) = (0, 1, 5, 19, 65, \ldots)$, defined with the help of the recursive relations $u_0 = 0$, $u_1 = 1$, $u_{n+1} = 5u_n - 6u_{n-1}$.

Hint. Note that we can rewrite the series $u_0 + u_1 z + u_2 z^2 + \ldots$ in the form

$$\begin{aligned}&\frac{z}{6z^2 - 5z + 1} = \frac{1}{1-3z} - \frac{1}{1-2z}\\ &= \left(1 + 2z + 2^2z^2 + 2^3z^3 + \cdots\right)\left(1 + 3z + 3^2z^2 + 3^3z^3 + \cdots\right).\end{aligned}$$

5. Let $n > 2$ be a fixed integer. Consider the series

$$f(z) = z^n + z^{n^2} + z^{n^3} + \cdots,$$

$$[f(z)]^2 = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

Show that the number of integer solutions of Fermat's equation $k_1{}^n + k_2{}^n = k_3{}^n$ is equal to c_m, when $m = k_3{}^n$.

Note: Unfortunately for the history of mathematics, no one has found a way to calculate c_m directly (although we now are sure of their values).

6. Calculate v_n, the number of different ways to divide a convex n-gon into triangles by nonintersecting diagonals (for instance, $v_5 = 5$, $v_6 = 9$).

Hint: Put $v_2 = v_3 = 1$ and prove the recursive formula

$$v_n = v_2 v_{n-1} + v_3 v_{n-2} + \cdots + v_{n-1} v_n,$$

where $n \geq 3$. Use this formula to prove that $(\alpha(z))^2 - \alpha(z) = z$, where $\alpha(z) = v_2 z + v_3 z^2 + v_4 z^3 + \cdots$, find $\alpha(z)$ and take the derivatives. *Answer:*

$$v_n = \frac{(2n-4)!}{(n-2)!(n-1)!}.$$

This sequence is called the *Catalan numbers.* ◘

Karate chop

The physics of tameshiwari

by A. Biryukov

TAMESHIWARI IS A KARATE term that means the testing of one's psychological training and of the skill to strike various objects with the hand. Karate came to the western world from Okinawa, Japan. It was developed in the sixteenth and seventeenth centuries, when in fear of rebellions, the governing powers confiscated all weapons from the people, including their ritual and kitchen knives. It was beyond the power of the peasants to fight the armed-to-the-teeth Samurais with bare hands, but they could repel a gang of bandits using karate.

Perhaps this explains the origin of tameshiwari, which is always interesting for spectators and produces the impression of a miracle upon the uninitiated. Today the skill of tameshiwari is most often shown in demonstrations and competitions of karate, where the targets are planks of certain sizes made of coniferous (soft) wood.

We consider in this article a simple physical model of a hand hitting a plank, which yields some estimates and advice, and evaluates the possible limits of athletic achievements in tameshiwari. To find a number of parameters for this model, we must solve several preliminary problems, which are interesting in themselves. However, to keep our train of thought running on the main line, we solve these problems in the appendixes at the end of the article.

Let a blow be struck with a fist of mass m arriving with speed v at the center of a plank of dimensions d, l, and h that lies on two supports (fig. 1). The fibers of the wood are parallel to the supports, which are separated by approximately the length of the plank l. One of the "secrets" of karate says that to enhance the effectiveness of the blow, one should apply force F to the accelerated fist just before the moment of contact and maintain it during the entire collision. We consider the deformation of the plank in the reference system shown in fig. 2. Let x_0 be the displacement of the plank's center from its equilibrium position. Assume that the breaking of the plank (signaled by the breaking of its surface) occurs at some critical value $x_0 = x_r$, when the stress σ (the force applied to a unit area of the plank's cross-section) at the plank's surface reaches some critical value σ_r, which depends on the strength of the material.

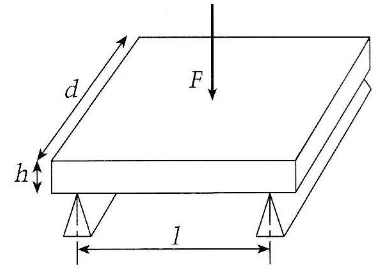


Figure 1

First we find the relationship between x_r and σ_r, which is determined by the elastic properties and geometry of the plank. The maximum bending and the maximum stress at the surface of the plank will take place at its center. In Appendix 1 we show that this stress is given by the formula

$$\sigma = \frac{Yh}{2R},$$

where R is the radius of curvature of the central line CC in the middle of the plank (fig. 2) and Y is Young's modulus for the type of wood.

Now we assume a particular shape for the deformed plank and take into consideration that its ends are fixed at the points $y = \pm l/2$, and the maximum displacement from equilibrium occurs at the center of the plank. Note that the exact shape

Art by Leonid Tishkov


Мастер Сен Саку
учит математика
решить задачи

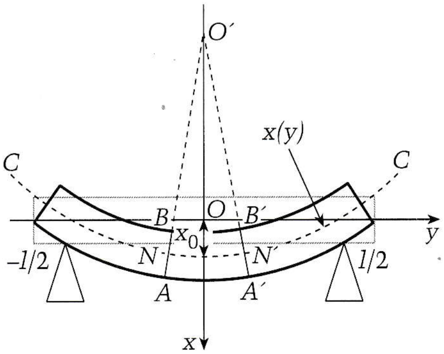


Figure 2

of the plank depends on the specific (and quite clearly understood) conditions of the interaction between the contact surface of the fist with the plank (in a correct blow, contact is made with the knuckles of the middle and index fingers). In our calculations we will use a practical formula based on experimental data, which makes it possible to obtain simple estimates.

Let's approximate the bending of the plank by a cosine between the points $y = \pm l/2$. In this case, the displacement x of any point along the central line depends on its coordinate y as

$$x(y) = x_0 \cos\left(\frac{\pi y}{l}\right).$$

Appendix 2 shows that the corresponding radius of curvature at the plank's center will be

$$R = \left(\frac{l}{\pi}\right)^2 \frac{1}{x_0}.$$

Plugging this into the formula for σ yields the stress on the surface of the plank at its middle when the plank's center is displaced by x_0:

$$\sigma = \frac{x_0 Y h \pi^2}{2l^2}.$$

This formula shows that breaking ($\sigma = \sigma_r$) occurs when the plank's center is shifted by

$$x_r = \frac{2\sigma_r l^2}{\pi^2 Y h}.$$

Now we model the elastic properties of the plank with a spring with a spring constant k, which is loaded by an external force. This spring constant is found in Appendix 3 to be

$$k = \frac{\pi^2 Y h^3 d}{3l^3}.$$

Having determined the necessary parameters, we return to the initial dynamic problem of a fist hitting a plank. The motion of the fist is described by Newton's second law:

$$mx'' = -kx + F,$$

where x henceforth means the displacement of the fist from the initial contact position with the plank, and the primes indicate differentiation with respect to time.

To simplify, we consider the force F, which is applied to the fist by the arm, to be constant. Substitutions yield the following solution:

$$x = A\cos\omega t + B\sin\omega t + \frac{F}{k},$$

which includes two arbitrary constants A and B. To find them, we specify the initial conditions: $x = 0$ and $x' = v$ at $t = 0$. Now we get

$$x = \frac{f}{\omega^2}(1 - \cos\omega t) + \frac{v}{\omega}\sin\omega t,$$

where $f = F/m$ has dimensions of acceleration, and $\omega = \sqrt{k/m}$ is the frequency of natural oscillation of the fist under the action of the elastic force of the plank.

The next step is to find the maximum displacement x_{max} of the fist for the given initial speed v and force F. By equating the time derivative of x to zero with the subsequent elimination of t, we get

$$x_{max} = \frac{f}{\omega^2}\left(1 + \sqrt{1 + \left(\frac{v\omega}{f}\right)^2}\right).$$

To obtain the conditions of breaking, this displacement must be set equal to x_r, which yields the equation

$$\frac{2\sigma_r h^2 d}{3Fl} = 1 + \sqrt{1 + \frac{\pi^2 Y h^3 v^2 dm}{3F^2 l^3}},$$

connecting the properties of wood and the geometry of the plank with the parameters of the collision.

We solve this equation for F, again using the parameters x_r and k:

$$F = \frac{kx_r}{2} - \frac{mv^2}{2x_r}.$$

For the plank to break, this force must be applied at the moment of contact to the fist moving with initial speed v. We can see that if the speed of the fist is large enough, the value of F becomes negative, so no force is needed to break the plank with a moving fist (in a similar way we need not apply force to a hammer when driving a small nail into wood). In this case the initial speed of the fist must be larger than

$$v = x_r\omega = \frac{2\sigma_r}{\pi\sqrt{3}}\sqrt{\frac{lhd}{mY}},$$

which is proportional to the square root of the plank's thickness h. By contrast, if the initial speed of the fist is zero, then this formula shows that to break the plank, the force must be no less than

$$F = \frac{kx_r}{2} = \frac{h^2\sigma_r d}{3l},$$

which is proportional to the square of the plank's thickness h. Therefore, to break a thicker plank, it is more practical to increase the speed of the blow, not its force.

Now let's solve the equation that determines the condition of the plank's breaking relative to its thickness h. It yields the thickness of the plank that can be broken for the given parameters of the blow:

$$h = \frac{3\pi^2 Y v^2 m}{8\sigma_r^2 l d}\left(1 + \sqrt{1 + \frac{64 F l^3 \sigma_r^3 d}{3\pi^4 Y^2 v^4 m^2}}\right).$$

Let's obtain some estimates, using the following experimental parameters for the wood: $E = 10^8$ N/m^2 and $\sigma_r = 5 \cdot 10^6$ N/m^2. The standard plank in tameshiwari has a width of 20 cm and a length of 30 cm. We assume $l = 25$ cm, because the ends of

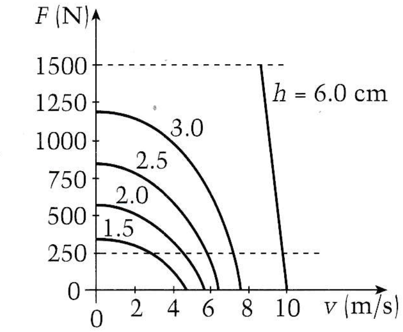


Figure 3

the plank (located beyond the supports) can be neglected. The mass of the fist is assumed to be 1 kg, and this number takes into account the forearm as well. Figure 3 shows the dependence of the force F on the initial speed v for various thicknesses h of the plank. If the combination of F and v corresponds to a point lying above the curve for a specified value of h, the plank will break.

Now we can evaluate the thickness of the plank that can be broken by a man. The force developed by the hand of a typical man is $F = 250$ N. Figure 3 shows that at $v = 0$ a typical man cannot break even a rather thin plank with a thickness of only 1.5 cm. To perform this deed, he must apply a force of about 300 N.

The experimental value for the maximum speed of the fist is about 10 m/s. Plugging $v = 10$ m/s and $F = 250$ N into the formula for h, we get the thickness of the plank: $h = 6$ cm. This value is rather large, and perhaps only experienced karate masters with excellent striking technique and psychological training can break such a thick plank. However, inquisitive readers can try to break a plank with a thickness of 2 cm, because the necessary values of force and speed can be achieved by the average person. In this process it is very important to follow the basic psychological "secret" of karate: Never doubt yourself.

Appendix 1

Let's find the stress on the surface of the plank. We consider two symmetrical cross-sections AB and $A'B'$ (fig. 2), which are normal to the line CC and separated by a small distance l_0 along this line. Consider the element $AA'B'B$. Due to its small value, we can approximate the curves AA', NN', and BB' by arcs with centers lying on the so-called axis of bending O', which is perpendicular to the page. The outer surface of the plank between points A and A' is stretched, while the inner surface between points B and B' is compressed. When bending is absent, the lengths of curves AA' and BB' are the same and equal to l_0 (the length of the central curve NN'), which retains its length during bending. Let R be the radius of curvature of the line NN'. Then $l_0 = R\alpha$, where α is the central angle subtending arc NN'. When the plank is not very thick—that is, when $h << R$, the length of curve AA' will be $l_1 = (R + h/2)\alpha$, and its elongation due to bending will be $\Delta l = l_1 - l_0 = h\alpha/2$. According to Hooke's law, the stress in the outer surface of the plank is

$$\sigma = Y\frac{\Delta l}{l_0} = \frac{Yh}{2R}.$$

Appendix 2

Let's find the radius of curvature of the surface of a bent plank at the middle point ($y = 0$). Recall that if R is the radius of curvature of any curve at a specified point, then the circle of radius R that passes through this point and whose center lies on the perpendicular to the curve at this point coincides (according to the definition of the radius of curvature) with the curve within a small distance of this point. When $|\pi y/l| <<$ 1, the function $x(y)$ becomes

$$x(y) = x_0 - \frac{x_0}{2}\left(\frac{\pi}{l}\right)^2 y^2.$$

Here we used the well-known approximation $\cos\gamma = 1 - \gamma/2$ for $|\gamma| << 1$.

The circle of radius R and center O' (fig. 2), which passes through the point $(x_0, 0)$ and which was considered in Appendix 1, is described by the equation

$$y^2 + (x - x_0 + R)^2 = R^2,$$

which can be easily solved to find the displacement $x(y)$:

$$x(y) = x_0 - R + R\sqrt{1 - \left(\frac{y}{R}\right)^2}.$$

Using another approximate formula, $\sqrt{1-\gamma} \cong 1 - \gamma/2$ for $|\gamma| << 1$, we get the following formula, which is correct for $|y/R| << 1$:

$$x(y) = x_0 - \frac{y^2}{2R}.$$

By comparing the two formulas for $x(y)$, we get the radius of curvature:

$$R = \left(\frac{l}{\pi}\right)^2 \frac{1}{x_0}.$$

Appendix 3

Let's find the dependence of the displacement x_0 of the center of a plank resting on two supports upon the external force F, which is distributed along the central fibers and directed downward. The mass of the plank will be neglected.

Due to the assumed symmetry, the force F is evenly distributed between the supports. We look at the cross-section through the plank at the plank's center (fig. 2) and consider the equilibrium condition for the left half of the plank. It is affected on the right by the external force $F/2$, which is applied near the edge and directed downward. This force is counterbalanced by the normal force of the left support. We can see that the sum of the torques relative to the plank's center will be determined only by the torque due to the left support:

$$\tau = \frac{Fl}{4}.$$

On the other hand, this torque is counterbalanced by the torques due to the tension and compression applied by the plank's right half on its left half in the plane of the cross-section. This torque can be derived from the formula for σ by modifying it to calculate the stress in the bulk of the plank along the y-axis. As follows from the derivation of this formula (Appendix 1), we must replace the displacement $h/2$ from line NN' corresponding to the point on the outer surface of the plank with the distance δ from this line $(-h/2 < \delta < h/2)$. In this case the stress in the bulk of plank will be

$$\sigma = \frac{Y\delta}{R}.$$

The total torque due to the elastic tension and compression forces relative to the plank's center will thus be equal to

$$\tau = \int_{-h/2}^{h/2} \delta\sigma d \cdot d\delta = \frac{Y}{R} d \int_{-h/2}^{h/2} \delta^2 d\delta = \frac{Yh^3 d}{12R}.$$

By plugging the value for the radius of curvature into this equation and equating the right-hand terms of the two formulas for τ, we get the relationship between the force F and the displacement x_0:

$$x_0 = \frac{3Fl^3}{\pi^2 Yh^3 d}.$$

This formula can be rewritten in the form $F = kx_0$, from which the formula for the spring constant k of the equivalent spring immediately follows:

$$k = \frac{\pi^2 Yh^3 d}{3l^3}.$$

Quantum **on deformation and strength:**

A. Borovoy, "Holding up under pressure," January 1990, pp. 30–32.

A. A. Dozorov, "Taking a flying leap," September/October 1990, pp.10–11.

A. Mineyev, "Trees worthy of Paul Bunyan," January/February 1994, pp. 4–10.

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Nascent non-Euclidean geometry

Revisiting Geometric Research on the Theory of Parallels

by N. I. Lobachevsky

IN THIS ARTICLE, WE REPRODUCE the first part of N. I. Lobachevsky's famous work Geometric Research on the Theory of Parallels, *which contains an elementary presentation of the foundations of non-Euclidean geometry. This work was published in Berlin in 1840 as a small book in German. It was from this book that many European mathematicians got to know Lobachevsky's ideas. For example, the prominent German mathematician Carl Friedrich Gauss (1777–1855) had two copies of this book, one of which was probably sent to him by Lobachevsky himself. Later, French and English translations were made from the German.*

The book can be naturally divided into three parts. The first part begins with a short list of propositions that are independent of the parallel postulate. This list doesn't embrace all such propositions; it only includes those that are required later.

*Five theorems that form the foundation of Lobachevsky's theory of parallels are presented in the first part. Two hypotheses are set forth in this part: One of them leads to Euclidean geometry (*ordinary geometry *in Lobachevsky's terminology), and the other leads to a new geometry, which Lobachevsky called* imaginary. *In the first part, the concept of the angle of parallelism and the corresponding function $\Pi(x)$ is introduced, and the dependence of the sum of the angles of a triangle on the parallel postulate is analyzed.*

In the second part, concepts of the boundary line and surface are introduced, and a theorem is proved stating that the geometry of the boundary surface coincides with Euclidean plane geometry. The third part discusses non-Euclidean trigonometry and a derivation of the basic equation

$$\tan\frac{1}{2}\Pi(x) = e^{-x/k}.$$

In the following reproduction of the first part of Geometric Research on the Theory of Parallels, *we have, for convenience, divided the text into sections, dropped some of the preliminary propositions mentioned above that are not used in the first part, and renamed propositions 16–25 as theorems 1–10.*

Introduction

In geometry I find certain imperfections that I hold to be the reason why this science, apart from the transition to analytic methods, can as yet make no advance from the state in which it has come to us from Euclid.

In order to discuss some of these imperfections, I consider the obscurity in the fundamental concepts of geometric magnitudes and in the manner and method of representing the measuring of these magnitudes, and finally the momentous gap in the theory of parallels, the filling of which has defied all efforts of mathematicians so far.

So as not to fatigue my reader with the multitude of theorems whose proofs present no difficulties, I prefix here only those of which a knowledge is necessary for what follows.

1. A straight line fits upon itself in all positions. By this I mean that during the rotation of the surface containing it, a straight line does not change its place if it goes through two unmoving points in the surface. In other words, if we turn the surface containing it about two points of the line, the line does not move.

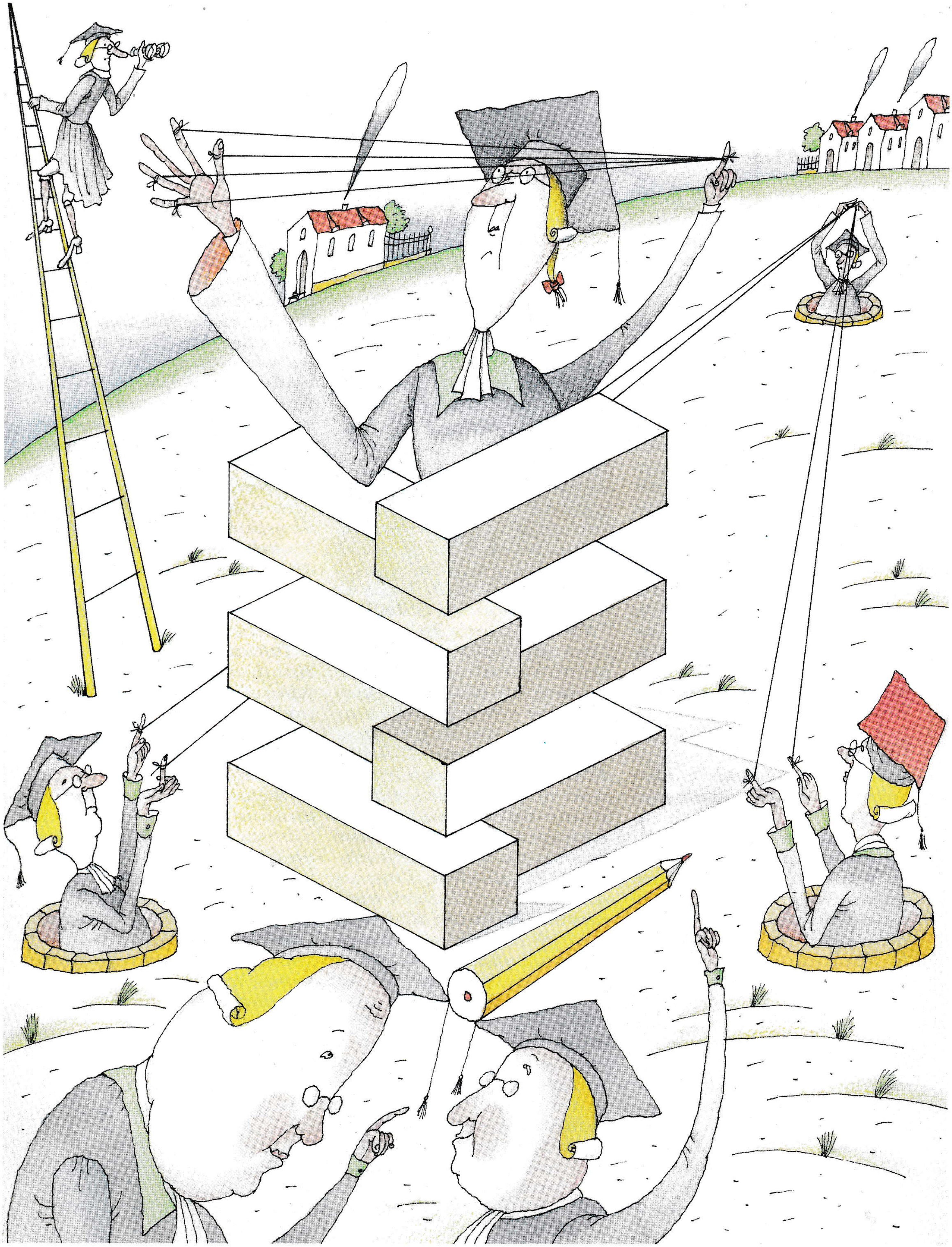
2. Two straight lines cannot intersect at two points.

