QUANTUM

JULY/AUGUST 1998

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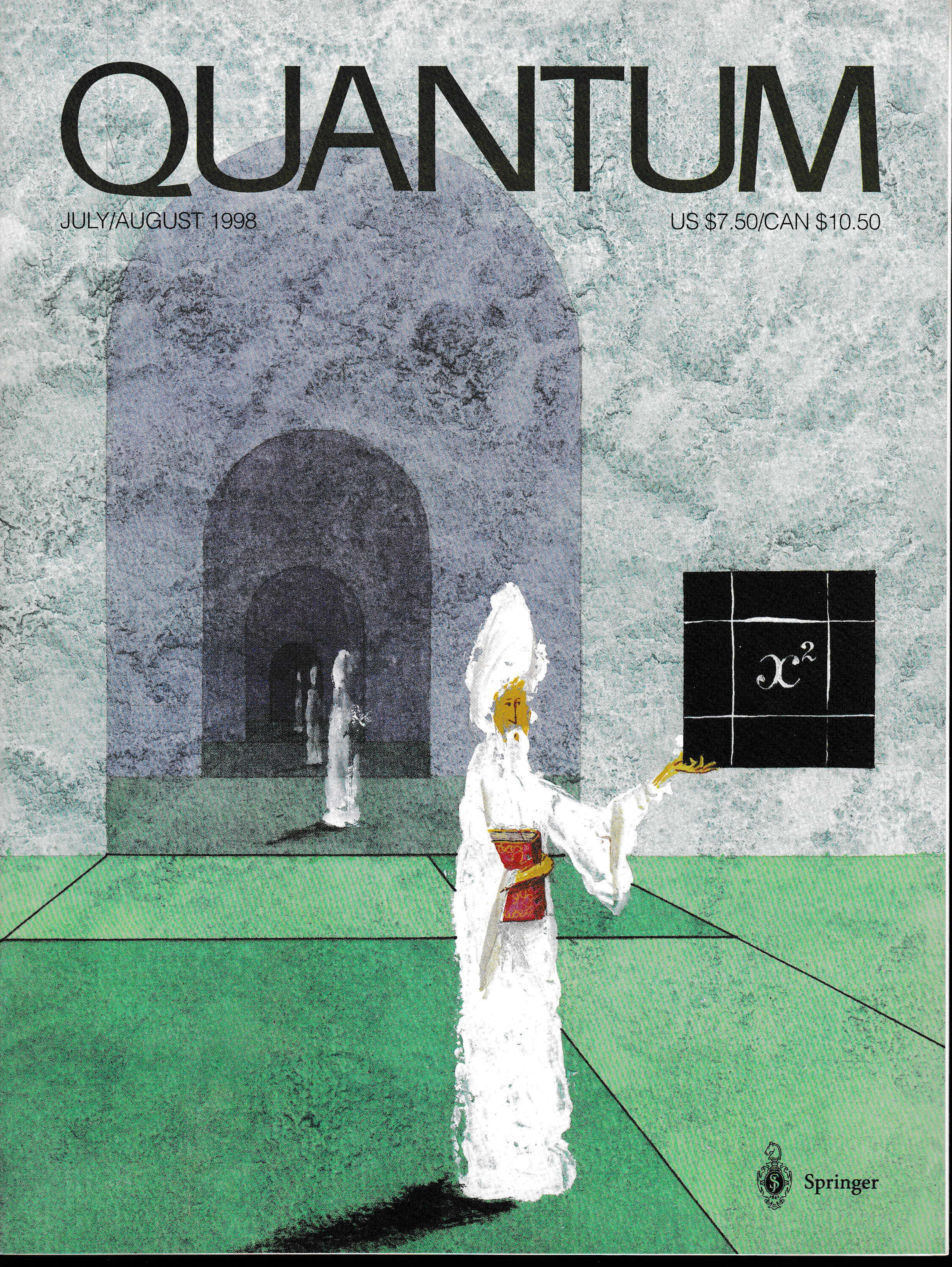