3. A straight line sufficiently extended both ways must go beyond all bounds, and in this way divides a bounded plane into two parts.

4. Two straight lines perpendicular to a third never intersect, regardless of how far they are extended.

5. A straight line always inter-

Art by V. Ivanyuk



sects another in going from one side of it to the other side. In other words, one straight line must intersect another if it contains points on both sides of the other line.

6. Vertical angles whose sides are continuations of the other angle's sides are equal. This holds for plane rectilinear angles (among themselves) as well as for dihedral angles.

7. Two straight lines cannot intersect if a third line intersects them at the same angle.

8. In a rectilinear triangle, equal sides lie opposite equal angles, and equal angles lie opposite equal sides.

9. In a rectilinear triangle a larger side lies opposite a larger angle. In a right triangle, the hypotenuse is larger than either of the other sides, and the two angles adjacent to it are acute.

10. Rectilinear triangles are congruent if they have a side and two angles equal, or two sides and the included angle equal, or two sides and the angle opposite the larger side equal, or three sides equal.

11. A straight line that stands at right angles to two other straight lines not in the same plane with it is perpendicular to all straight lines drawn through the common intersection point in the plane of the other two lines.

Parallel lines

The following presents the other theorems with their explanations and proofs.

Theorem 1. *All straight lines in a plane that pass through a point can be divided, with reference to a given straight line in the same plane, into two classes—intersecting and nonintersecting. The boundary lines between these classes are called the* parallels *to the given line.*

From point A (fig. 1) drop a perpendicular AD onto line BC, to which we draw a perpendicular AE. In the right angle EAD, either all straight lines that go out from point A will intersect DC, as for example AF, or some of them, like the perpendicular AE, will not intersect DC. In the uncertainty whether the perpendicular AE is the only line that does not intersect DC, we will

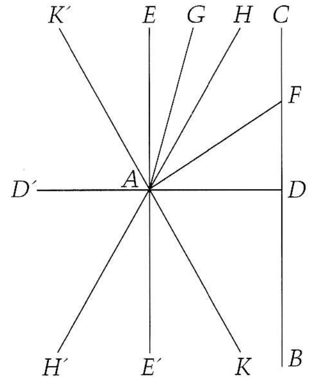


Figure 1

assume it may be possible that other lines exist, for example AG, that do not intersect DC, regardless of how far they are extended. In passing over from the intersecting lines, like AF, to the nonintersecting lines, like AG, we must come upon a line AH parallel to DC, a boundary line on one side of which all lines AG do not intersect DC while on the other side every straight line AF intersects DC.

Angle HAD between the parallel HA and the perpendicular AD is called the parallel angle, which we will here designate by $\Pi(p)$ if $AD = p$. If $\Pi(p)$ is a right angle, the extension AE' of the perpendicular AE will be parallel to the extension DB of DC. Furthermore, we note that in regard to the four right angles, which are made at point A by the perpendiculars AE and AD and their extensions AE' and AD', every straight line that passes through the point A, either itself or at least its extension, lies in one of the two right angles that are closest to BC, so that excepting the parallel EE', all others, if they are sufficiently extended both ways, must intersect the line BC.

If $\Pi(p) < (1/2)\pi$, then on the other side of AD, a line AK will also lie parallel to the extension DB of the line DC and making the same angle $DAK = \Pi(p)$, so that under this assumption we must also distinguish two *sides of parallelism*.

All remaining lines or their extensions within the two right angles closest to BC belong to the set of those that intersect if they lie within the angle $HAK = 2\Pi(p)$ between the parallels. On the other hand, they belong to the set of nonintersecting lines if they lie on the other sides of the parallels AH and AK, in the opening of the two angles $EAH = (1/2)\pi - \Pi(p)$ and $E'AK = (1/2)\pi - \Pi(p)$ between the parallels and EE', the perpendicular to AD. On the other side of the perpendicular EE', the extensions AH' and AK' of the parallels AH and AK will likewise be parallel to BC. The remaining lines belong, if they are in the angle $K'AH'$, to those that intersect, but if they are in the angles $K'AE$ and $H'AE'$, they belong to those that don't intersect.

In accordance with this, for the assumption $\Pi(p) = (1/2)\pi$, the lines can be only intersecting or parallel, but if we assume that $\Pi(p) < (1/2)\pi$, then we must allow two parallels, one on the one side and one on the other side. In addition, we must group the remaining lines into nonintersecting and intersecting.

For both assumptions it serves as the mark of parallelism that the line becomes intersecting for the smallest deviation toward the side where the parallel lies, so that if AH is parallel to DC, every line AF intersects DC, however small angle HAF may be.

Theorem 2. *A straight line maintains the characteristic of parallelism at all its points.*

It is given that AB (fig. 2) is parallel to CD and that AC is perpendicular to CD. We will consider two points taken at random on the line AB and its extension beyond the perpendicular.

Let the point E lie on the side of the perpendicular on which AB is looked upon as parallel to CD. From the point E, drop a perpendicular EK on CD, and draw EF in such a way that it falls within the angle BEK.

Connect the points A and F by a straight line, whose continuation then (by theorem 1) must intersect

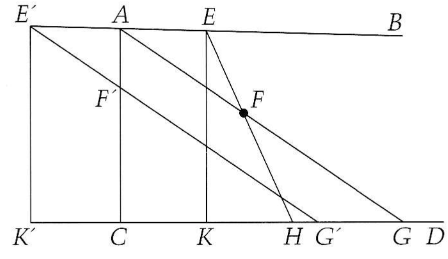


Figure 2

CD somewhere in G. Thus we get a triangle ACG, into which the line EF goes. Now since EF, from the construction, cannot intersect AG or EK a second time (proposition 2), then it must meet CD somewhere at H (proposition 3).

Now let E' be a point on the continuation of CD. Draw the line $E'F'$, making angle $AE'F'$ so small that $E'F'$ intersects AC somewhere at F'. Then, making the same angle with AB, draw the line AF from A, whose continuation will intersect CD at G (theorem 1). Thus we get a triangle AGC, into which the continuation of line $E'F'$ goes. Because this line now cannot intersect AC a second time and also cannot intersect AG, since angle $BAG = BE'G'$ (proposition 7), then it must intersect CD somewhere at G'.

Therefore, from whatever points E and E' the lines EF and $E'F'$ emanate, and however little they may diverge from the line AB, they will always intersect CD, to which AB is parallel.

Theorem 3. *Two lines are always mutually parallel.*

Let AC be a perpendicular on CD, to which AB is parallel (fig. 3). If we draw from C the line CE, making any acute angle ECD with CD, and drop the perpendicular AF from A onto CE, we obtain a right triangle ACF, in which AC, being the hypotenuse, is greater than the side AF (proposition 9).

Next we make $AG = AF$ and slide the figure $EFAB$ until AF coincides with AG, such that AB and FE take the positions AK and GH, and such that $\angle BAK = \angle FAC$. Consequently, AK must intersect the line DC somewhere at K (theorem 1), thus forming a triangle AKC, on one side of which the perpendicular GH intersects the

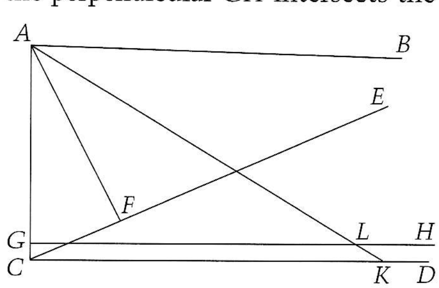


Figure 3

line AK at L (proposition 3), and thus determines the distance AL of the intersection point of the lines AB and CE on the line AB from point A.

It follows that CE will always intersect AB, regardless of how small the angle ECD may be. Consequently, CD is parallel to AB (theorem 1).

The sum of the angles of a rectilinear triangle

Theorem 4. *In a rectilinear triangle the sum of the three angles cannot be greater than two right angles.*

Suppose in the triangle ABC (fig. 4) the sum of the three angles is equal to $\pi + \alpha$. In the case where the sides are not equal, choose the smallest side BC, halve it at D, draw the line AD from A through D, and make the continuation of it, DE, equal to AD. Then join the point E to the point C by the straight line EC. In the congruent triangles ADB and CDE, $\angle ABD = \angle DCE$, and $\angle BAD = \angle DEC$ (propositions 6 and

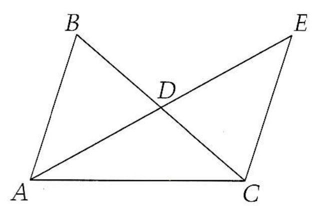


Figure 4

10). It follows that the sum of the three angles in triangle ACE must be equal to $\pi + \alpha$. Furthermore, the smallest angle BAC (proposition 9) of triangle ABC in passing over into the new triangle ACE has been divided into $\angle EAC$ and $\angle AEC$. Continuing this process of continually halving the side opposite the smallest angle, we must finally obtain a triangle in which the sum of the three angles is $\pi + \alpha$, but in which there are two angles each of which in absolute magnitude is less than $(1/2)\alpha$. However, because the third angle now cannot be greater than Π, α must either be zero or negative.

Theorem 5. *If in any rectilinear triangle the sum of the three angles is equal to two right angles, then this is also the case for every other triangle.*

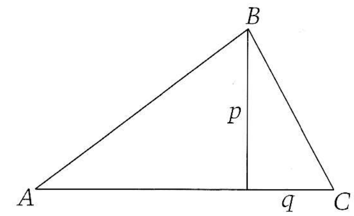


Figure 5

If in any rectilinear triangle ABC (fig. 5) the sum of the three angles is π, then at least two of its angles, A and C, must be acute. From the vertex of the third angle B, we drop the perpendicular p onto the opposite side AC. This divides the triangle into two right triangles, in each of which the sum of the three angles must also be π, since it cannot in either be greater than π, and in their combination not less than π.

So we obtain a right triangle with the perpendicular sides p and q, and from this a quadrilateral whose opposite sides are equal and whose adjacent sides p and q are at right angles (fig. 6).

By repetition of this quadrilateral we can make another with sides np and q, and finally a quadrilateral $ABCD$ with sides at right angles to each other, such that $AB = np$, $AD = mq$, $DC = np$, and $BC = mq$, where m and n are any whole numbers. Such a quadrilateral is divided by the diagonal DB into congruent right triangles BAD and BCD, in each of which the sum of the three angles is π.

The numbers m and n can be taken sufficiently large for the right triangle ABC (fig. 7), whose perpendicular sides $AB = np$ and $BC = mq$, to enclose within itself another given (right) triangle BDE, with the

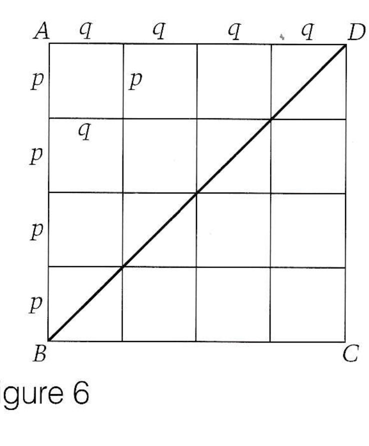


Figure 6

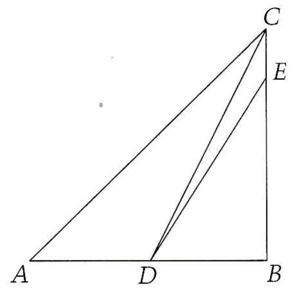


Figure 7

right angles coinciding. Drawing the line DC, we obtain right triangles of which every successive two have a side in common.

The triangle ABC is formed by the union of the two triangles ACD and DCB, in neither of which can the sum of the angles be greater than π. Consequently, the sum of the angles must be equal to π so that the sum in the compound triangle may be equal to π.

In the same way, triangle BDC consists of the two triangles DEC and DBE. Therefore, the sum of the three angles in DBE must be equal to π, and in general this must be true for every triangle, since each can be cut into two right triangles.

From this it follows that only two hypotheses are allowable: Either the sum of the three angles in all rectilinear triangles is equal to π or this sum is less than π.

Theorem 6. *From a given point we can always draw a straight line that makes an angle as small as we choose with a given straight line.*

From the given point A (fig. 8) we drop the perpendicular AB onto the given line BC. We then randomly put the point D on BC, draw the line AD, make $DE = AD$, and draw AE.

In the right triangle ABD, let the angle $ADB = \alpha$. Then in the isosceles triangle ADE the angle AED must be less than or equal to $(1/2)\alpha$

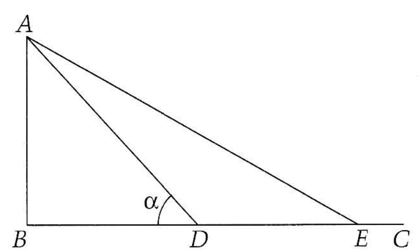


Figure 8

(proposition 8 and theorem 5). Continuing in the same manner, we finally obtain such an angle AEB that is less than any given angle.

Theorem 7. *If two perpendiculars to the same straight line are parallel to each other, then the sum of the three angles in a rectilinear triangle is equal to two right angles.*

Let the lines AB and CD (fig. 9) be parallel to each other and perpendicular to AC. Draw from A the lines AE and AF to the points E and F, which are taken on the line CD at any distances $FC > EC$ from C.

Suppose that in the right triangle ACE the sum of the three angles is equal to $\pi - \alpha$ and that in triangle AEF this sum is equal to $\pi - \beta$. Then in triangle ACF it must equal $\pi - \alpha - \beta$, where α and β cannot be negative.

Further, let $\angle BAF = a$ and $\angle AFC = b$ so that $\alpha + \beta = a - b$. Now by revolving the line AF away from the perpendicular AC, we can make the angle a between AF and the parallel AB as small as we choose, and we can also diminish the angle b. Consequently, the two angles α and β can have no other magnitude than $\alpha = 0$ and $\beta = 0$.

It follows that in all rectilinear triangles the sum of the three angles is either π and that the parallel angle $\Pi(p) = (1/2)\pi$ for every line p, or for all triangles this sum is less than Π and $\Pi(p) < (1/2)\pi$.

The first assumption serves as foundation for *ordinary geometry* and *plane trigonometry*.

The second assumption can likewise be admitted without leading to any contradiction in the results, and founds a new geometric science, to which I have given the name *Imaginary Geometry*, and which I intend here to expound as far as the development of the equality between the

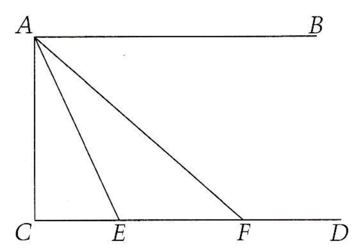


Figure 9

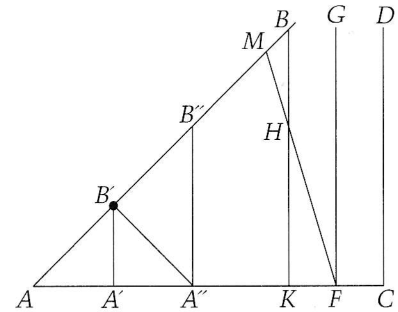


Figure 10

sides and angles of the rectilinear and spherical triangle.

Analysis of the angle of parallelism

Theorem 8. *For every given angle α there is a line p such that $\Pi(p) = \alpha$.*

Let AB and AC (fig. 10) be two straight lines that make the acute angle a at their point of intersection A. Take at random a point B' on AB, and from this point drop $B'A'$ at right angles to AC. Make $A'A'' = AA'$, then construct at A'' the perpendicular $A''B''$, and continue in this manner until a perpendicular CD that no longer intersects AB is obtained. This must happen, because if in triangle $AA'B'$ the sum of all three angles is equal to $\pi - a$, then in triangle $AB'A''$ it equals $\pi - 2a$, in triangle $AA''B''$ it is less than $\pi - 2a$ (theorem 5), and so forth, until it finally becomes negative and thereby shows the impossibility of constructing the triangle.

It may happen that the perpendicular CD is the one such that for all points nearer to A, the perpendicular to AC intersects AB. But even if CD is not this perpendicular, such a perpendicular must exist, as we pass from those that intersect AB to those that do not.

Now draw from point F a line FH that makes with FG an acute angle HFG on the side where point A lies. From any point H on the line FH, drop onto AC the perpendicular HK whose continuation consequently must intersect AB somewhere at B and thus makes a triangle AKB into which the continuation of FH enters and must therefore intersect the hypotenuse AB somewhere at M. Since angle GFH is arbitrary and can be taken as small as we wish, then

FG is parallel to AB and $AF = p$ (theorems 1 and 3).

We can easily see that as p decreases, angle α increases, and for $p = 0$, it approaches the value $\pi/2$. As p grows, angle α decreases, and it continually approaches zero for $p = \infty$.

Since we are at liberty to choose what angle we will understand by the symbol $\Pi(p)$ when the line p is expressed by a negative number, we will assume that $\Pi(p) + \Pi(-p) = \pi$, an equation that holds for all values of p, positive as well as negative, and for $p = 0$.

Relative position of parallel lines

Theorem 9. *The farther parallel lines are extended on the side of their parallelism, the more they approach one another.*

If to the line AB (fig. 11) two perpendiculars $AC = BD$ are constructed and their endpoints C and D are joined by a straight line, then the quadrilateral $CABD$ will have two right angles at A and B and two acute angles at C and D (theorem 7) that are equal to one another, as we can easily see by imagining the quadrilateral superimposed upon itself so that BD falls upon AC and AC upon BD.

Next we halve AB and erect at

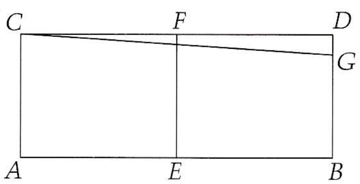


Figure 11

the midpoint E the line EF perpendicular to AB. This line must also be perpendicular to CD, since the quadrilaterals $CAEF$ and $FDBE$ fit one another if we place one on the other in such a way that the line EF remains in the same position. Therefore, CD cannot be parallel to AB, but the parallel to AB for the point C, namely CG, must incline toward AB (theorem 1) and intersect from the perpendicular BD a part $BG < CA$. Since C is a random point in the line CG, it follows that CG itself nears AB the farther it is extended.

Theorem 10. *Two straight lines that are parallel to a third are also parallel to each other.*

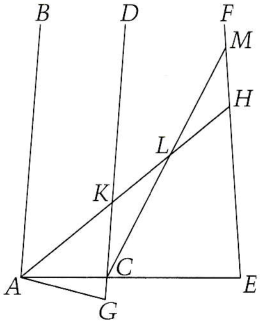


Figure 12

We first assume that the three lines AB, CD, and EF (fig. 12) lie in one plane. If two of them in order, AB and CD, are parallel to the outermost one EF, then AB and CD are parallel to each other. To prove this, drop from any point A of the outer line AB onto the other outer line FE the perpendicular AE, which will intersect the middle line CD at some point C (proposition 3) at an angle $DCE < \pi/2$ on the side toward EF, the parallel to CD (theorem 7). A perpendicular AG drawn from the same point A upon CD must fall within the opening of the acute angle ACG (proposition 9). Every other line AH from A drawn within the angle BAC must intersect EF, the parallel to AB, somewhere in H, regardless of how small the angle BAH may be. Consequently, the line CD in triangle AEH will intersect the line AH somewhere in K, since it is impossible for it to meet EF. If AH emerged from point A within the angle CAG, then it must intersect the continuation of CD between the points C and G in the triangle CAG. Therefore, AB and CD are parallel (theorems 1 and 3).

If both the outer lines AB and EF were assumed parallel to the middle line CD, then every line AK from the point A drawn within the angle BAE would intersect CD somewhere at the point K, regardless of how small the angle BAK may be. On the continuation of AK take at random a point L and join it with C by the line CL, which must intersect EF somewhere in M, thus making a triangle MCE. The continuation of AL within the triangle MCE can intersect neither AC nor CM a second time. Consequently, it must meet EF somewhere in H. Therefore, AB and EF are mutually parallel.

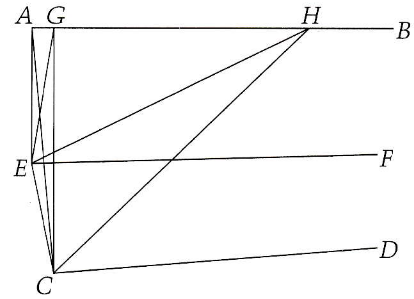


Figure 13

Now let the parallels AB and CD (fig. 13) lie in two planes whose intersection line is EF. From a random point E on EF drop a perpendicular EA onto one of the two parallels (for example, upon AB), then from A, the foot of the perpendicular EA, drop a new perpendicular AC onto the other parallel CD and join the endpoints E and C of the two perpendiculars by the line EC. The angle BAC must be acute (theorem 7). Consequently, a perpendicular CG from C to AB intersects it at the point G on the side of CA on which the lines AB and CD are considered parallel. Every line EH (in the plane $FEAB$), regardless of how little it diverges from EF, belongs with the line EC to a plane that must intersect the plane of the two parallels AB and CD along some line CH. This latter line intersects AB somewhere at the very point H that is common to all three planes through which the line EH must also pass. Consequently, EF is parallel to AB.

In the same way, we can show the parallelism of EF and CD.

Therefore, the hypothesis that a line EF is parallel to one of the two other parallels, AB and CD, is the same as considering EF as the intersection of two planes in which two parallels, AB and CD, lie. Consequently, two lines are parallel to one another if they are parallel to a third line although the three are not coplanar.

The last theorem can thus be expressed as follows:

Three planes intersect in lines that are all parallel to each other if the parallelism of two is presupposed.

Coalescing droplets

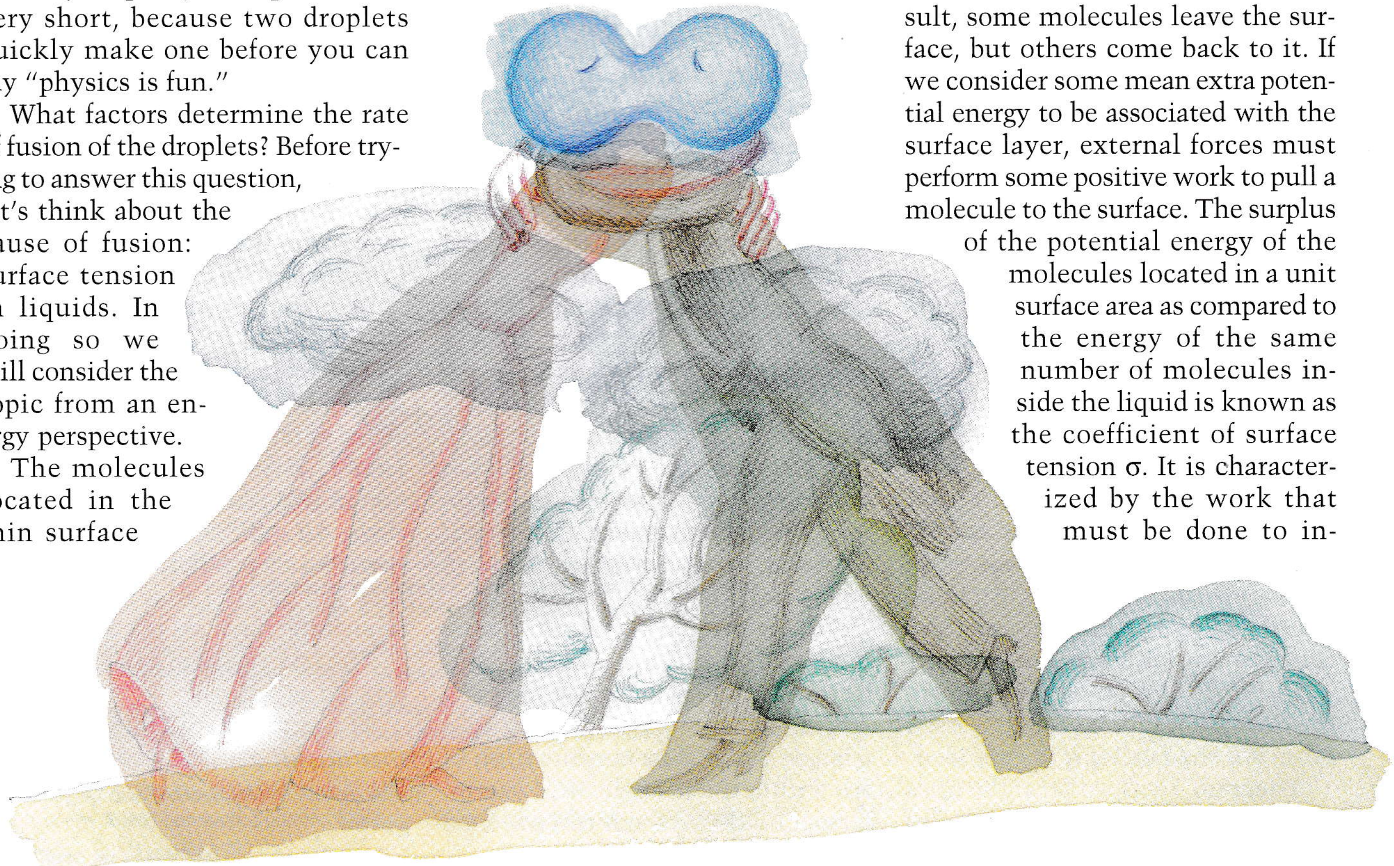
by A. Varlamov

DID YOU EVER RUMINATE in your early days over a bowl of chicken soup topped with golden droplets of grease? If you had no appetite at the time, you surely played with these droplets in the bowl, observing how they slowly joined together and assumed circular shapes.

Similar observations can be made by watching droplets of mercury from a broken thermometer fuse together (but please don't try this, because mercury is extremely toxic!) However, in the case of the mercury droplets, the experiment is very short, because two droplets quickly make one before you can say "physics is fun."

What factors determine the rate of fusion of the droplets? Before trying to answer this question, let's think about the cause of fusion: surface tension in liquids. In doing so we will consider the topic from an energy perspective.

The molecules located in the thin surface layer "live" in particular conditions. The point is that they have neighboring siblings only on one side, in contrast to the molecules located within the liquid that are surrounded by similar molecules. At not very small distances, the molecular interaction results in attraction. This means that if the potential energy of two molecules located far from each other (at an infinite distance) is taken to be zero, it will be negative at smaller distances. To a first approximation, the absolute value of the potential energy of a molecule can be taken to be proportional to the number of its closest neighbors.



We can see that the potential energy of the surface molecules is higher than that of the internal molecules, because the latter have more neighbors. Another factor, which also elevates the potential energy of the surface molecules, is the decrease in molecular concentration when approaching the surface.

It goes without saying that the molecules in a liquid are not stable. On the contrary, they are involved in constant thermal motion. As a result, some molecules leave the surface, but others come back to it. If we consider some mean extra potential energy to be associated with the surface layer, external forces must perform some positive work to pull a molecule to the surface. The surplus of the potential energy of the molecules located in a unit surface area as compared to the energy of the same number of molecules inside the liquid is known as the coefficient of surface tension σ. It is characterized by the work that must be done to in-

Art by Virginia Smith

crease the surface area of the liquid by one unit. Of course, this definition of σ is equivalent to the common one as the force affecting a unit length of liquid surface.

It is known that among all possible states of a system, the stable one corresponds to the minimum potential energy. In particular, the surface of a liquid tends to assume a shape that minimizes the surface energy for the given conditions. Thus, a droplet assumes a spherical shape in conditions when gravity can be neglected, because this shape minimizes the surface area and is therefore the most economical in terms of energy. Similarly, it is advantageous for two or more droplets to fuse together to make a single drop, because the surface of the resulting drop will be smaller than the total surface of the original droplets (a fun proof to try on your own), and the surface energy of this single drop will be correspondingly smaller.

Now let's return to the question we began with: What factors determine the fusing time of two droplets? This problem arose long ago. And it was worthwhile, because the theory of droplet fusion did not result from idle curiosity. On the contrary, it had profound practical applications, particularly in explaining the technology of powder metallurgy, where pressed metal grains are thermally fused with substances that have unique properties.

In 1944 the Soviet physicist Y. I. Frenkel proposed a very simple model of fusing that became a cornerstone of the principles of powder metallurgy. The main idea of Frenkel's work will be our tool to evaluate the fusing time of liquid droplets. The simplest approach is through energy considerations.

Let two identical droplets make contact at some moment. A narrow "neck" is formed at the contact point (figure), which gradually grows until fusion is completed. What happens with the energy in this process?

The total "energy deposit" of the two-droplet system consists of the extra energy ΔE_S, which is equal to

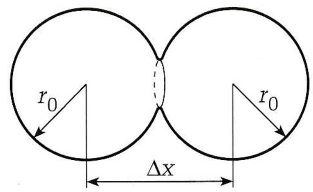


Figure 1

the difference between the surface energies of the initial and final states, or in other words, to the difference of the energies of two individual droplets of radius r_0 and of one compound drop with radius r:

$$\Delta E_S = 8\pi\sigma r_0^2 - 4\pi\sigma r^2.$$

Since the total volume of the droplets does not change during fusion,

$$\frac{4}{3}\pi r^3 = 2\frac{4}{3}\pi r_0^3,$$

from which we get

$$r = r_0\sqrt[3]{2}.$$

Thus,

$$\Delta E_S = 4\pi\sigma\left(2 - 2^{2/3}\right)r_0^2.$$

According to Frenkel, this extra energy must be spent on the work performed against the forces of viscous friction that arise during fusion. Let's evaluate this work.

We find the force of viscous friction using a formula found by the legendary English physicist and mathematician Sir George Gabriel Stokes (1819–1903) for a ball of radius R moving in a viscous fluid with velocity v:

$$F = 6\pi\eta R v.$$

The dimensional coefficient η in this formula (known as Stokes' law) is called the viscosity coefficient, or just viscosity. It characterizes the ability of a fluid to slow the relative motion of adjacent fluid layers. The force of viscous friction, which is generated during the fusion of liquid droplets, depends only on the viscosity of the liquid, the linear size of the droplets, and the rate of fusion. Thus, to estimate the order of magnitude of these forces, we may safely use Stokes' formula, making the following substitutions. We insert r_0, the radius of the droplets, in place of R; v will be the rate of fusion; and η will be the viscosity of the liquid. Thus, we get the following formula for the force of viscous friction:

$$F = 6\pi\eta r_0 v.$$

Note that the displacement of the liquid during fusion has the same order of magnitude as the radius of a droplet: $\Delta x = r_0$. Therefore, the friction forces perform the work

$$W = F\Delta x = 6\pi\eta r_0^2 v.$$

This formula shows that the quicker the fusion, the more energy is needed to overcome the larger viscous forces. However, the amount of energy is limited by the value ΔE_S, which determines the duration of the droplet fusion (Frenkel fusion time). Estimating it as $v = r_0/\tau$, and assuming $W = \Delta E_S$, we get

$$6\pi\eta r_0^2 \frac{r_0}{\tau} = 4\pi\sigma\left(2 - 2^{2/3}\right)r_0^2,$$

or

$$\tau \cong \frac{r_0\eta}{\sigma}.$$

Equipped with this formula, we consider some examples. For water droplets, $r_0 \cong 1$ cm, $\sigma \cong 0.1$ N/m, and $\eta \cong 10^{-3}$ kg/m · s, so the fusion time is about 10^{-4} s. However, this time is far greater for the more viscous glycerin ($\eta \cong 1$ kg/(m · s)). Therefore, the fusion time for droplets of the same radius can vary widely, depending on the viscosity and surface tension of the liquid. Moreover, in contrast to surface tension, viscosity depends strongly on temperature, so the fusion time can vary to a large extent even for the same liquid. ◘

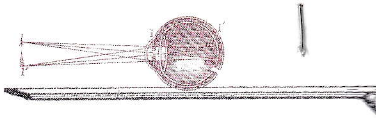
Quantum on surface tension and drops:

Y. Bruk, M. Zelnikov, A. Stasenko, "Wobbling nuclear drops," January/February 1997, pp.12–17.

Y. Bruk, A. Stasenko, "Drops for the crops," March/April 1994, pp. 10–13.

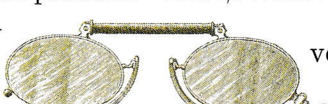
A. Mitrofanov, "Bubbles in puddles," July/August 1995, pp. 4–7.

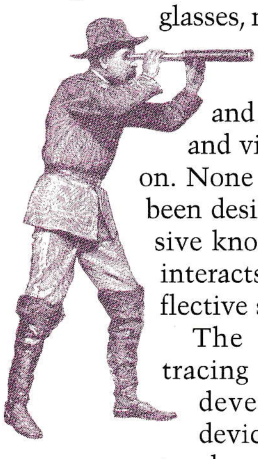
Can you trace



THE SUBJECT OF THIS KALEIDOSCOPE brings together many scientists: the author of the laws of planetary motion, the mathematician known for his famous theorems, the creator of classical mechanics, the ingenious experimentalist who carried out very difficult experiments to determine the speed of light in various media, and many other celebrities. Their scientific interests were diverse, but all of them tried to decide if it was possible to trace the paths of light beams and how to do it correctly.

The importance of this problem becomes apparent by a simple enumeration of optical devices: eyeglasses and magnifying glasses, microscopes and telescopes, various film projectors and cameras, binoculars and video cameras, and so on. None of these could have been designed without extensive knowledge of how light interacts with lenses and reflective surfaces.





The importance of ray tracing extends beyond the development of optical devices. The models and concepts advanced in optics fertilized other scientific fields, as evidenced by terms such as *electron microscope*, *neutron mirror*, and *optical computer.*

The following set of problems shows both the beauty of ray diagrams and the logic of the optical laws on which they are based.

Problems and questions

1. The setting Sun illuminates a lattice fence through a gap between the clouds. Why are there no vertical rods in the shadow cast by the fence onto a wall, while the shadows of the horizontal rods are clearly seen? All rods have the same diameter.

2. Figure 1 shows the region of complete visibility of a straight object in a plane mirror (shaded with straight lines) and the region of partial visibility (shaded with curved lines). Where is the object located?

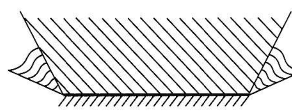


Figure 1

3. Solar rays strike a vertical screen after reflecting from a large horizontal mirror. A king from a chess game is placed on the mirror. What is the size of its shadow on the screen?

4. Why are the illuminating mirrors of microscopes usually concave?

5. A piece of plate glass is placed between a luminous point and an eye. Draw the ray diagram and obtain the image of this point.

6. The distance between an object and its image formed by a thin lens is $0.5F$, where F is the focal length of the lens. Is the image virtual or real?

7. A converging lens produces an image of a source at point S' on the principal optical axis. The locations of the lens's center and foci are known. Using a ray diagram, locate the source S, if $OF < OS'$.

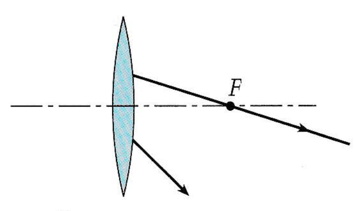


Figure 2

8. Using a ray diagram, find the position of a point of light, if after refraction in the lens, the two rays travel as shown in figure 2.

9. Figure 3 shows object AB and its image $A'B'$ formed by a thin lens.

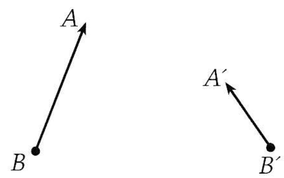


Figure 3

Using a ray diagram, locate the lens and its foci.

10. The image of some rectilinear continuous object AB consists of two semi-infinite parts, one real and one virtual. Looking at figure 4, restore the position of the object.

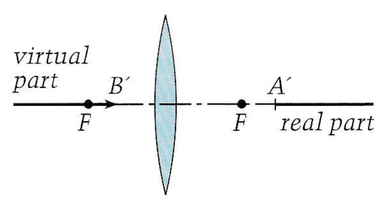


Figure 4

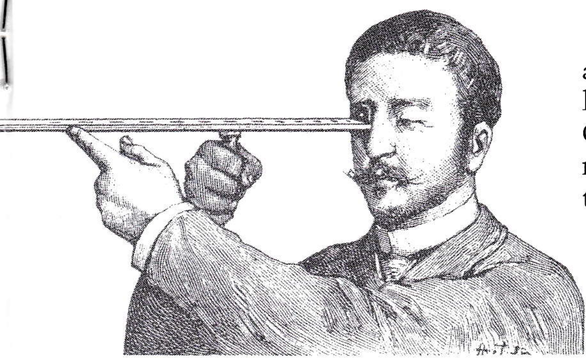
11. Is it possible to take a photograph of an image?

12. At what location on the optical axis of a converging lens should a point source be placed so that neither the source nor its image can be seen simultaneously from any point?

13. How are two lenses (one converging, another diverging) positioned if parallel rays remain parallel after passing through them?

14. Draw the image of an object in the optical system consisting of a converging lens and a plane mirror located in the focal plane of the lens. The object is set in front of the lens

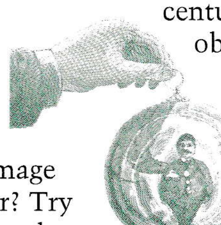
ce the rays?



a distance between one and two focal lengths.

15. Why can a telescope launched in space detect much dimmer stars than a ground-based telescope of the same diameter?

Microexperiment

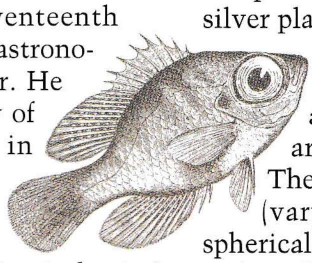


Place two mirrors at right angles to each other in the corner of a room. What does your image look like in such a mirror? Try to find a place in the room where you cannot see your image.

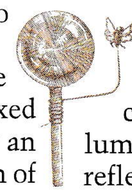
It is interesting that . . .

. . . Even in the late Renaissance, optical phenomena and vision were treated as mysterious and suspicious subjects. This may explain why even such an outstanding optician as Francesco Maurolico did not dare publish his main work until 1575, the year of his death.

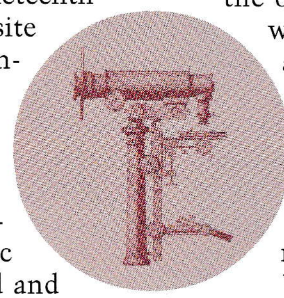
. . . The first scientifically correct ray tracing in an eye was made at the beginning of the seventeenth century by the great astronomer Johannes Kepler. He developed the theory of image construction in optical systems and introduced such fundamental notions as the "focus" and the "optical axis."



. . . The telescope invented by Galileo was considered a miracle, and people came by the dozens to look through it. The salary of Galileo was doubled after he donated a model of the device to the Venetian senate.



. . . A simple microscope made of a magnifying glass fixed on a support was replaced by an intricate device with a system of lenses in the seventeenth century, an invention almost simultaneous with the telescope. Credit for its invention probably also belongs to the Dutch. However, such a microscope could not compete with a magnifying glass until the nineteenth century, when composite objectives were invented.



. . . In his effort to improve the refracting telescope, Isaac Newton invented and made a device "using a concave metal instead of glass objective"—that is, a concave mirror. For the invention of the reflecting telescope, Newton became a member of the London Royal Society in 1672.

. . . Similar to many outstanding scientists, Jean Foucault invented original instruments, including astronomical devices. For example, to produce reflecting telescopes, he developed a very important method of silver plating glass.

. . . A sharp image in the eye of a fish is formed just as in cameras when they are adjusted for sharpness. The fish do not accommodate (vary) the curvature of the spherical cornea as humans do, but rather shift the cornea back and forth with specialized muscles.

. . . To improve image quality, the objectives of modern cameras are composed of several lenses of various types of glass. The design is so intricate that computers must be used for the calculations. Purely geometrical ray tracing cannot account for all the loss of luminous flux during the numerous reflections on the lens surfaces.

. . . In recent times the amount of astronomical data has increased immensely, mainly due to the work of the record-holder 10-meter multiple-mirror Keck telescope on Mauna Kea, Hawaii, and also due to the orbiting Hubble telescope, which has a mirror with a diameter of 2.4 meters.

—A. Leonovich

ANSWERS, HINTS & SOLUTIONS ON PAGE 53

Quantum on lenses, mirrors, and ray tracing:

V. Dubrovsky and A. Savin, "What harmony means," January/February 1993, pp. 32–35.

A. Eisenkraft and L. D. Kirkpatrick, "Mirror full of water," July/August 1994, pp. 32–34.

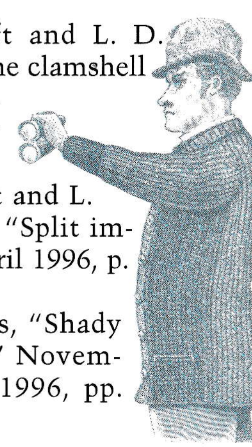
A. Zilberman, "A little lens talk," May/June 1994, pp. 35–37.

S. Semenchinsky, "Focusing on the fleet," September/October 1993, pp. 28–30.

A. Eisenkraft and L. D. Kirkpatrick, "The clamshell Mirrors," September/October 1992, p. 26.

A. Eisenkraft and L. D. Kirkpatrick, "Split image," March/April 1996, p. 36.

C. W. Bowers, "Shady computations," November/December 1996, pp. 34–36.



PHYSICS CONTEST

The eyes have it

by Arthur Eisenkraft and Larry D. Kirkpatrick

The history of the living world can be summarised as the elaboration of ever more perfect eyes within a cosmos in which there is always something more to be seen.
—Pierre Teilhard de Chardin (1881–1985)

SIX OF THE 30 PHYLA OF ANImals have eyes that can produce images. These mere six dominate the animal kingdom with over 95 percent of the population of animals on Earth. It is no surprise that eyes provide such a distinct advantage for survival. The blind species have to nudge up to another object to detect its presence. Does this object present itself as an obstacle, a potential food, or a potential predator? Animals with eyes have a remote sensing apparatus that allows them to avoid obstacles and predators and survey the environment for food.

The complexity of the human eye confounded Darwin. In *On the Origin of Species*, he wrote, "To suppose that the eye, with all its inimitable contrivances for adjusting the focus to different distances, for admitting different amounts of light and for the correction of spherical and chromatic aberration, could have formed by natural selection, seems, I freely confess, absurd in the highest possible degree." But Darwin's concept of adaptation and natural selection guided a steady stream of biologists who have collectively depicted a series of 40 steps, each a small advantage over the prior, which describe the evolutionary trail of the eye. You can read about this journey in Richard Dawkins's *Climbing Mount Improbable*.

A lens can be crudely modeled as a triangular prism atop a cube resting on an inverted prism. Three parallel rays of light, carefully placed, will converge at a single point. The design of a good lens requires us to "smooth" the sides of the prisms and cube so that all rays of light, undergoing refraction, will converge at a single point—the focus (figure 1.)

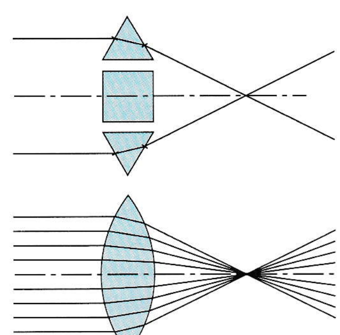


Figure 1

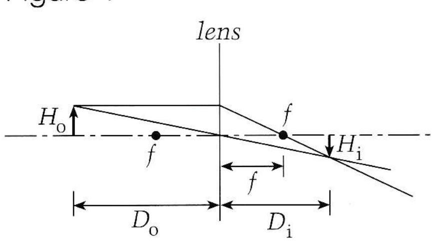


Figure 2

Using these rays of light, we can create ray diagrams—a graphical means of determining the location and orientation of the image of an illuminated object (see Kaleidoscope, page 28). One ray of light emanating from the tip of the object travels parallel to the principal axis, refracts through the lens, and travels through the focus. A second ray of light leaves the object and travels through the near focus emerging after refraction parallel to the principal axis. A third ray travels through the center of the lens, and if the lens is thin, is displaced negligibly (figure 2).