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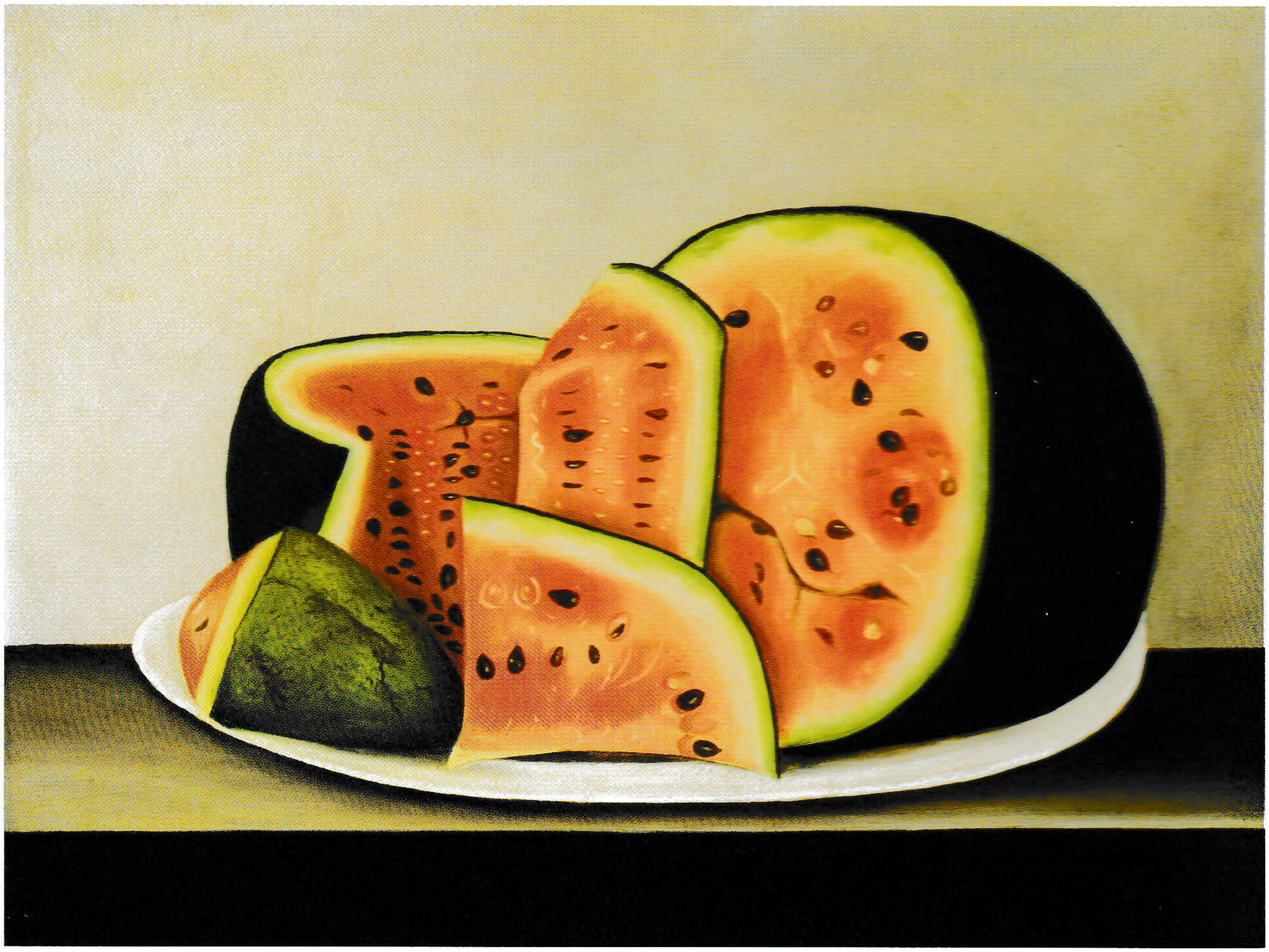

Watermelon on a Plate (nineteenth century) American

Watermelon—a staple of the summer cookout. Most consider it delicious and easy to prepare, but anyone who's delved beneath the dark jade rind knows that it can be a demanding fruit.