A look at similar triangles yields two equations:

$$\frac{H_i}{H_o} = \frac{D_i}{D_o},$$

$$\frac{H_i}{H_o} = \frac{D_i - f}{f}.$$

Combining these yields the lens equation, $1/f = 1/D_o + 1/D_i$. A more formal proof requires us to use Snell's law to calculate the change in angle of the light as it enters (and then leaves) the glass.

Assuming that light emanates from point O and is brought to focus at point I inside the glass as shown in figure 3, we can draw a

Art by Tomas Bunk


STARRY EYES
Z
LOOK
WINDOW 2000
BLIND EYE
FOCUS
LEFT BRAIN - RIGHT BRAIN
PANVISION
THIRD EYE
EYE OPENER LIGHT
TOMAS BUNK

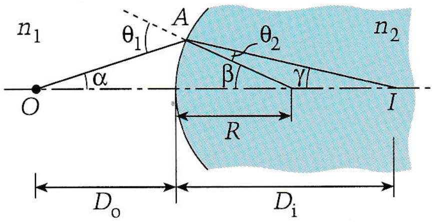


Figure 3

radial line from the center of curvature of the lens to the interface at point A. We can assume that the two media have indices of refraction n_1 and n_2. Thus we have the following equations:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$
$$\theta_1 = \alpha + \beta,$$
$$\beta = \theta_2 + \gamma.$$

Assume that all angles are small because α is small. We can then use the approximation that $\sin\theta = \theta$ in radians:

$$n_1\theta_1 \cong n_2\theta_2,$$
$$n_1(\alpha + \beta) \cong n_2(\beta - \gamma),$$
$$\beta(n_2 - n_1) \cong n_1\alpha + n_2\gamma.$$

The arc length is equal to the radius multiplied by the angle subtended (in radians). Therefore, for small angles we have

$$\alpha \cong \frac{AB}{D_o},$$
$$\beta = \frac{AB}{R},$$
$$\gamma \cong \frac{AB}{D_i},$$

and

$$\frac{n_1}{D_o} + \frac{n_2}{D_i} = \frac{n_2 - n_1}{R}.$$

Completing a parallel derivation for the ray of light leaving n_2 and refracting at a concave surface (that is, a double convex lens—convex from each side) into n_1, we emerge with the lensmaker's equation:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

We have set $n_1 = 1$ for air and $n_2 = n$ for glass. The R_2 is now negative for the double convex lens. If the lens is thick, the proof becomes a bit more tedious and yields an equation that includes the thickness d of the lens:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} - \frac{n-1}{n}\cdot\frac{d}{R_1R_2}\right).$$

The contest problems for this month focus on different elements in our story.

A. In the human eye, much of the imaging is due to the cornea, since the light must travel from air ($n = 1.00$) to the cornea ($n = 1.376$) before reaching the lens. The purpose of the lens is to change its focal length for accommodation. Assuming that the fixed image distance (lens to retina) is 2.50 cm, calculate the focal length of the lens/cornea system when the object distance is 20 cm and when it is 20 m.

B. If the human eye were constructed of a fixed focal length lens and moved it for accommodation (as a fish does), what distance would the lens have to move to accommodate the object distances of 20 cm and 20 m? (By the way, mollusks expand or contract the entire eye, and birds of prey change the curvature of the cornea for accommodation.)

C. Describe what would happen to the image of a candle if

(1) the top half of the lens were covered,

(2) the candle were much larger than the lens diameter.

D. A student completes a lens lab and records the data shown in figure 4. Calculate the focal length, graph the data with D_o on the x-axis and D_i on the y-axis, and derive the lens equation from the graph.

E. The prism from our crude lens would disperse the light into the familiar spectrum. A lens that distorts in this way is said to have chromatic aberration. In the VIII International Physics Olympiad (East Germany, 1975), students were asked to find the conditions for a thick lens such that the focal length would be the same for two different wavelengths. Please solve this problem and discuss the practical limitations of your solution with different types of lenses.

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington, VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space.

Warp speed

Our contest problem on superluminary velocities in the November/December issue of *Quantum* was quite a hit. Art Hovey and nine students in his honors physics class at the Amity Regional High School in Woodridge, Connecticut, sent in correct solutions. The students are Steve Boyle, Stephanie Brelsford, Sid Govindan, Alex Kaloyanides, Maya Roberts, Ted Rounsaville, Samira Saleh, Mirosla Volynskiy, and Aaron Webber. Scott Wiley, a teacher at Weslaw High School in Texas, also submitted a correct solution.

A. The experimental data obtained from the photographs of the microquasar indicate that the ejecta had angular velocities of 17.6 and 9.0 milliarcseconds (mas) per day. We can obtain the transverse velocities knowing that $v = \omega r$ and converting units:

$$\begin{aligned} v &= \omega r \\ &= \left(17.6\,\frac{\text{mas}}{\text{day}}\right)\left(3.86\times10^{20}\ \text{m}\right) \\ &\times \left[\frac{1\ \text{as}}{10^3\ \text{mas}}\right]\left[\frac{1^\circ}{3600\ \text{as}}\right] \\ &\times \left[\frac{2\pi\ \text{rad}}{360^\circ}\right]\left[\frac{1\ \text{day}}{24\ \text{h}}\right]\left[\frac{1\ \text{h}}{3600\ \text{s}}\right] \\ &= 3.81\times10^8\ \text{m/s}. \end{aligned}$$

object distance D_o (cm)	90	80	70	60	50	40	30	20	10
object distance D_i (cm)	8.8	8.9	9.0	9.2	9.5	10	11	13	40

Figure 4

Notice that this speed is greater than the speed of light by 27 percent. The corresponding speed for the other ejectum is $1.95 \cdot 10^8$ m/s, well below the speed of light.

B. Figure 5 shows the geometry of the situation that can produce such superluminary speeds. (Un-

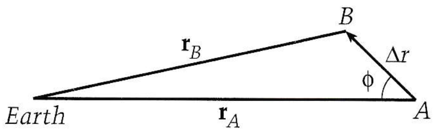


Figure 5

fortunately, the labels for $\mathbf{r}_A$ and $\mathbf{r}_B$ were interchanged in the figure that accompanied the contest problem.) An object is moving from point A to point B with a speed $v = \Delta r/\Delta t$. The light that leaves the object when it is located at point A takes a time

$$t_A = \frac{r_A}{c}$$

to reach Earth. The signal from point B originates a time Δt later and therefore arrives at Earth at

$$t_B = \frac{r_B}{c} + \Delta t.$$

Thus, the difference in the arrival times of the two signals on Earth is

$$\Delta t_0 = t_B - t_A = \frac{r_B - r_A}{c} + \Delta t.$$

Because the distance to the object is extremely large compared to Δr, the directions to points A and B are nearly parallel and

$$r_A - r_B \cong v\Delta t \cos\phi.$$

Therefore,

$$\Delta t_0 = \Delta t\left(1 - \frac{v}{c}\cos\phi\right) = \Delta t(1 - \beta\cos\phi).$$

C. The observed transverse velocity for this motion is given by

$$v_\perp = \frac{\Delta r \sin\phi}{\Delta t_0} = \frac{\Delta r \sin\phi}{\Delta t(1-\beta\cos\phi)} = v\frac{\sin\phi}{1-\beta\cos\phi}$$

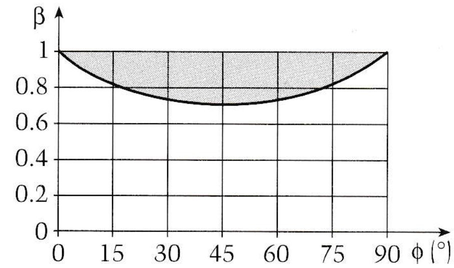


Figure 6

or

$$\beta_\perp = \frac{\beta\sin\phi}{1-\beta\cos\phi}.$$

D. We can find the minimum value of β for which we can observe $\beta_\perp \geq 1$ by looking at

$$\beta\sin\phi = 1 - \beta\cos\phi.$$

This tells us that

$$\beta = \frac{1}{\sin\phi + \cos\phi}.$$

Because $\sin\phi + \cos\phi$ has a maximum at $\phi = \pi/4$, $\beta_{min} = 1/\sqrt{2}$ for the angle $\phi = 45°$.

E. See shaded region in fig. 6.

CONTINUED FROM PAGE 7

ing the 10-volt scale it reads $V_2 = 2.6$ V. What would it read using the 100-volt scale? It is known that under constant illumination a solar cell is just a source of emf coupled with a large series resistance. (A. Zilberman)

P264

Oscillating circuit. A capacitor in a circuit with an open switch S (fig. 2) is charged to a potential V_0. Then the switch is closed, and after some time the current stops flowing. What should V_0 be in order to charge the capacitor to the steady-state voltage $V_{SS} = 1$ V with its polarity opposite to the initial polarity? Assume the emf of each battery to be $\mathscr{E} = 1.5$ V and the diodes to be ideal. (A. Kirkinsky)

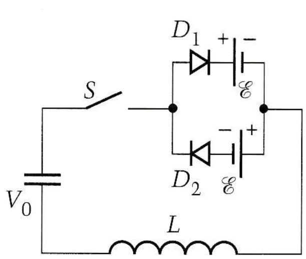


Figure 2

P265

The backward spectacles. A teacher turned to the chalkboard and watched his students in the reflection in his spectacles. He saw two images of the same student who sat 5 m from him: one image was formed at a distance of 5 m, another at 5/7 m. When the teacher turned to the students, he saw the image of the same person at a distance of 2.5 m. Find the refractive index of the glass of the spectacles. (V. Belonuchkin)

ANSWERS, HINTS & SOLUTIONS ON PAGE 49

A rotating capacitor

by A. Stasenko

VAGUELY AWARE OF THE custom originated in the early days of electricity that electric fields are stored in capacitors and magnetic fields are stored in current-carrying coils, an inquisitive girl named Trudy one day placed a capacitor in a magnetic field. "Something new should happen," she thought, and her intuition did not disappoint her.

Trudy made a capacitor with two long coaxial cylinders with approximately the same radii a and b, so their difference (axial clearance) $b - a = l$ was much less than either radius, which is to say $l << a < b$. She suspended the capacitor in a vertical magnetic field in such a way that it could rotate about its own (also vertical) axis without friction (fig. 1). The inner cylinder had a charge $+q_0$, and the outer cylinder had a charge $-q_0$. As a result, there was a radial electric field E_r between the capacitor plates. Unfortunately, nothing particularly interesting occurred in this setup.

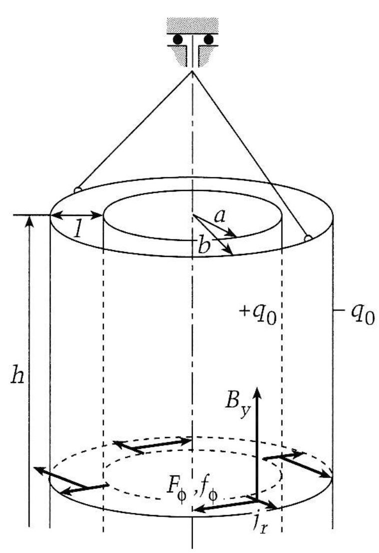


Figure 1

"What can I do to instill life into this dead junk?" the inquisitive child thought ruefully. As usual, an idea struck her. Trudy filled the space between the plates with a conductor, so the capacitor became leaky and a radial current began to flow from the inner to the outer cylinder. At this point, we should recall that every charge that moves across the lines of magnetic force is affected by the Lorentz force, which is perpendicular to two vectors: the velocity **v** of the charge and the magnetic field **B**. Therefore, this force is tangent to the circle with a magnitude of

$$F_\phi = ev_r B_y.$$

The indices stress the fact that all three vectors are mutually perpendicular.

If the concentration of moving charges is n, then every cubic centimeter of the conductor between the plates will be affected by the Lorentz force, which can be called the *volume force density*

$$f_\phi = nF_\phi = nev_r B_y.$$

Since this force is tangential, the entire capacitor will rotate. The rotation of the cylindrical conductor and the plates attached to it will accelerate under the total force acting on the entire volume $2\pi alh$. This force is

$$f_\phi \cdot 2\pi alh = nev_r B_y \cdot 2\pi alh.$$

To obtain this formula, Trudy recalled with satisfaction that it was a prophetic thought to make a very small clearance between the plates—otherwise she would have had to integrate the force, and she didn't want to waste such a powerful tool on trifles.

Thus, the acceleration of the capacitor can be described by Newton's second law written as follows:

$$m\frac{dv_\phi}{dt} = (nev_r)(2\pi ah) \cdot lB_y.$$

The expression in the first parentheses is the current density $j_r = ne_0 v_r$, the second parentheses contains the plate's area $S = 2\pi ah$ (it is almost identical for both plates, again due to the small clearance between them). Multiplying the current density by the cross-sectional area yields the total electric current $I = j_r S$. Thus,

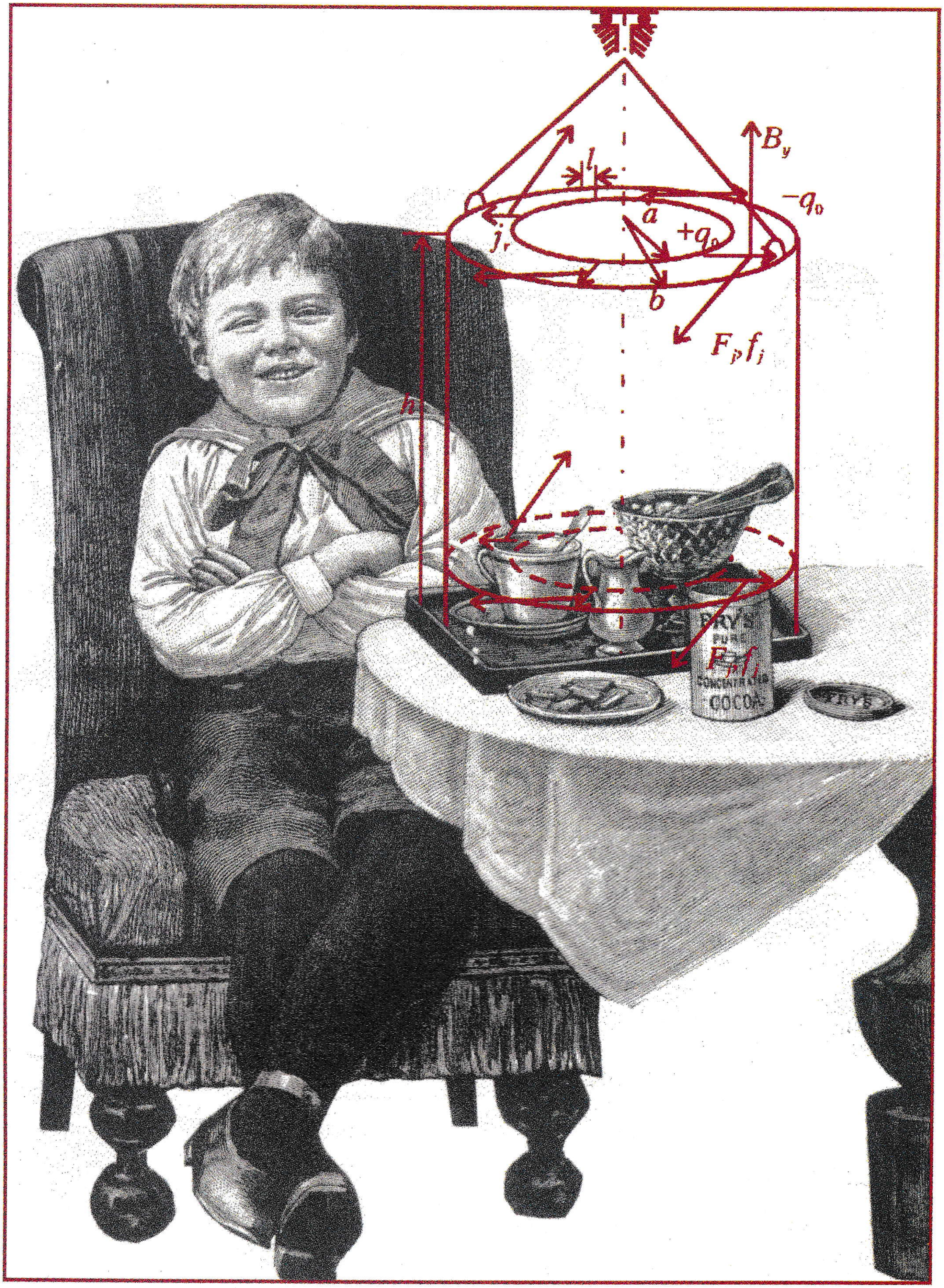
$$d(v_\phi m) = lB_y(Idt).$$

In this equation the parentheses again help us to see its physical nature. On the right, the parentheses show the change of the positive charge on the inner plate:

$$dq = -Idt,$$

where the minus sign corresponds to the fact that the radial current decreases the positive charge. We see

Art by Vera Khlebnikova



B_y
l
$-q_0$
a
j_r
$+q_0$
b
$F_p f_i$
h
FRY'S
PURE
CONCENTRATED
COCOA
$F_p f_i$
FRY'S

that the increase in the capacitor's (linear) momentum $d(mv_\phi)$ is proportional to the decrease in the capacitor's charge. Therefore, when the capacitor loses all of its charge, its (rotational) momentum will reach the maximum value

$$mv_\phi = lB_y q_0.$$

In the case of rotation, physicists do not use the terms *momentum* and *impulse*, because the center of mass of our capacitor doesn't move. Instead, physicists use the notions of *torque* (force multiplied by the lever arm a—that is, the distance from the axis of rotation) and *angular momentum* (linear momentum mv multiplied by the same distance a). The corresponding equation for the rotation of the capacitor in terms of torque and angular momentum is

$$mv_\phi a = alB_y q_0.$$

However, our inquisitive Trudy focused on another point: Where could the torque and angular momentum be taken from? Initially, the capacitor had neither of them and did not rotate. According to the basic laws of nature, angular momentum cannot arise from nothing. Only one conclusion could be made: The electromagnetic field (characterized by E_r and B_y) had the angular momentum hidden within its intricate structure, and it transferred this momentum to the capacitor! During discharge, the electric field decreased, and the angular momentum of the electromagnetic field was gradually transferred to initiate and accelerate the rotation of the capacitor. Finally, when the electric field disappeared, the combined electromagnetic field no longer existed (although its magnetic component was the same), and the capacitor had acquired the largest angular velocity.

"What a strange world we live in," thought the future physics giant. "It seems that electromagnetic fields have the mechanical attributes of *torque* and *momentum densities*! What would Sir James Maxwell think?" ◘

AT THE BLACKBOARD III

Numerical data in geometry problems

by S. V. Ovchinnikov and I. F. Sharygin

NUMERICAL DATA ARE IRRELEVANT to most geometry problems encountered in school. As a rule, these problems can be solved for the general case using symbolic notation, and then numerical data can be substituted in the resulting formula. However, in some cases, using particular numerical data allows us to obtain a simpler solution. In this article, we consider several examples that demonstrate how numerical data can affect the solution of a problem.

Problem 1. Chord AB subtends a $120°$ arc of a circle. Point C lies on this arc, point D lies on the chord AB, $AD = 2$, $BD = 1$, and $DC = \sqrt{2}$. Find the area of triangle ABC.

Notice that $\angle ODC = 90°$. Indeed, extend segment CD until it intersects the given circle (fig. 1). Since $AD \cdot BD = CD \cdot DE$, we find that $DE = CD = \sqrt{2}$ and therefore $OD \perp CE$. The radius of the circle can be found from triangle AOF: it is $\sqrt{3}$. Now in 30°-60°-90° triangle AOF, $AF = (1/2)AB = 3/2$, and so $OF = \sqrt{3}/2$ and $AO = \sqrt{3}$. Also, $DF = AD - AF = 1$, and so triangle ODF is also 30°-60°-90°. Thus, $\angle CDA = 30°$ and $\angle CDB = 150°$. Now the area of triangles ADC and CDB can be found from two sides and the angle between them:

$$S_{\triangle ADC} = \frac{1}{2}(CD)(AD)\sin 30° = \frac{\sqrt{2}}{2}$$

and

$$S_{\triangle CDB} = \frac{1}{2}(CD)(DB)\sin 150° = \frac{\sqrt{2}}{4}.$$

Therefore,

$$S_{\triangle ABC} = \frac{3}{4}\sqrt{2}.$$

We see that the numerical data in this problem are chosen very carefully. If we try to solve the problem in the general form—that is, use arbitrary numbers as the initial data–the solution turns out to be extremely tedious. Even so, we invite the reader to try to solve the problem in general.

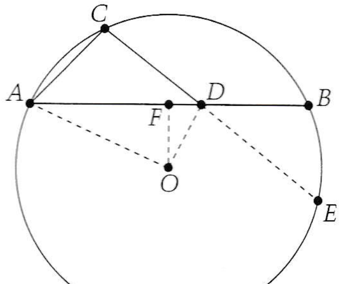


Figure 1

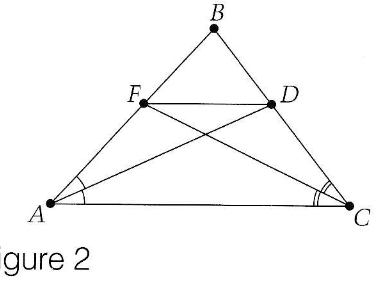


Figure 2

We consider next a problem in which the numerical values of the initial data are of no importance and are used only at the final stage of the solution when they are substituted in the resulting general formula.

Problem 2. In a triangle ABC, AD is the bisector of angle BAC and CF is the bisector of angle ACB (point D lies on side BC and point F, on side AB of the triangle). Find the ratio of the areas of triangles ABC and AFD if $AB = 21$, $AC = 28$, and $CB = 20$.

Denote the length of sides AB, BC, and CA by c, a, and b, respectively (fig. 2). We employ a useful technique for working with angle bisectors in a triangle. If $BD = m$ and $DC = n$, we know that $mk/n = b/c$ (this well-known result is called the *angle bisector theorem*). So we can find a number k such that $m = bk$, $n = ck$. Then $m + n = bk + ck = a$, so $k = a/(b + c)$, and we find

$$BD = \frac{ab}{b+c}, DC = \frac{ac}{b+c}.$$

Similarly, we find

$$AF = \frac{bc}{a+b}.$$

Now, triangles ABD and ABC have a common altitude from A, so the ratio of their areas is the ratio of their bases. That is,

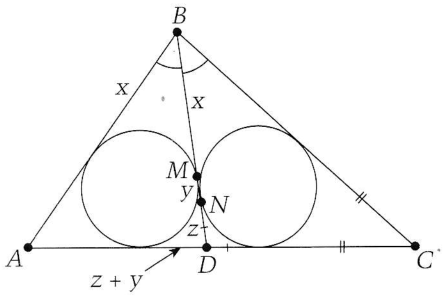


Figure 3

$$\frac{S_{\triangle ABD}}{S_{\triangle ABC}} = \frac{BD}{BC} = \frac{c}{b+c}.$$

Similarly,

$$\frac{S_{\triangle AFD}}{S_{\triangle ABD}} = \frac{AF}{AB} = \frac{b}{(a+b)},$$

$$\frac{S_{\triangle ABC}}{S_{\triangle AFD}} = \frac{(a+b)(b+c)}{bc}.$$

Substitute the numerical values of a, b, and c in the last formula to obtain

$$\frac{S_{\triangle ABC}}{S_{\triangle AFD}} = 4.$$

Thus, the desired ratio is 4.

The following problem can be solved in its general form. However, the solution is much simpler if numerical values are substituted at the proper moment.

Problem 3. In a triangle ABC, $AB = 3$, $BC = 4$, $AC = 5$, and BD is the bisector of angle ABC. Circles are inscribed in triangles ABD and BCD, which touch BD at points M and N, respectively. Find the length of segment MN.

We see from the values of the length of the triangle's sides that the triangle is a right triangle. This fact can induce one to solve the problem using metric properties of right triangles.

However, this problem has a simple solution for an arbitrary triangle ABC. Denote AB by c, BC by a, and AC by b (fig. 3). As in problem 2, the "angle bisector" theorem leads to

$$AD = \frac{bc}{a+c}, \quad CD = \frac{ab}{a+c}.$$

We introduce the unknowns $x = BM$, $y = MN$, and $z = ND$. Since tangents drawn from a point to a circle are equal, we have the following system of equations:

$$\begin{cases} x + y + \left(\dfrac{ab}{a+c} - z\right) = a, \\ x + \left(\dfrac{bc}{a+c} - z - y\right) = c. \end{cases}$$

We subtract the second equation from the first to obtain

$$2y + \frac{ab}{a+c} - \frac{bc}{a+c} = a - c,$$

from which we get

$$y = \frac{1}{2}\frac{(a-c)(a+c-b)}{a+c}.$$

We then substitute the numerical values for a, b, and c to obtain the final result: $MN = 1/7$.

Now let's consider several problems in which specially selected numerical values make a solution much simpler.

Problem 4. In a triangle ABC, $AB = 4$, $AC = \sqrt{17}$, and $BC = 5$. A point D is taken on side AB such that $AD = 1$. Find the distance between the centers of the circles circumscribed around triangles DBC and ADC.

The solution can be found easily if we guess that CD is the altitude in triangle ABC. In this case, the centers of the circles circumscribed around triangles DBC and ADC are the midpoints of sides AC and BC. Therefore, the desired distance is the length of the midline parallel to AB, which is 2 (fig. 4). The fact that CD is the altitude follows from the equation

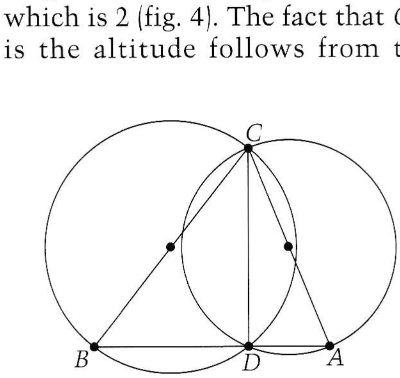


Figure 4

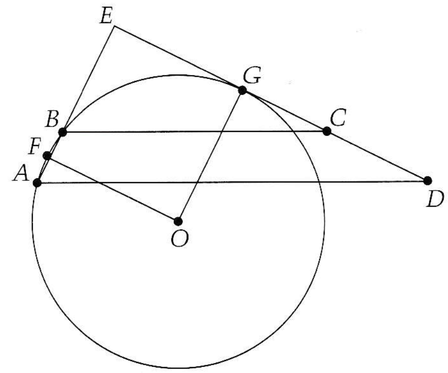


Figure 5

$$BC^2 - AC^2 = BD^2 - DA^2 = 8.$$

(In fact, it is a standard theorem of advanced geometry that point C is on the perpendicular to AC at D if $BC^2 - AC^2 = BD^2 - AD^2$. Its proof, using the Pythagorean theorem, is not difficult.)

However, how can we guess that CD is the altitude? This observation is the key point to our solution, and all other reasoning is easy. It is impossible to specify a universal method for making guesses. However, an accurate drawing can help us make a guess in geometry problems, whether the data are symbolic or numerical. If the data are numerical, the drawing must be made using these data. For example, if we make an accurate drawing for the above problem, it is not difficult to guess that CD is the altitude.

Problem 5. In a trapezoid $ABCD$, the bases $AD = 39$ and $BC = 26$ and the legs $AB = 5$ and $CD = 12$ are given. Find the radius of a circle that passes through points A and B and is tangent to side CD or its extension.

We extend sides AB and CD to their point of intersection to form triangle AED (fig. 5). From similar triangles BEC and AED, we get $BE = 10$ and $CE = 24$. If the drawing is accurate enough, it is not difficult to notice that $\angle AED = 90°$. This fact can be proved easily: $AE^2 + ED^2 = AD^2$. Let O be the center of the circle passing through points A and B and tangent to ED at a point G. Draw a perpendicular OF to AB. Then F is the midpoint of chord AB.

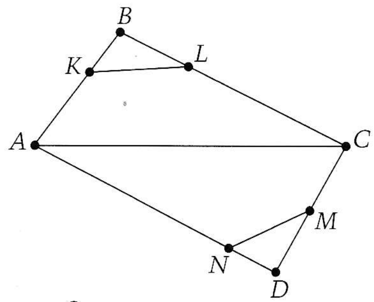


Figure 6

It is clear that $OG = EF = 12.5$. Therefore, the radius of the circle is 12.5.

In the following problems, numerical data manifest themselves in a peculiar way. At first glance, it is not obvious how these values can influence the solution. However, in the process of solving the problem, it becomes clear that the result can be obtained only for those particular values. In problems of this kind, a line of attack on the problem is evident, though it is not immediately apparent whether this line gives a result.

Problem 6. Consider a convex quadrilateral $ABCD$ with unit area. Points K, L, M, and N are given on the respective sides AB, BC, CD, and DA of this quadrilateral such that

$$\frac{AK}{KB} = 2,$$
$$\frac{BL}{LC} = \frac{1}{3},$$
$$\frac{CM}{MD} = 1,$$

and

$$\frac{DN}{NA} = \frac{1}{5}.$$

Find the area of hexagon $AKLCMN$.

The ratio of the areas of triangles KBL and ABC (fig. 6) is[1]

$$\frac{BK \cdot BL}{AB \cdot BC} = \frac{1}{12}.$$

The ratio of the areas of triangles MND and ADC is also

$$\frac{DN \cdot DM}{AD \cdot CD} = \frac{1}{12}.$$

Thus, the sum of the areas of triangles KBL and MND is 1/12, from which the required area is calculated as 11/12.

The solution rests on the equality of the calculated ratios of areas. The problem would be much more difficult in general.

Problem 7. Three points A, B, and C are connected by straight roads. A square field abuts a segment of road AB with side $(1/2)AB$. Another square field abuts the segment of road BC with side BC, and a rectangular forest abuts road CA. The length of the forest is CA, and its width is 4 km. The area of the forest is 20 km^2 greater than the sum of the areas of the square fields. Find the area of the forest.

At first sight, the data seem to be insufficient to solve the problem. However, let us make certain calculations. Denote by c, a, and b the sides AB, BC, and CA of triangle ABC. Then, $4b = 20 + c^2/4 + a^2$. By the triangle inequality, $b \le a + c$. Express b in terms of a and c, and substituting in this inequality, we have:

$$\frac{1}{4}\left(20 + \frac{c^2}{4} + a^2\right) \le a + c.$$

This can be rewritten in the form:

$$\left(\frac{c}{2} - 4\right)^2 + (a-2)^2 \le 0.$$

Since squares cannot be negative, we get $c = 8$ km, $a = 2$ km, and $b = a + c = 10$ km. Thus, the area of the forest is 40 km^2.

Our line of attack is perhaps the only possible way of solving this problem. The fact that this way yields an unambiguous result is a consequence of specially selected values of the initial data. In fact, the three points A, B, and C turn out to lie on a line, which makes the given data sufficient for solving the problem.

Problem 8. Orthogonal projections of a triangle ABC onto two perpendicular planes are equilateral triangles with unit sides. Find the perimeter of triangle ABC if it is given that $AB = \sqrt{5}/2$.

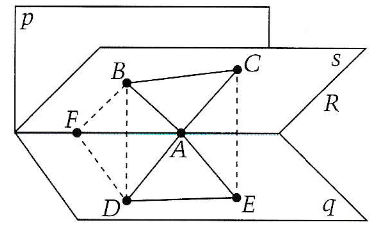


Figure 7

Denote the given planes by p and q (fig. 7). Without loss of generality, we may assume that one of the vertices of the triangle, let's say A, lies on the line of intersection of planes p and q (denoted by R in fig. 7). Since the projections of sides AB and AC onto planes p and q are equal, points A and C lie in the bisector plane (denoted by s in fig. 7) of the dihedral angle formed by planes p and q.[2] Let D and E be the projections of points B and C onto plane q. We draw a perpendicular BF from point B to line R, and note that $\angle BFD = 45°$.

In further manipulations, we could use an arbitrary value for AB, say a, and solve the problem in its general form. However, the use of the particular numerical value $AB = \sqrt{5}/2$ makes the solution much simpler.

Indeed, we find from triangle ABD that $BD = 1/2$ (angle $ADB = 90°$, and AB and AD are known). Since $\angle BFD = 45°$, we have $FD = BD = 1/2$, and $\angle AFD = 90°$, so $\angle FAD = 30°$ in triangle AFD. Now we can obtain the following relations: $AE \perp AF$, $\angle CAE = 45°$, $\angle CEA = 90°$, and $AC = \sqrt{2}$. From the right trapezoid $CBDE$, we have $(CE - BD)^2 + DE^2 = BC^2$. Therefore, $BC = \sqrt{5}/2$, and the perimeter of triangle ABC is $\sqrt{5} + \sqrt{2}$.

In this problem, the numerical values of the initial data considerably reduced the amount of computations, yet in some cases the use of numerical values makes calculations more complex and can even hide a simple geometrical sense of a problem.

[1]We use the fact that the ratio of the areas of two triangles with a common angle is equal to the ratio of the products of the sides adjacent to this common angle.

[2]The reader is invited to prove this, for example by showing that point B must be equidistant from planes p and q.

Exercises

1. Orthogonal projections of a plane quadrilateral onto two mutually perpendicular planes are squares with sides of length 2. One of the diagonals of the given quadrilateral is $\sqrt{14}$. Find the other diagonal.

2. In acute triangle ABC, $AB = 15$, $BC = 10$, and $\angle BAC$ is arccos $(7/9)$. A circle is circumscribed around triangle ABC, and a chord BE is drawn through a point D that lies on AC and such that $AD = 9$. Find the area of triangle AEC.

3. A triangle ABC with sides 5, 2, and $\sqrt{22}$ is given. A point D on side AC is chosen such that $BD = 3$. Find the distance between D and the center of the circle circumscribed around triangle ABC.

4. A sphere is inscribed in a regular tetrahedron $SABC$ with edges of length a. A point M is chosen on the edge SA such that $AM = MS$, and a point N is chosen on the edge BC such that $2CN = NB$. Line MN intersects the given sphere at points P and Q. Find the length of segment PQ.

5. In a triangle ABC with sides $AB = \sqrt{3}$ cm, $BC = 4$ cm, and $AC = \sqrt{7}$ cm, BD is a median. Circles are inscribed in triangles ABD and BDC that touch BD at points M and N, respectively. Find the length of segment MN.

6. Two circles of radius r are tangent to each other. In addition, each of them is tangent externally to a third circle of radius R at points A and B, respectively. Find r if $AB = 12$ cm and $R = 8$ cm.

7. An equilateral triangle ABC with a side of length 3 is inscribed in a circle. Point D lies on this circle, and chord AD has a length of $\sqrt{3}$. Find the length of chords BD and CD.

8. A triangle ABC of unit area is given. Point P, Q, and R are chosen on medians AK, BL, and CN, respectively, such that

$$\frac{AP}{PK} = 1,$$

$$\frac{BQ}{QL} = \frac{1}{2},$$

and

$$\frac{CR}{RN} = \frac{5}{4}.$$

Find the area of triangle PQR.

9. The volume of a rectangular parallelepiped is 150 cm^3, its surface area is 280 cm^2, and the perimeter of its base is 40 cm. Find the dimensions of the parallelepiped.

10. A sphere of radius R is inscribed in a dihedral angle of 60°. Find the radius of a sphere inscribed in the same angle such that it is tangent to the first sphere if it is known that the line that connects the centers of both spheres forms an angle of 45° with the edge of the dihedral angle.

11. A triangle ABC with sides $AB = 4$, $BC = 3$, and $AC = 5$ is given. A point D is chosen on side AB such that $DB = 7/8$. A circle is drawn through points C, D, and B, which intersects AC at point E. Find the length of segment BE.

Feedback

Conquering division by 7

The article in the March/April issue titled "Divide and Conquer!" by Ruma Falk and Eyal Oshry, notes that "the number 7, for example, is notorious for evading an efficient (divisibility) criterion." True, but there is one (simple?) criterion that is at least 1500 to 2000 years old and will be of interest to the *Quantum* readership.

The Sabbatical year is the name given to every seventh year during which the ancient Hebrews could not sow their fields or prune their vineyards (Exod. 23:10–11; Lev. 25:2–7; and more). There were other prohibitions and prescriptions. Which years are Sabbatical years (that is, which years are evenly divisible by 7)?

In the Babylonian Talmud (Aboda Zarah:9b), redaction circa A.D. 500, we find the following given without proof (paraphrased in English as given by W. M. Feldman in *Rabbinical Mathematics and Astronomy*, Hermon Press, New York, Second American Edition, 1965, originally published in England in 1931):

"Multiply the hundreds in the given number by 2, and add the product to the tens and units of that number. If that sum is divisible by 7, the whole number is divisible by 7."

Thus, for example, 5759, the current year in the Hebrew calendar, is not a Sabbatical year, since $(57 \cdot 2) + 59 = 173$, which is not (evenly) divisible by 7. On the other hand, the forthcoming 5761 is a Sabbatical year, since $(57 \cdot 2) + 61 = 175$, which is divisible by 7.

Feldman's verbal "proof" of the algorithm is not satisfying. It is easy to develop a proof using modern modulus arithmetic notation. Perhaps the reader would care to do this.

Harvey J. Hindin
Dix Hills, New York

Weightlessness in a magic box

by A. Dozorov

THE STATE OF WEIGHTLESSNESS IS ATTAINED during free fall or while flying. A satellite in an orbit, a hurled stone, and a jumping person experience this state. A load attached to a cord has no weight in free flight, so it doesn't stretch the cord. One can easily make a device that demonstrates how weightlessness "works."

Such a device is shown in figure 1. In the "normal" state, the bob (B) stretches the thread and the elastic plate (EP) bends and separates the electric contacts $C1$ and $C2$. In this stationary state, the lamp (L) is not lit. If the device is thrown upward, the bob becomes weightless, and it doesn't stretch the thread. Therefore, the elastic plate straightens, the contacts close the circuit, and the lamp turns on. The lamp is on only when the whole setup is weightless. Note that the condition of weightlessness is attained when the device moves both upward and downward.

The tuning screw (S) can regulate the position of the contacts so that they are slightly apart when the device is stationary. The device should be housed in a transparent box (fig. 2).

Now for a few practical hints. The battery ($\mathscr{E}$) can be large or small, but it is better to make a large battery compartment so that either a small or a large battery can be used. Because the battery must be occasionally replaced with a new one, the battery compartment should be made on the outside of the device with two holes drilled for the connecting wires. The elastic plate can be made of any thin elastic metal bar, even from half of a razor blade.

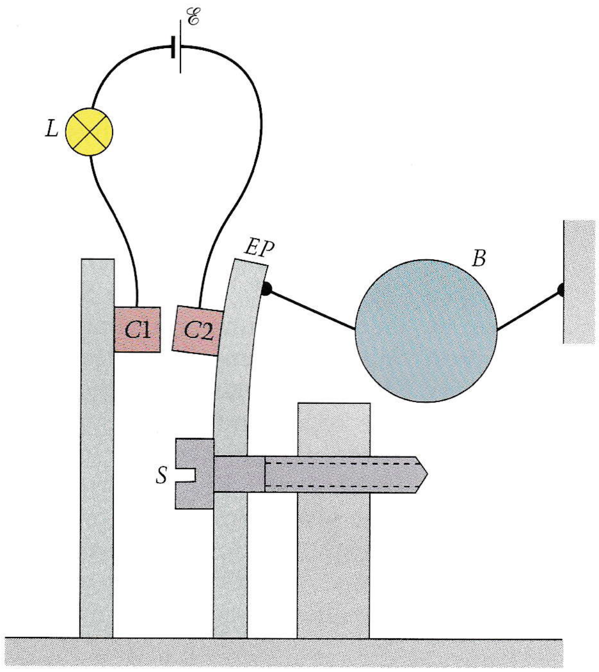


Figure 1

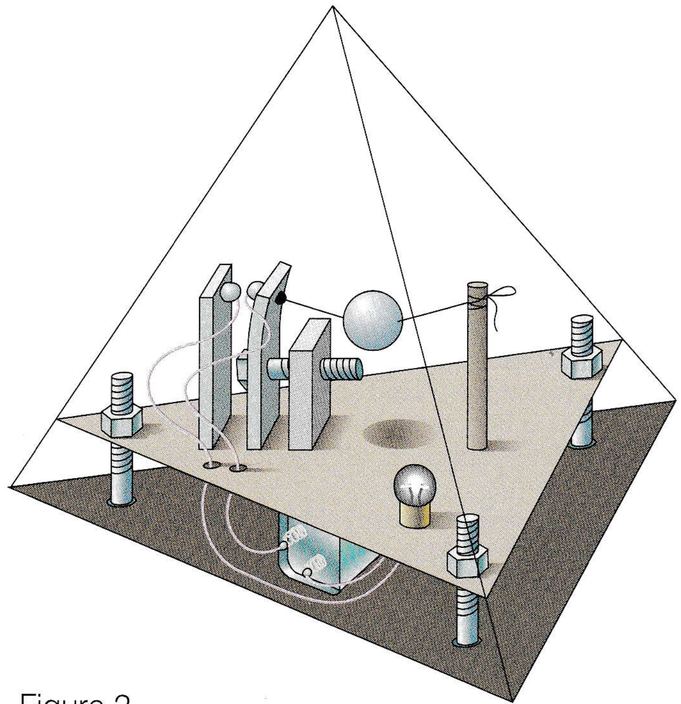


Figure 2

Magnetic personality

by V. Kartsev

WHEN 43-YEAR-OLD DANISH professor Hans Christian Ørsted sent off a thin pamphlet of four pages in 1820, scientists in France, Switzerland, England, and Russia realized that the papers touched on both scientific and universal human problems. How were they to treat the author of these pages, and how were they to evaluate his toil? Who was this person—a scientist, dreamer, romantic, or just a lucky man? And what was his profession—physics, chemistry, pharmacology, philosophy, or poetry?

These were not simple questions. They linger even now, so let's go back in time more than two centuries and visit a town called Rudkøbing on the remote Danish islet of Langeland. Hans Christian was the son of a poor apothecary. Hans and his brother Anders received their elementary education from sundry places: a town barber taught them German, and his wife taught them Danish; a minister explained the rules of grammar and familiarized the brothers with history and literature; a land surveyor showed them how to add numbers; and a student told them all about minerals.

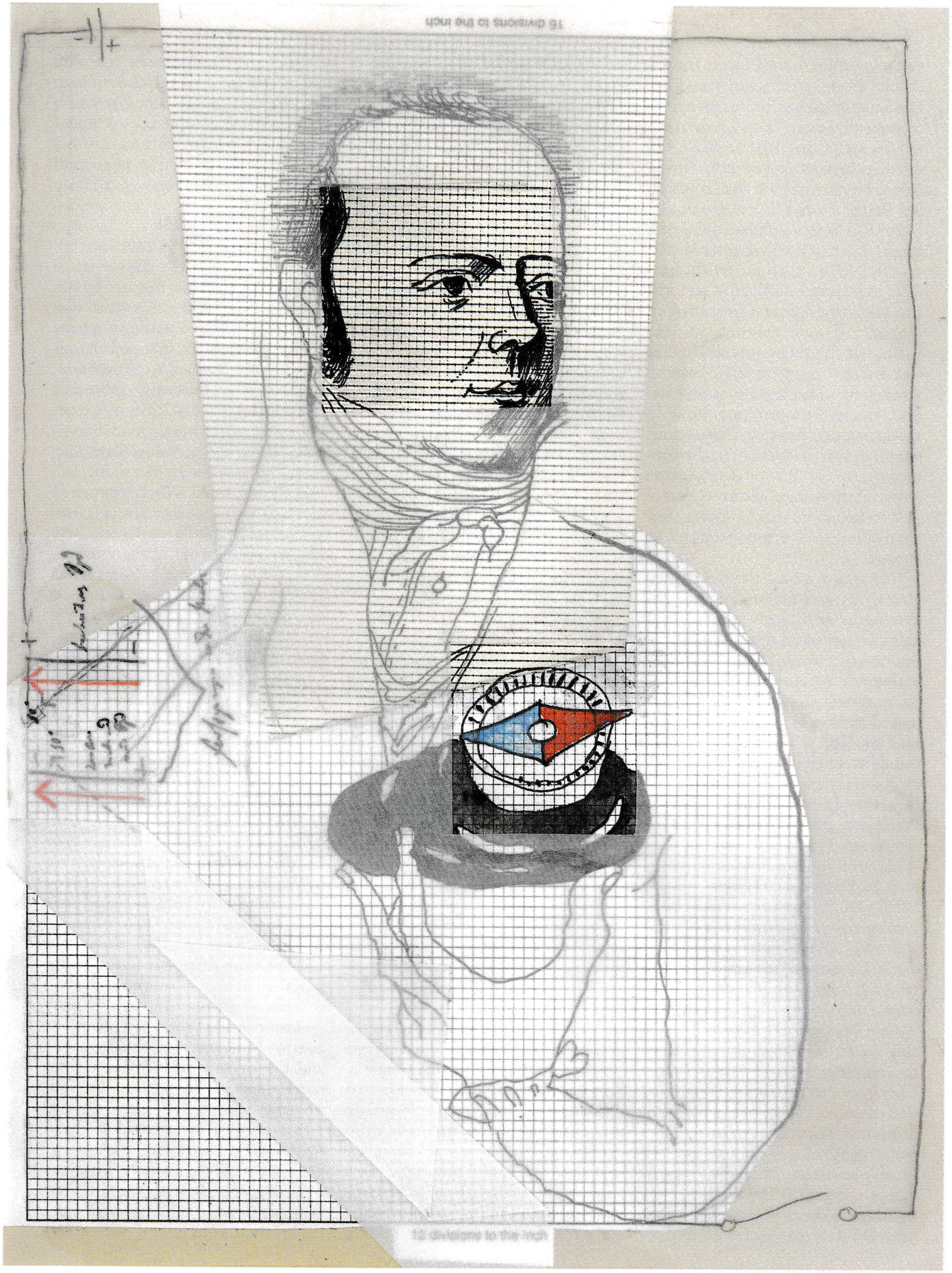
At the age of 12, Hans was already lured by science. Alas, instead of getting a more formal education, he worked at his father's drugstore. There he was taken with medicine, which for a while became his favorite subject, overshadowing chemistry, history, and fine arts. He decided to enter the University of Copenhagen, but doubts tormented him: What should he study? Hans Christian began to learn everything: medicine, physics, astronomy, philosophy, and poetry.

Ørsted led a happy life at his alma mater. Later, he wrote that to achieve absolute freedom, a youth must enjoy himself in the kingdom of reason and imagination, where struggle goes hand in hand with freedom of thought and phrase, and where a defeated person is given a chance to rise and struggle again. He lived in a world with room for modest victories, the conquest of new knowledge, and the chance to correct previous mistakes. In 1797 he earned a gold university medal for his essay "The Boundaries of Prose and Poetry." His next paper, which received similar praise, was on the properties of alkalis, and his doctoral dissertation was devoted to medicine. Although his accomplishments were impressive, he risked jeopardizing his scientific career by compromising his depth of professional skill by cultivating a variety of interests.

Meanwhile, a new age began. In the vortex of the French and American revolutions, a new perception of the world arose and old dogmas were rejected by new morals and reasoning. The Industrial Revolution produced an unstoppable flow of practical innovations. The nineteenth century proclaimed new ways of thought and life through novel social and political ideas, modern philosophy, art, and literature. Hans was enchanted by this new world. He decided to go where the major scientific and philosophic problems were being solved. Alas, Denmark was no more than a European province, and Ørsted did not want to spend the rest of his life there. Fortunately, his talent, persistence, and luck molded into a happy alloy: After a brilliant defense of his thesis, Hans was sent by the university to study in France, Germany, and Holland. There he attended various lectures on such topics as how problems of physics could be solved with the help of poetry and mythology. Although he enjoyed the lectures of brilliant philosophers, he did not forsake the experimental approach to studying physical phenomena.

Ørsted was deeply influenced by the philosophies of Georg Hegel (1770–1831) and Friedrich Schelling (1775–1854), especially Schelling's idea on the universal connections of all phenomena. This idea validated his wide-ranging scientific interests: According to the philosophy of the time, all subjects were interdependent. Ørsted became obsessed with the idea of universal connections, and quickly found a consonant soul who shared his views and was equally unfocused and romantic. It was the German physicist Johann Ritter (1776–1810), inventor of the storage battery, who was a dreamer

Art by Yury Vashchenko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and a fountain of mad ideas. For example, from astrological considerations he "deduced" that an epoch of new discoveries in electricity would arrive in 1819 or 1820. Indeed, this was to happen, and it would be spurred by Ørsted himself, but Ritter did not live to see it.

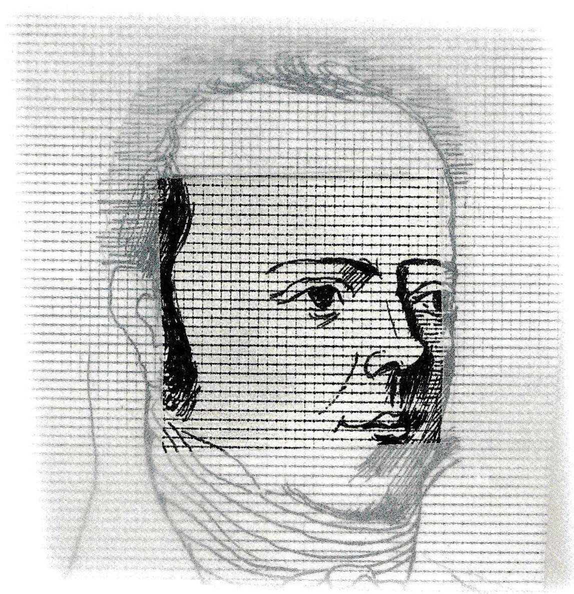
In 1813 Ørsted published *A Study of the Identity of Chemical and Electrical Forces*, where for the first time in history he officially set forth the idea of a connection between electricity and magnetism. He wrote, "It should be tested whether electricity . . . acts in some way on a magnet." The logic was simple: Electricity can produce light (a spark), sound (the crack that accompanies a spark), and heat (in the connecting wire). Wasn't this an example of the universal connection of physical phenomena? Couldn't electricity produce some magnetic effect as well?

The idea of a connection between electricity and magnetism, which originated from noticeable similarities in the attraction of tiny objects to amber and of iron filings to magnets, had been tossed around before, and many brilliant European minds were captivated by it. As early as 1747, this idea was discussed in St. Petersburg, Russia, by academician Franz Aepinus (1724–1802). The Frenchman François Arago (1786–1853) spent many years collecting mystical phenomena about ships, treasures, and other mysterious events in which he tested this alleged connection.

One day the French battleship *La Raleign* appeared in Palma, the major port of Majorca, Spain. The ship was in such a poor state that it could barely make it to the moorage. When the ship's mate came ashore, a group of famous French scientists (including Arago) stepped aboard. They realized that the ship had been damaged by lightning. While the rest of the committee ruefully sighed about the burnt masts, Arago rushed to the compasses and saw what he expected: The polarity of some of the magnetic needles was reversed, so that the needles pointed in opposite directions.

A year later Arago examined the wreckage of a Genoan ship that had been ruined on the rocks of Algerian shores. Again Arago found that the needles of the compasses had changed polarity. In the pitch black night the captain had directed the ship northward away from danger, or so he had thought. In reality, the ship had rushed in the opposite direction, toward the rocks.

At last Arago had found his treasure! It lay in the hold of the merchant ship: a dinner set that had traveled to North America. Lightning had melted the pieces, and some of them had turned into very strong magnets—evidence of a connection between lightning and magnetism.

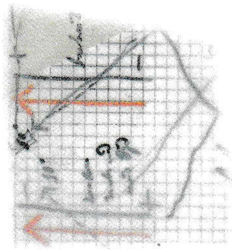


The most famous and daring experiment with lightning was performed by Benjamin Franklin in America and by Michael Lomonosov and Georg Richmann in Russia, who discovered that lightning was just a giant electric spark. Nowadays this fact seems trivial, but Richmann sacrificed his life for this knowledge. Arago collected much data attesting to the connection between lightning and magnetism. He felt that he was nearing a very important discovery, so he must have felt both joy and disappointment when he heard his longstanding problem was solved. Ørsted had found the answer.

Accidentally on purpose?

On February 15, 1820, Ørsted, now a professor at the University of Copenhagen, gave a lecture to his students. As usual, the lecture was amply illustrated by demonstrations. In addition to standard chemical equipment, there were other devices on the laboratory table: an electrical source with wires connected to its terminals, and a compass. When Ørsted closed the circuit, the needle in the compass jerked and turned. When he opened the circuit, the needle returned to its initial position. This was the first experimental proof of the connection between electrical and magnetic phenomena that had been overlooked for years by many scientists.

At first glance, this discovery seems straightforward. Ørsted demonstrated to his students one more piece of evidence for the universal connection between physical phenomena. Why is the story then viewed with skepticism? Why were there so many discussions about the circumstances of this discovery? Because students who were actually at the famous lecture told quite another story. They said Ørsted had intended to demonstrate the *thermal* effect of electric current. To do this, he used a wire and an electric source, and it was merely by chance that a compass sat nearby. Moreover, it was a student, and not the esteemed professor, who noted the needle's slight jerking and rotating. Students said that Ørsted was sincerely astonished and delighted. However, in his later papers, Ørsted wrote, "Everybody who visited my lecture is witness to the fact that I announced the result of this experiment beforehand. Thus, the discovery was not made by chance, as professor Hilbert would like to conclude from the expressions that I used in the first description of the discovery."



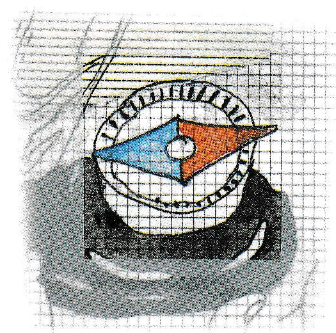
Does it really matter whether Ørsted's discovery was accidental or intended? And what is an "accidental" discovery? However accidental it may have seemed, several circumstances had already set the stage for it to happen. Was it accidental that a chemist, Ørsted, gave a lecture on electricity? Certainly not. In Ørsted's time, electricity was a comparatively new field.[1]

Little was known on the nature of electricity, and no particular training was necessary to study it. Therefore, many scientists and engineers could conduct experiments on electricity, including physicists, chemists, and mechanics. The devices were also very simple and could be produced in any workshop. Thus, there were no "accidental" devices on Ørsted's table and no "impromptu" topics at his lecture. The electric devices used at the time were few: voltaic piles, wires, frog legs, magnets, and compasses.

English physicist Sir William Bragg (1890–1971), the inventor of X-ray diffraction analysis of crystals, said that it's no surprise Ørsted made his discovery by chance. The real miracle is that 20 years passed between the invention of the voltaic pile and this discovery. Dozens of laboratories had everything needed to demonstrate the connection between electricity and magnetism: voltaic piles, wires, and compasses. These objects were placed close to each other on thousands of occasions. It was inevitable that a magnetic needle would be placed in the vicinity of a wire connecting the terminals of the voltaic pile. *Someone* should have noticed the jerk of the needle!

Still, no less than 20 years passed before this chain of events really happened. An unknown student at Ørsted's lecture played the historical role by glancing at the compass at the right moment. His role may be compared with that of the sailor who cried to Christopher Columbus that he saw the New World.

Was it so accidental that Ørsted was involved in this striking discovery? Couldn't similar devices have been so luckily arranged and tuned in another laboratory? Yes. But in this case the odds were in Ørsted's favor, because Ørsted was among a small group of researchers looking for connections between physical phenomena.



Let's return to the essence of Ørsted's discovery. The deflection of the magnetic needle in Ørsted's spectacular demonstration was actually rather small. In July 1820 Ørsted repeated the experiment with more powerful electric sources, resulting in a much more pronounced effect. He found that the thicker the wire, the stronger the needle's deflection.[2]

In addition, Ørsted observed a paradoxical effect that was at odds with the classical Newtonian concepts of action and reaction. The force affecting the needle was not directed to the wire, but acted perpendicular to it! In Ørsted's words, the magnetic effect of electrical current was similar to a circulatory motion around the wire. The needle never pointed to the wire; rather, it was directed tangentially to imaginary circles around it. It looked like invisible magnetic forces around the current-carrying wire affected the needle of the compass. This explains why Ørsted was astonished and why others responded with distrust and mockery. To back himself up, his four-page pamphlet carefully listed his witnesses without omitting a single detail of their scientific merits.

Strictly speaking, Ørsted did not provide a correct theoretical explanation of his experiment. However, he set forth a critical idea on the vortical nature of electromagnetic phenomena. For a long time, the concept of electromagnetic vorticity was not shared by most scientists, who believed that the forces acting between a current-carrying wire and a magnetic needle were just conventional forces of attraction and repulsion, similar to Newtonian forces of universal attraction and Coulombian forces between electrical charges. Thus, Ørsted not only proved the connection between electricity and magnetism, but he happened upon a new mystery that could not be explained by known physical concepts and laws.

News travels

Ørsted's four-page pamphlet was published on July 21, 1820. After this date, word spread unusually fast for the moderate pace of nineteenth-century science. After only a few days the pamphlet appeared in Geneva at the same time Arago was there. A glance at the paper was enough for him to realize that Ørsted had found a solution to the problem which had been a headache for him and other scientists for so long. Reaction to Ørsted's experiment was so strong that one of the demonstration's spectators rose and exclaimed the phrase that became

[1]In 1800 Italian physicist Alessandro Volta (1745–1827) invented the first reliable and continuous source of electric current, the voltaic pile. It allowed experiments on electricity to be conducted much more efficiently.

[2]Isn't this a prototype of an ammeter? Now it's clear that the thicker wire in Ørsted's experiment had a smaller electrical resistance, so it carried a stronger current.

famous: "Gentlemen, it is the revolution!" Deeply astonished, Arago returned to Paris. He rushed to the nearest session of the French Academy, where on September 4, 1820, he gave an oral report on Ørsted's experiments. The academicians asked for a full-scale demonstration of Ørsted's experiment. It was done at the next session on September 22.

André-Marie Ampère (1775–1836) paid particular attention to Arago's report. Perhaps he felt at that moment that it was his turn to take the baton from Ørsted. Similar to Arago and Ørsted, he had waited for this decisive moment for 20 long years. And now, on September 4, 1820, the clock struck. Ampere realized that he must act. It took him only two weeks to report on his study, in which he advanced his own idea and supplied the experimental arguments: All magnetic phenomena can be explained by electrical ones. Thus a new science was born, electrodynamics, which theoretically coupled electrical and magnetic phenomena. Forty years later electrodynamics became an integral part of Maxwell's electromagnetic field theory, which remains our reliable compass in an ocean of electrical phenomena.

After his famous discovery, a cornucopia of honors poured down on Ørsted. He became a member of many celebrated scientific societies, including the Royal Society of London and the French Academy. In Great Britain he was awarded the Copley Medal, and in France he was granted 3000 gold francs, a prize established by Napoleon for the greatest discoveries in electricity.

Accepting all these honors, Ørsted never forgot that the new age required modern approaches to teaching science. He founded a society and a literature magazine in Denmark to promote both science and the fine arts, he delivered lectures especially for women, and he supported Hans Christian Andersen, his namesake and the future author of fairy tales. In short, Ørsted became a national hero.

Unfortunately, Ørsted did not long enjoy his triumph. He died on March 9, 1851. He was buried at night, and 200,000 people took part in the funeral procession. Scientists, state representatives, members of the royal family, diplomats, students, and ordinary people all considered his death a private loss. ◘

Quantum on electromagnetic phenomena:

Kaleidoscope: "Electromagnetic induction," March/April 1991, pp. 32–33.

I. Slobodetsky, "Direct current events," March/April 1992, pp. 52–55.

A. Varlamov, "How does electric current flow in a metal?," September/October 1992, pp. 49–50.

A. Byalko, "Backtracking to Faraday's law," January/February 1994, pp. 20–23.

S. Murzin, M. Trunin, and D. Shovkun, "Beyond the reach of Ohm's law," November/December 1994, pp. 24–29.

J. Wylie, "Magnetic monopoly," May/June 1995, pp. 4–9.

A. Stasenko, "Magnet, charges, and planets," May/June 1997, pp. 42–45.

A. Mitrofanov, "Can you see the magnetic field?," July/August 1997, pp. 18–22.

Bulletin Board

Aeronautics academy

Embry-Riddle Aeronautical University will operate its Summer Academy at its Daytona Beach, Florida, campus from June 18 to August 18, offering educational programs for students ages 12–18 who want to learn about aviation and aerospace in a fun, relaxing atmosphere.

This year's courses, some of which may be taken for college credit, are Aerospace Summer Camp, Aviation Career Education Specialization, Engineering Technology Academy, Flight Exploration, and SunFlight. Application is required **by June 1.**

Summer Academy program include housing at Embry-Riddle's new Student Village residential housing complex, on-campus meals, classroom instruction, and educational materials. Students in some camps will take field trips that may include the beach, water park, a major theme park, air traffic control centers, Kennedy Space Center, U.S. Space Camp, the Museum of Science and Industry, and Patrick Air Force Base. Transportation for off-campus activities is provided by Embry-Riddle.

Aspiring astronauts and scientists will learn about NASA programs, space shuttle operations, and the history of space flight during the four-week Aerospace Summer Camp. The course fosters a basic understanding of space and space technology through classroom lectures, field trips, guest speakers, and hands-on projects. Tuition is $2,950.

In the one-week Aviation Career Education Specialization Academy, students who have an interest in aviation or previous aviation experience may learn in-depth from aviation professionals about the career areas that interest them the most: air traffic control, avionics, engineering, flight, or maintenance. The course includes dual one-hour flight time, classroom instruction, and field trips. Tuition is $600.

Engineering Technology Academy introduces the design, building, and testing of aircraft-related components. Hands-on activities in composites, sheet metal, and welding will be conducted in the project lab during the one-week program. A computer-aided drafting project is followed by demonstrations in the wind tunnel and stereolithography labs. The course includes dual one-hour flight time, classroom instruction, an observation flight, and other field trips. Tuition is $600.

Flight Exploration is a one-week introduction to flying and flight training. Students practice flight maneuvers and get acquainted with how an airplane responds to cockpit commands. Participants learn how to comply with aviation regulations and how to analyze weather conditions. The course includes flight and ground lab instruction, flight fees, field trips, and a logbook to record flight hours. Tuition is $1,000.

Three SunFlight programs also offer flight instruction. In the three-week SunFlight Solo Camp, the goal is for all qualified students to solo by the end of the program. Tuition is $2,950. Students in the eight-week SunFlight Private Pilot Camp earn their private pilot certificate. Tuition is $9,200. Participants in the SunFlight Instrument Camp earn their instrument rating in eight weeks. Tuition is $8,700. The SunFlight courses include flight fees, field trips, and flight, ground lab, and simulator instruction.

For registration details, and a brochure, call the Embry-Riddle Summer Academy at (800) 359-4550 or (904) 226-7648 or write to Embry-Riddle Aeronautical University, Division of Continuing Education, 600 S. Clyde Morris Blvd., Daytona Beach, FL 32114-3900. More information can be found at the web site www.erau.edu/dce.

The full monty

Yes, this month's CyberTeaser (B264 in this issue) was a thinly disguised version of the "Monty Hall problem," made famous by Marilyn vos Savant in her "Ask Marilyn" column in the September 9, 1990, issue of *Parade* magazine. (We are awarding the winners who made note of this fact with an extra *Quantum* button and honoring them with an asterisk next to their names in the winners list that follows.) We were initially hesitant to run this much-debated problem precisely because of its notoriety, but we feel that it has taken on the patina of a classic and as such should periodically be reintroduced for new generations of students to experience.

Rather than rehash the controversial and colorful history of this problem, we offer some resources for further study. Marilyn vos Savant claims to have received upward of 10,000 letters in response to her answer to the problem. The debate became so widespread that on July 21, 1991, the *New York Times* ran a front-page article on the subject ("Behind Monty Hall's Doors: Puzzle, Debate and Answer," by John Tierney), complete with an interview with Monty Hall himself. (For readers who may not know, Monty Hall was the host of the long-running TV game show *Let's Make a Deal.*)

The problem exhibited great staying power, as people continued to debate it years later. Indeed, *Quantum* got into the act by publishing the article "Generalizing Monty's Dilemma (whether to stick with a choice or a switch)," by John P. Georges and Timothy V. Craine, in the March/April 1995 issue.

Quantum's resident *Mathematica* wondercow, Dr. Mu, a.k.a. Don Piele, paid homage to the Monty problem in a 1996 article in *Mathematica in Education and Research* 5(1), available on the web at http://www.telospub.com/catalog/MATHEMATICA/MiER/Vol5No5/Piele/piele51.html. An excellent discussion of the problem is available at the Virtual Laboratories in Probability and Statistics at http://www.math.uah.edu/stat/monty/index.html, which includes an interactive game and simulation.

. . . And the winners are

Bruno Konder (Rio de Janeiro, Brazil)
Theo Koupelis* (Wausau, Wisconsin)
Anastasia Nikitina (Pasadena, California)
Jerold Lewandowski (Troy, New York)
John E. Beam* (Bellaire, Texas)
Xi-An Li (Middlebury, Vermont)
Christopher Franck (Redondo Beach, California)
Hana Bizek (Argonne, Illinois)
Liubo Borissov (New York, New York)
Joshua Zucker (Mountain View, California)

Congratulations! Each of the winners will receive a *Quantum* button and a copy of the March/April issue. Everyone who submitted a correct answer in the time allotted was entered in a drawing for a copy of *Quantum Quandaries,* our collection of the first 100 *Quantum* brainteasers.

Message from afar

by David Arns

Once upon a weeknight dreary,
I beheld an image smeary,
Captured by a telescope that's been in space from days of yore,
As I sat with eyelids drooping,
A strange and unexpected grouping
Of celestial objects caught my eye like none had done before—
I knew I had to find out more.

I didn't know what I was seeing,
But I thought, "Another being
From another galaxy, perhaps an alien 'Signal Corps,'
Created this configuration
To confer some information
To a random listener. Yes, surely that is what it's for!"
Thus I let my fancy soar.

Then I stopped and gripped the table,
Forced my thoughts to be more stable,
Realizing I would need some proof, some evidence, and more.
So I called to book the Hubble—
To my surprise, I had no trouble
Getting seven hours' observation time, that day at four.
Now I'd give them proof galore!

So I made my observations,
Measurements, and calculations,
Disbelief and wonder nearly left me breathless on the floor.
This was proof beyond ignoring—
Sweat was from my brow outpouring—
I could see my name in scientific journals evermore!
(I'd been a no-name heretofore.)

Five weeks, and almost all was ready,
(I'd show those stuck-up folks at SETI!)
I merely had to translate all these symbols I had grabbed before.
Already I had seen a pattern:
The spectrogram's bright lines were scatterin'
In ways that shocked, amazed, bewildered, stunned, and shook me to the core
A *message* from a distant shore!

Methodically, I put together
Facts and data, heedless whether
Days were passing, pizza mould'ring, knocks and calls outside my door.
Finally, it was translated,
And I stood aghast, deflated:
The message from afar, for which I'd launched into my eight-week chore,
Read only, "Made in Singapore."

ANSWERS, HINTS & SOLUTIONS

Math

M261

Note that $f(f(x)) = 1 - 2x$ is one-to-one. That is, it has the property that if $a \neq b$, then $f(f(x_1)) \neq f(f(x_2))$. If a function $f(x)$ such as described in the problem existed, it would have to have the same property. Indeed, if $a \neq b$, and $f(x_1) = f(x_2)$, then $f(f(x_1)) = f(f(x_2))$, which is not true. Therefore, $f(x)$ is one-to-one, and such a function on the real line must be monotonic: that is, for all real numbers a and b, $a > b$ implies either $f(x_1) > f(x_2)$ or $f(x_1) < f(x_2)$.

We will show that if $f(x)$ is monotonic, then $f(f(x))$ is monotonic increasing, which is not true of the function given in the problem. Indeed, consider two cases:

(1) $f(x)$ is monotonically decreasing. Then, $x_1 < x_2$ implies $f(x_1) > f(x_2)$ and $f(f(x_1)) < f(f(x_2))$;

(2) $f(x)$ is monotonically increasing. Then, $x_1 < x_2$ implies $f(x_1) < f(x_2)$ and $f(f(x_1)) < f(f(x_2))$.

M262

We will solve a more general problem. Denote by x_n the number of possible ways to select a subset of n first natural numbers that doesn't include three consecutive integers. What is x_1? We can either choose the empty set or the set $\{1\}$ ("singleton 1"): Neither of these sets contains three consecutive integers. Thererore, $x_1 = 2$. By a similar direct count, we find that $x_2 = 4$ and $x_3 = 7$.

Suppose we know the values $x_1, x_2, \ldots, x_{n-1}$ (for $n > 3$). Let's see how we can obtain from these the value x_{n+1}. The subsets that satisfy the

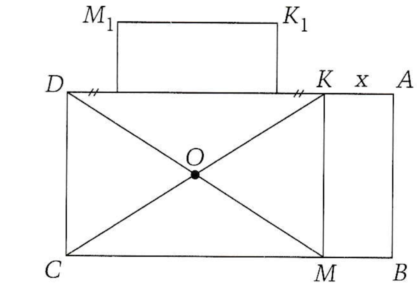


Figure 1

conditions of the problem either contain n or do not contain n. Those that do not contain n are subsets of the set $\{1, 2, 3, \ldots, n-1\}$, and these are counted by x_{n-1}.

We must add to this the number of subsets that do contain n. These again split into two types: those that contain $n-1$ and those that do not. Subsets that contain n and $n-1$ cannot contain $n-2$, and thus are counted by x_{n-3}. It remains to count the subsets that contain n but not $n-1$ (and satisfy the condition of the problem). Any such subset can be formed by choosing a subset of $\{1, 2, 3, \ldots, n-2\}$ and including the number n as well. Thus these are counted by x_{n-2}, and $x_n = x_{n-1} + x_{n-2} + x_{n-3}$.

By direct computation, we now find that the sequence of values x_i is 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, and $x_{11} = 927$.

M263

Consider rectangle $ABCD$ in which $AB = 10$ and $AD = 20$. We can try to make the rectangle fit into a circle by cutting off a rectangular slice $AKMB$ and placing it on top of $KMCD$ as shown in figure 1. But how much of a slice must we cut off? That is, how long is AK?

Suppose $AK = x$, and let us try to find x so that M_1 and K_1 lie on the circle circumscribing rectangle $KMCD$ (whose center is at point O in figure 1). Then, since equal chords are equidistant from the center of a circle, the distances from O to KM and K_1M_1 are equal. This condition leads to the equation

$$\frac{20 - x}{2} = 5 + x,$$

whose solution is $x = 10/3$.

We must still verify that the new figure will fit into a circle of radius 19.5. That is, we need to verify the inequality

$$10^2 + (20 - x)^2 < (19.5)^2.$$

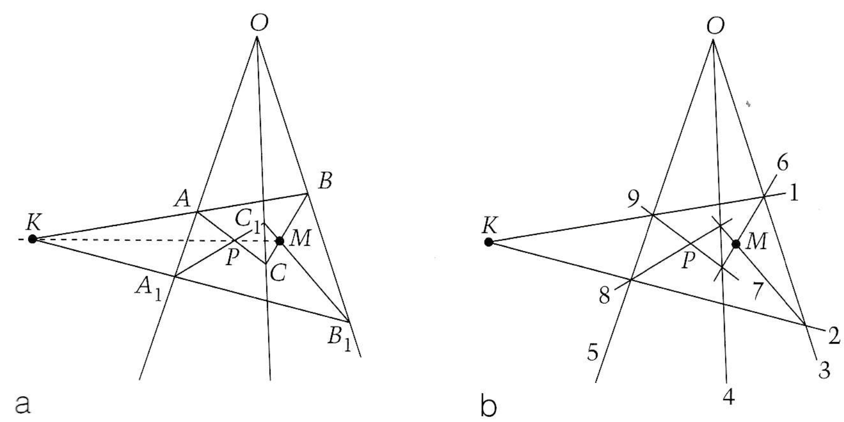


Figure 2

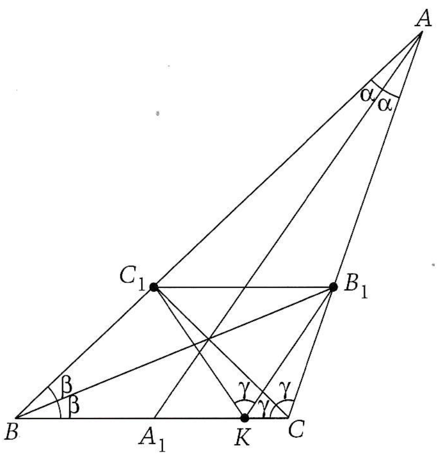


Figure 3

We substitute 10/3 for x, multiply both sides by 9, and divide by 25 to obtain the inequality

$$36 + 100 < 9(3.9)^2 \text{ or } 136 < 138.89.$$

Note: This result can be improved. It is possible to divide the given rectangle into two pieces that fit into a smaller circle—for example, one of radius 19.4. But even this small improvement is difficult. The reader is challenged to obtain it.

M264

We can accomplish the construction by finding a point P on segment KM that is closer to M than the length of our straightedge. We do this by reproducing the figure 2a, which demonstrated Desargues' theorem. Figure 2b shows the construction. The numbers in the figure show the order in which the line segments are to be drawn. To draw line 1, for example, we use our short straightedge to draw a short segment in any direction at all from K, then slide the straightedge along the short segment to lengthen it, and so on. This technique produces an arbitrarily long line in the given direction. Note that lines 1 and 2 must be drawn sufficiently "close," and that if point P is still too far from M, the construction can be repeated to find a point P' on segment PM, and so on.

To prove Desargues' theorem, we return to the problem. Figure 2b shows a method for finding a point P on the segment KM. The numbers in the figure show the order of drawing the lines. The first two lines should be drawn such that they are sufficiently "close" to each other. Here we employ the fact that arbitrarily long segments can be drawn using a short straightedge by sliding it gradually along the segment. The construction is justified by the Desargues theorem just proved. If point P is still too far from M, then the construction can be repeated.

M265

Denote the angles of the given triangle ABC (figure 3) by 2α, 2β, and 2γ. Through B_1, draw a line parallel to AA_1 and denote by K its points of intersection with BC. Angle $AA_1C = \alpha + 2\beta$ (it is an exterior angle of triangle BAA_1). It is given that we have $\angle AA_1C = \angle AC_1B_1$. However, $\angle AC_1B_1$ is an exterior angle of triangle BC_1B_1. So we have

$$\begin{aligned}\angle BB_1C_1 &= \angle AC_1B_1 - \beta \\ &= \alpha + 2\beta - \beta = \alpha + \beta,\end{aligned}$$

and

$$\angle KB_1C = \angle A_1AC = \alpha.$$

Now we find

$$\begin{aligned}\angle BB_1K &= \angle BB_1C - \angle KB_1C \\ &= (2\alpha + \beta) - \alpha = \alpha + \beta = \angle BB_1C_1.\end{aligned}$$

Therefore, triangles BB_1C_1 and BKB_1 are congruent (by ASA). Thus, triangle C_1B_1K is isosceles, and

$$\begin{aligned}\angle C_1KB_1 &= (1/2)(180° - \angle C_1B_1K) \\ &= (1/2)[180° - (2\alpha - 2\beta)] \\ &= 90° - (\alpha + \beta) = \gamma = \angle C_1CB_1.\end{aligned}$$

This implies that points C_1, B_1, C, and K lie on a circle. In this circle, equal chords are subtended by inscribed angles C_1B_1K and C_1KB_1. So these two angles must be equal, and triangle C_1B_1K is equilateral. Therefore, $\gamma = 60°$ and $\angle BCA = 120°$.

Physics

P261

The acceleration of a person who stepped onto the first band increased the person's momentum from 0 to Mv_1. Note that a passenger who moves to the second band does not borrow any momentum from the first band, because the passenger leaves it perpendicular to its motion. Thus, the force we are looking for can be expressed via the increase of the system's momentum per unit time:

$$F_1 = NMv_1 = 1600 \text{ N}.$$

Similarly, for the second band we must know the increase of the passenger's speed after arriving from the first band: This value is half of the previous one. Therefore,

$$F_2 = NM(v_2 - v_1) = 800 \text{ N}.$$

P262

The table shows that the temperature doesn't depend linearly on time. Therefore, we must take into account the heat loss to the surrounding air, which is proportional to the temperature difference between the jar and the air. Conservation of thermal energy gives us

$$c\Delta T = W\Delta t - \alpha(T - T_0)\Delta t,$$

where c is the heat capacity of the jar with all its contents, T its temperature, t is the time, $W = IV$ is the power of the heater, and α is a proportionality constant.

Because this equation has two unknown values, c and α, we take two different temperatures T_1 and T_2 and calculate

$$k_1 = \left(\frac{\Delta T}{\Delta t}\right)_1$$

and

$$k_2 = \left(\frac{\Delta T}{\Delta t}\right)_2$$

near these temperatures. Now we have two equations

$$ck_1 = W - \alpha(T_1 - T_0)$$

and

$$ck_2 = W - \alpha(T_2 - T_0),$$

from which we get

$$c = W\frac{(T_2 - T_0) - (T_1 - T_0)}{k_1(T_2 - T_0) - k_2(T_1 - T_0)}.$$

Plugging the data from the top row of the table into this equation, we get the heat capacity of the jar with the water only:

$$c_1 \cong 770 \text{ J/K}.$$

Using the data from the bottom row of the table, we get the heat capacity of the jar with the metal sample:

$$c_2 \cong 890 \text{ J/K}.$$

Therefore, the heat capacity of the sample is

$$c = c_2 - c_1 \cong 120 \text{ J/K}.$$

Note that this is a small difference of two rather large values. Accordingly, there may be a large error (from 100 to 130 J/K) in the answer.

P263

We denote the series resistance for the 10-volt and 100-volt scales by $R_2 = 10R_1$ and $R_3 = 100R_1$, respectively. If the emf of the solar cell is $\mathscr{E}$ and its internal resistance is r (figure 4), we get

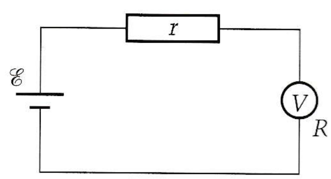


Figure 4

$$V_1 = \frac{\mathscr{E}}{r + R_1} R_1 = \frac{\mathscr{E}}{1 + \frac{r}{R_1}},$$

$$V_2 = \frac{\mathscr{E}}{1 + \frac{r}{10R_1}},$$

$$V_3 = \frac{\mathscr{E}}{1 + \frac{r}{100R_1}}.$$

By solving these simultaneous equations, we get the answer:

$$V_3 \cong 3.6 \text{ V}.$$

P264

Figure 5 shows that in diode D_1 the electric current flows only from left to right (along the arrow in the diode's schematic icon), and in diode D_2 it flows only to the left. Thus, this circuit is an oscillating circuit with a constant voltage source that is always opposed to the current.

Let's start our analysis of the closed circuit from the moment when the electric current is zero, the voltage across the capacitor has just assumed a maximum value V_n, and the corresponding charge is $q_n = CV_n$ (the upper plate is positively charged). In the following half-cycle the capacitor will discharge to zero, and then it will be recharged with the opposite polarity. The current through diode D_1 performs work in passing through the battery. After the recharging half-cycle, the capacitor will have a charge q_{n+1}, so the charge $q_n + q_{n+1}$ will pass through the battery (the polarity of the capacitor changes during this process).

According to the conservation of energy, the decrease in the energy of the capacitor's electric field equals the work performed:

$$\frac{q_n^2}{2C} - \frac{q_{n+1}^2}{2C} = (q_n + q_{n+1})\mathscr{E},$$

from which we get

$$\frac{q_n}{2C} - \frac{q_{n+1}}{2C} = \mathscr{E},$$

or

$$V_n - V_{n+1} = 2\mathscr{E}.$$

Thus, in a half-cycle the voltage across the capacitor will drop by $2\mathscr{E} = 3$ V. This process will go on until the voltage becomes less than $\mathscr{E} = 1.5$ V at zero current. According to the statement of the problem, the final voltage is 1 V (of reverse polarity), so the initial voltage on the capacitor can be (in volts):

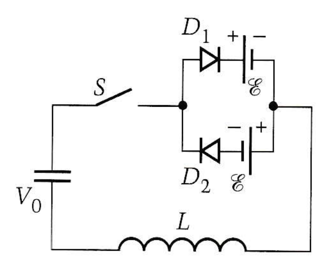


Figure 5

$$V_0 = 4 + 6n,$$

where $n = 0, 1, 2, \ldots$

However, this is not the end of the story. This series of solutions was obtained under the assumption that the polarity of the capacitor changes after every half-cycle. Still another case is possible, when in the last half-cycle the charge drops from q_{N-1} to q_N without changing polarity. In such a case the charge $q_{N-1} - q_N$ passes through the battery, so conservation of energy assumes the form

$$\frac{q_{N-1}^2}{2C} - \frac{q_N^2}{2C} = (q_{N-1} - q_N)\mathscr{E},$$

from which we get

$$V_{N-1} + V_N = 2\mathscr{E}.$$

Thus, $V_{N-1} = 2\mathscr{E} - V_N = 2$ V (the upper plate is negative), $V_{N-2} = V_{N-1} + 2\mathscr{E} = 5$ V (the upper plate is positive), and so on. The initial voltage in this case must be

$$V_0 = 5 + 6n,$$

where $n = 0, 1, 2, \ldots$

Thus, the problem has two series of solutions:

$$V_0 = \begin{cases} 4 + 6n \\ 5 + 6n, \end{cases}$$

where $n = 0, 1, 2, \ldots$

P265

The refractive index can be obtained from the lens maker's equation:

$$P = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad (1)$$

provided we know the optical power P of the lens and the radii R_1 and R_2 of its spherical surfaces. The optical power of a lens is equal to the inverse of its focal length in meters.

Let's consider how the student's images are formed when the teacher turns to the board. One image is formed by the rays reflected from the proximal (that is, nearest the eye) surface of the lens. Another image is created by the rays that passed through the lens forward and

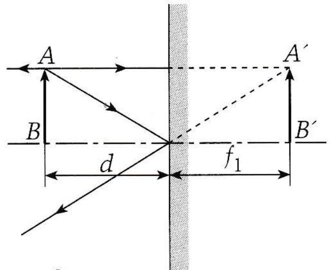


Figure 6

backward, having reflected from the distal spherical surface.

According to the statement of the problem, the distance to one of the images equals the distance to the object (the student): $f_1 = d = 5$ m. This is characteristic of a flat mirror (figure 6). Thus, as a first step, it is natural to look for a solution with a lens that has a flat proximal surface, so that

$$\frac{1}{R_1} = 0.$$

The image at the distance $f_2 = 5/7$ m is formed by the optical system lens-mirror-lens. This system can be replaced by a single equivalent lens. Its optical power must be equal to the algebraic sum of the optical powers of all the elements of our optical system:

$$P + \frac{2}{R_2} + P = 2P + \frac{2}{R_2}.$$

Then, according to the lens equation,

$$\frac{1}{d} - \frac{1}{f_2} = 2P + \frac{2}{R_2}. \qquad (2)$$

The minus sign is inserted because the image is virtual.

When the teacher looks directly at the student through his glasses, he sees the student's virtual image at the distance $f_3 = 2.5$ (figure 7). In this case the lens equation reads

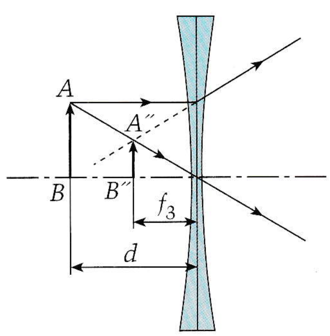


Figure 7

$$\frac{1}{d} - \frac{1}{f_3} = P. \qquad (3)$$

Equations (2) and (3) yield

$$P = -\frac{1}{5} \text{ diopters}$$

and

$$\frac{1}{R_2} = -\frac{2}{5}\frac{1}{\text{m}}.$$

As expected, we obtained a convex mirror, which means that the real lens is flat-concave. Plugging the values for P, $1/R_1$, and $1/R_2$ into (1) yields $n = 1.5$.

If we assume that after reflection from the proximal surface of the lens the image is formed at the distance of 5/7 m, and make the respective calculations, we get a strange result: $n = 0.75$. Prove this on your own. Of course, the refractive index of glass is larger than 1, so this answer is wrong. However, can you imagine a situation where the second solution could be correct?

Brainteasers

B261

Such a polygon does exist. For example, rotate an equilateral triangle around its center by an angle less than 60°, and consider the intersection of the original triangle with its image (fig. 8). When rotated about the center of the original triangle by 120°, it returns to its original position, yet it has neither line symmetry nor point symmetry.

B262

Ten students missed the first problem, 20 missed the second, 30 missed the third, and 40 missed the fourth. The sum of these numbers is 10 + 20 + 30 + 40, and the union of these four sets contains at most this many students. Since no student solved all four problems (each student missed at least one), these four sets "cover" all the 100 contestants, which means that no two of them overlap. That is, each student in the contest missed exactly one problem. Consider the 70 students who solved the third problem: 10 of them didn't solve the first problem, 20 didn't solve the second, and 40 didn't solve the fourth. Therefore, 70 – 40 = 30 students were awarded a prize.

B263

Any convex heptagon has diagonals of two kinds: "short" and "long" ones. A short diagonal connects two vertices that are one vertex apart (such a diagonal has one vertex on one of its sides and four vertices on the other side). A long diagonal connects two vertices that are two vertices apart (such a diagonal has two vertices on one of its sides and three vertices on the other side). The total number of diagonals is 14. Drawing all of them, we obtain seven triangles, each of which has two short and one long diagonal as its sides. When we remove a short diagonal, two triangles disappear. To remove all seven triangles, we must remove at least four diagonals. Therefore, the greatest number of diagonals that can be drawn in such a way that no triangles consisting of diagonals are originated is 10.

B264

If the participant retains the original choice, he wins only if this choice was the right one. However, if he chooses another box, he wins if the original choice was wrong. In essence, the alternative is of one box and the two remaining boxes. Therefore, it is better to change the box selected, which doubles the chances of winning.

B265

A hint: What would occur if the bottle was filled with glass? Having cracked this nut, "take" the glass out of the bottle.

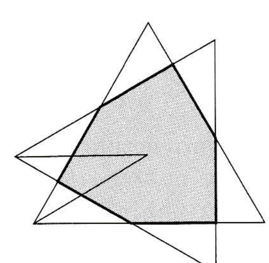


Figure 8

Kaleidoscope

1. See figure 9.

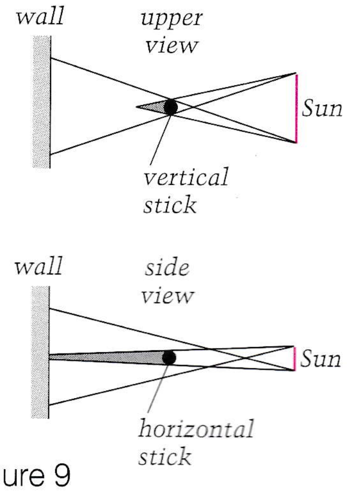


Figure 9

2. See figure 10.

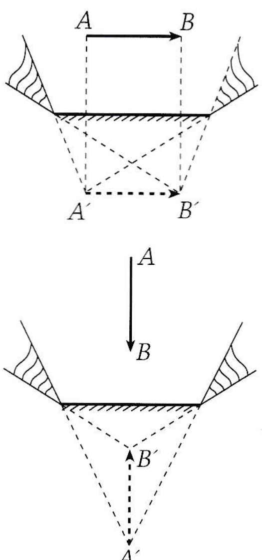


Figure 10

3. If the screen is located farther from the king than $h \tan \alpha$ (α is the angle of incidence of solar rays), then the length of the shadow will be $2h$ (fig. 11). If the screen is located nearer, then the shadow will be shorter.

4. In order to direct more light to the object.

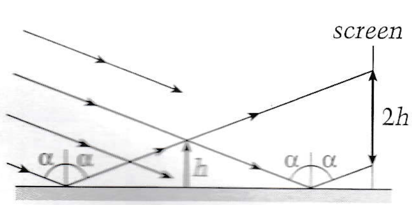


Figure 11

5. See figure 12.

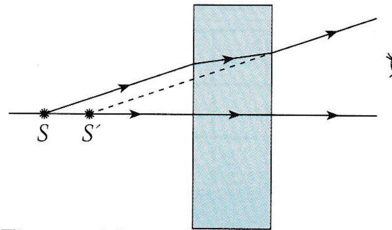


Figure 12

6. The image is virtual.

7. See figure 13. The source is located at the point S_1 if the image is real, and at point S_2 if it is virtual.

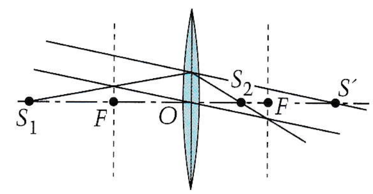


Figure 13

8. See figure 14.

9. See figure 15.

10. A variant of the ray diagram is given in figure 16.

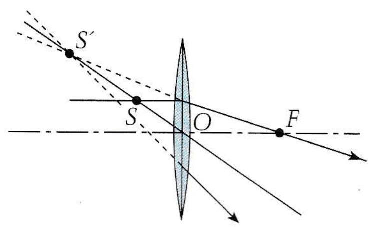


Figure 14

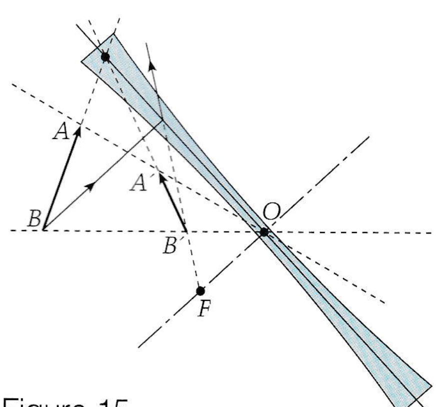


Figure 15

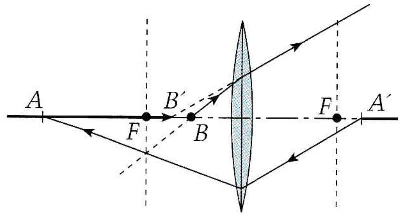


Figure 16

11. Yes. In this case the lens of the camera works like the lens of the eye.

12. The source of light must be placed closer to the lens than twice the focal length, otherwise regions will exist where both the source and its image can be observed simultaneously (figure 17).

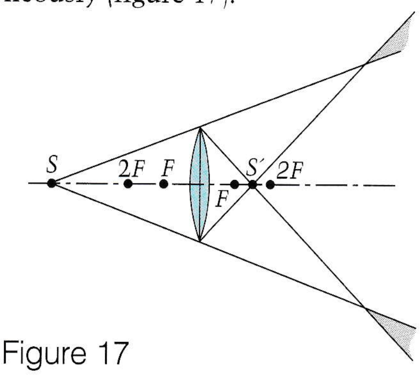


Figure 17

13. An example is shown in figure 18.

14. See figure 19.

15. There is no absorption of light by the atmosphere in space. In addition, the background is much dimmer, and atmospheric flicker is absent. Also, exposure time is not limited to nighttime. Thus, the factors that impede the detection of dim stars from Earth are eliminated or drastically moderated in space.

Microexperiment

Due to double reflection, your image will not be "reversed" left and right. If the room is rectangular, you will see yourself in the mirror from any point.

CONTINUED ON PAGE 57

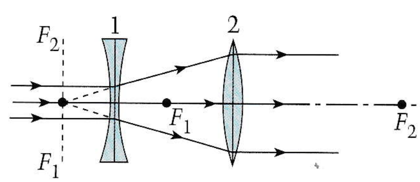


Figure 18

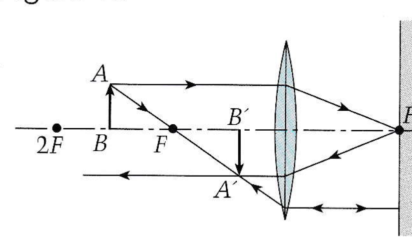
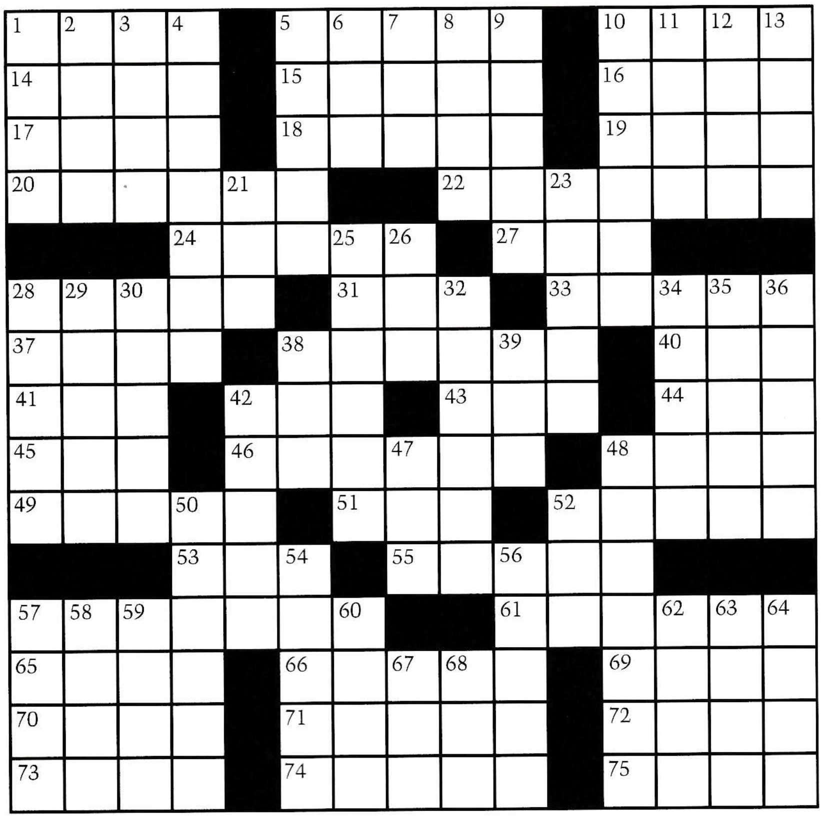


Figure 19

Criss × cross science

by David R. Martin



Across

1 Acid's opposite
5 ___ ganglia
10 Semitic fertility god
14 Dry
15 Gram-negative aerobic bacteria
16 Arm bone
17 Microbiologist ___ Dubos
18 Turn back
19 Retina cells
20 Anthropologist Montagu
22 Anthropologist Hooton
24 ___ acid (It's in apple juice)
27 Actor's signal
28 Type of cell or wind
31 Unhappy
33 Sirius and Antares
37 Nerve cell arm
38 Silica, alumina, and lime
40 Threading device
41 Sheep disease
42 ___ soda (sodium carbonate)
43 Copper sulfide
44 Songlike poem
45 Computer language
46 Neptune's moon
48 Moslem ruler
49 Type of gas
51 Approval
52 Type of salt
53 ___ Fleming
55 Relaxes
57 Beriberi researcher
61 Psychologist Kline
65 Tiny insect
66 Look for water
69 No difficulty
70 It's one for hydrogen: abbr.
71 Type of gas law
72 Eye muscle
73 Active person
74 Dissident
75 Orderly

Down

1 Actress Theda ___
2 Mars
3 Hyperbolic function
4 '72 physiol. Nobelist
5 A gemstone
6 High card
7 Absorb
8 To shelter
9 Pale purple
10 '60 physiol. Nobelist
11 Healing plant
12 "No ifs, ___, or buts . . ."
13 "___ but not least"
21 Sense organ
23 Deteriorates
25 Oceanographer Columbus O'Donnell
26 Engine shaft
28 Astronomer Carl
29 Element 8 compound
30 Direction finder
32 Decipher
34 Elementary units
35 ___ waves (3 kHz–3 THz)
36 ___ whale
38 Automobile
39 Woman of habit
42 Vaporized water
47 Crag
48 ___-Barr virus
50 Astrophysicist Ambartsumian
52 Zeta follower
54 Zenith's opposite
56 ___'s law of refraction
57 Mild oath
58 Interested in
59 Fonda or Seymour
60 Point of minimum disturbance
62 Lagomorpha member
63 Largest continent
64 Aves' home
67 Internet site
68 ___ number (of motor oils)

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE MARCH/APRIL PUZZLE

N	E	N	E	■	B	R	O	T	H	■	O	P	A	L
E	T	A	L	■	I	O	R	G	A	■	F	A	L	A
C	U	B	E	■	S	O	F	A	R	■	F	L	A	G
K	I	R	C	H	O	F	F	■	M	A	S	E	R	S
■	■	■	T	I	N	■	■	B	O	R	E	■	■	■
S	C	A	R	E	S	■	J	U	N	C	T	I	O	N
E	A	R	E	D	■	J	O	R	I	S	■	L	X	I
E	M	I	T	■	I	O	N	I	C	■	F	L	I	T
D	E	E	■	A	S	I	A	N	■	B	L	A	D	E
S	O	L	I	T	O	N	S	■	L	O	I	T	E	R
■	■	■	B	O	T	T	■	■	E	S	P	■	■	■
D	Y	N	A	M	O	■	W	A	V	E	F	O	R	M
R	E	I	D	■	P	R	I	M	E	■	L	I	E	U
U	L	N	A	■	E	O	S	I	N	■	O	S	O	S
G	L	E	N	■	S	T	E	R	E	■	P	E	S	T

COWCULATIONS

Think twice, code once

by Dr. Mu

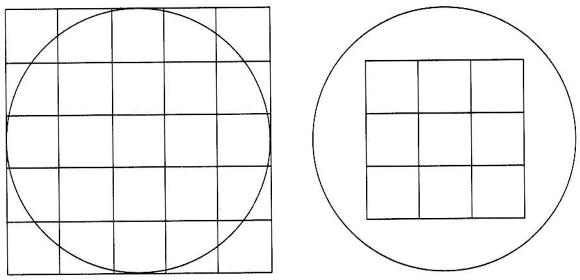
WELCOME BACK TO COWCULATIONS, THE column devoted to problems best solved with a computer algorithm. Last fall, a fierce windstorm swept across southern Wisconsin and toppled a couple of fine oak trees on the farm. Both trees were in the prime of their life, and it seemed a waste to cut them up for firewood. As luck would have it, one of the trees flattened the sick-calf shed and a couple of sick calves. This spring we are planning to rebuild the shed, so we need plenty of good, stiff four-by-fours for support beams. The trees will be put back into service after we rip them up, of course in an efficient way.

Let's examine our options before we power up the ripsaw. Remember the old adage, "Measure twice, cut once." In computer programming, this means: Before you rip off some brute force code, do a bit of thinking

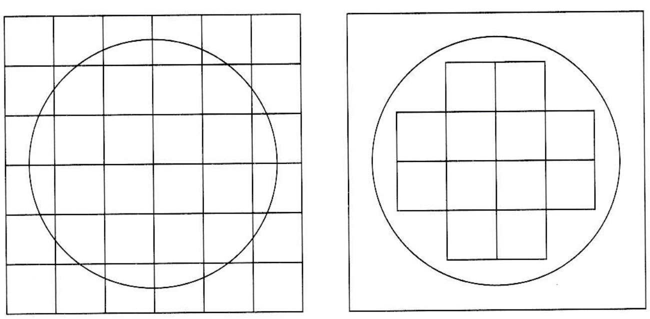
Art by Mark Brenneman

and planning on how best to solve the problem. The programmer's motto should be, "Think twice, code once."

One plan might be to design a rectangular array of horizontal and vertical lines, 4 inches apart, that span a 20×20 square array, assuming the tree has a base diameter of 20 inches. With this plan, we would be able to extract 9 perfect four-by-fours as shown below on the right:



Another plan might be to offset the first plan by 2 inches vertically and horizontally. Plan 2 yields two rows of four boards each from the center, and two rows of two boards each, for a total of 12 four-by-fours. That's a 33-percent improvement over the first plan:



But wait! Hold that ripsaw! Maybe there is still a better way to cut the boards. This suggests a problem, which, you guessed it, is the next Challenge Outta Wisconsin.

COW 16

Write a program that will find the largest number of $n \times n$ boards that can be cut from a tree of diameter 20, by making horizontal and vertical cuts only. It is not required to keep all the boards together as you normally do when you cube an onion. For example, you could rip planks horizontally, and then cut each one up individually into four-by-fours. You must be able to make all the cuts with a rip saw—no 90-degree turns, of course. Report the largest number of four-by-fours and the largest number of two-by-twos. If you can graphically show cut lines, you are really cool.

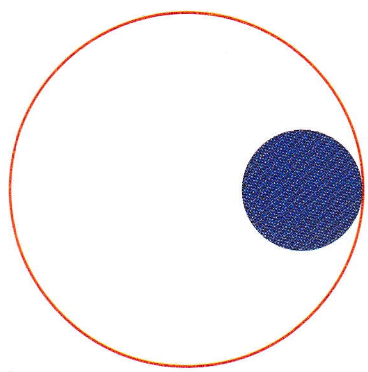
Take the bottom of the tree.
Measure once, twice, or three.
When you think you've got it right.
Rip those boards with all your might.
—Dr Mu

COW 14

In Cow 14 you were asked to write a program that would animate Hula Hoops that contact each other. The purpose of this problem was to illustrate the ease with which complicated graphical objects can be animated in *Mathematica*. This was intended to be fun, rather than a challenging programming problem. Of course, without a tool such as *Mathematica*, it would be an impossible problem to discuss in the pages of this magazine.

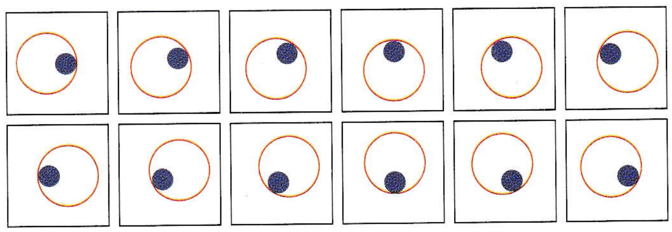
We begin by drawing a blue disk centered at (1, 0) with radius 1, (**Disk[{1,0},1]**), and surrounding it with a red circle centered at (–1, 0) and radius 3, (**Circle[{–1,0},3]**). This is done by wrapping the **Graphics** command around each object and **Show**ing it.

```
red = RGBColor[1, 0, 0];
blue = RGBColor[0, 0, 1];
Show[Graphics[{blue, Disk[{1, 0}, 1]}],
  Graphics[{red, Thickness[0.01],
Circle[{-1, 0}, 3]}],
AspectRatio -> Automatic]
```



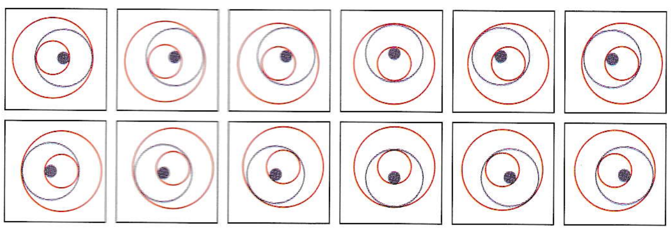
Next, we move the center of the blue disk along the circular path $(x, y) = (\cos t, \sin t)$ as t goes from 0 to 2π in steps of $\pi/6$. At the same time we move the center of the red circle along the circular path $(u, v) = (\cos(t + \pi), \sin(t + \pi))$. Notice that the center of the circle is 180 degrees ahead of the center of the disk at all times. This is animated in *Mathematica* as follows:

```
Animate[{x, y} = {Cos[t], Sin[t]};
  {u, v} = {Cos[t + Pi], Sin[t + Pi]};
    Show[Graphics[{blue, Disk[{x, y}, 1]}],
    Graphics[{red, Thickness[0.01],
      Circle[{u, v}, 3]}],
  AspectRatio -> Automatic, Frame -> True,
    FrameTicks -> None,
PlotRange -> {{-5, 5}, {-5, 5}}], {t,
  0, 2 Pi, Pi/6}])],
```



Once you understand this principle, it is a simple step to add another hoop 180 degrees ahead of the last one. Just use the same path (x, y) as we did for the disk, but draw the circle with a radius of 5, which is the **Circle[{x,y}, 5]**. The next hoop is 180 degrees ahead of this one, so its center moves on the (u, v) path with a radius of 7, which is **Circle[{u,v},7]**. Putting this altogether into one animation in *Mathematica*, we have a very skillful cow.

```
Animate[{x, y} = {Cos[t], Sin[t]};
  {u, v} = {Cos[t + Pi], Sin[t + Pi]};
    Show[Graphics[{blue, Disk[{x, y}, 1]}],
    Graphics[{red, Thickness[0.01],
      Circle[{u, v}, 3]}],
    Graphics[{blue, Thickness[0.01],
      Circle[{x, y}, 5]}],
    Graphics[{red, Thickness[0.01],
      Circle[{u, v}, 7]}],
  AspectRatio -> Automatic, Frame -> True,
    FrameTicks -> None,
PlotRange -> {{-8, 8}, {-8, 8}}], {t,
  0, 2 Pi, Pi/6}])],
```



Hope you enjoyed the exercise.

And finally . . .

Send in your solutions to COW 16 via email to drmu@cs.uwp.edu.

By the time this column reaches you, the USA Computing Olympiad (USACO) will have sent its first team to the Baltic Olympiad in Informatics, held April 16–18 in Riga, Latvia. The team consisted of six high school students chosen by their ranking in the first two Internet competitions held in the fall and winter by the USACO. The team members were: Po-Shen Loh, 16, from James Madison Memorial High School, Madison, Wisc.; Daniel Wright, 18, from Longmont, Colo.; Percy Liang, 16, from Mountain Pointe High School, Phoenix, Ariz.; Reid Barton, 15, from Arlington, Mass.; Kenn Hamm, 16, from The Albany Academy, Albany, N.Y.; and Jon McAlister, 17, from Langham Creek High School, Houston, Tex. Congratulations to them all.

You can meet the team and see how they ranked at the Baltic Olympiad by going to the USACO web site at http://www.usaco.org. Click on 1999 and then the Baltic Olympiad. You can also view the type of programming problems they faced.

If you are interested in learning more about any international science or mathematics olympiad, go to http://olympiads.win.tue.nl/.

CONTINUED FROM PAGE 53

When things fall apart

Exercise 1. Yes, it will! (It makes a nice physics trick.) The pressure in the stream is lower, according to Bernoulli's equation, so the outside air forces the ball to stay in the stream. Even if you blow the ball sideways (gently!), it tends to stay over the nozzle.

Exercise 2. No, a system of charges can never be in stable equilibrium if they interact via electrostatic forces only. To prove this, let us consider a positive charge. For the equilibrium to be stable, the charge, when slightly moved, must experience a force directed toward the original position of the charge. This means that the electric field lines in the vicinity of the *positive* charge must be directed *toward* the positive charge, which, of course, is impossible, according to Gauss's law. Similar reasoning applies to a negative charge.

Exercise 3. No, this equilibrium is unstable. Imagine that the balloon is moved down a bit. The water pressure increases, the volume of the balloon decreases, and the buoyancy force decreases, so the balloon begins to sink further. If, on the other hand, the balloon is moved slightly up, it ascends to the surface.

Exercise 4. The loop has two equilibrium positions: one stable and one unstable. Try to draw them both and classify them.

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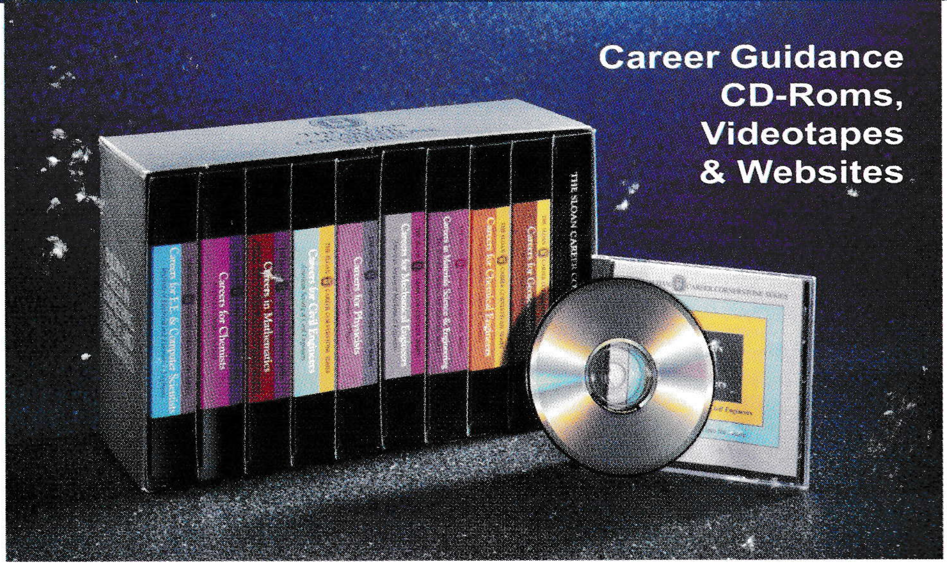

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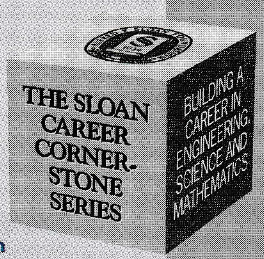

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