The challenge of selecting a ripe melon is a science unto itself. Thumping is thought to reveal acoustic clues to the maturity of the fruit, others prefer to pursue olfactory indications of peak flavor. But nothing can surpass a core sample of the pink flesh within. The challenge, however, is far from over. Assuming you're abiding by the parliamentary procedures of picnics, you will have set aside your cutlery and taken hold of a half-moon wedge of melon with both hands. The trick, as we all know, is to maximize the amount of fruit ingested while minimizing the amount of juice dribbled onto your shirt. And there's the question of the seed disposal—to spit or not to spit, what would Miss Manners do?

Hopefully, after reading this culinary overview, you're ready to sink your teeth into an even more confounding puzzle concerning watermelons. It leads off an article that contains a cornucopia of problems that explore how appearances can be deceiving. Make sure you have your napkin ready before turning to page 34.

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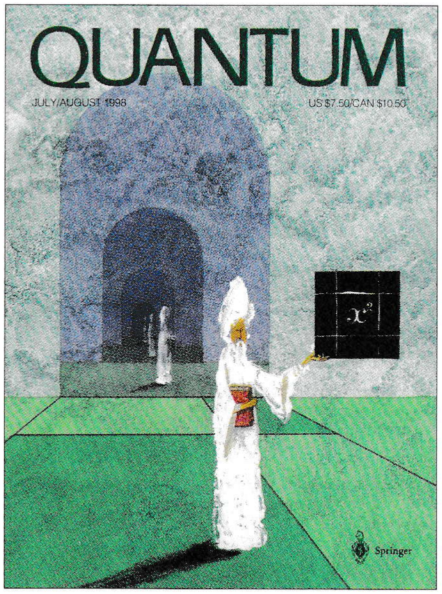
JULY/AUGUST 1998 VOLUME 8, NUMBER 6



Cover art by P. Chernusky

The ghostly personage on our cover is perhaps an ancient wise man coming forth through the portals of time to share with us his knowledge of the origins of algebra. Look for him again on page 26, where he expounds the contribution of one Muhammad ibn Musa al-Khwarizmi.

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FEATURES

DEPARTMENTS

Readers respond

Money trumps image

We enjoy hearing from our readers by any mode of communication. Lloyd Kannenberg e-mailed us (at quantum@nsta.org) the following reaction to last issue's Front Matter:

> I was amazed at the allegation in "Enough Nerdiness" (Front Matter, May/June 1998) that the nerd image of scientists deters young people from choosing careers in science and technology.
>
> The same image has been around for a very long time; why then would its negative impact only manifest itself now? A far more likely reason that bright young people no longer choose technological careers is that today they stand to earn *far* more by going into management or finance. It is true that there has always been a gap between salaries of the management and of the technical staff of a typical company; but in recent years this gap has widened into a chasm (I will say nothing about relative working conditions, but think: Does the manager who decided to put the technical staff into Dilbert-style cubicles work in a cubicle himself?).
>
> Young people aren't stupid. They can apply their smarts in any of a broad spectrum of careers. Why should they take what amounts to a vow of poverty by following the technological path? Nerd image? Bah! Since when are managers and venture capitalists sexy?

Whoops

Eric E. Wickey of New York City also chose e-mail to contact us about a questionable calculation in a previous issue:

> I just found your magazine, and I think it's great. However, I think I found an error in the March/April 1998 issue in the article "Symmetry in Algebra," on page 43, example 3. We're asked to solve simultaneously this system of equations:

$$\begin{cases} xy = 6 \\ yz = -2 \\ zx = 10. \end{cases}$$

> Multiplying gives $x^2y^2z^2 = -120$, right? To go any further means taking the square root of a negative number, which is imaginary.
>
> The solution on page 52 gives the right sides of these equations as 6, 15, and 10, which when multiplied gives a square root ±30, and that works out well.
>
> Anyway, thanks for a great magazine.

Well, Eric, we're glad you found our magazine, and we thank you for your kind words and for keeping us honest.

Oops

In Physics Contest in the January/February 1998 issue on page 32, middle column, the net force should be 768 N rather than 588 N. Our thanks to Victor Mazmanian for pointing out this regrettable mistake.

What's going on?

Summer study ... competitions ... new books ... ongoing activities ... clubs and associations ... free samples ... contests ... whatever it is, if you think it's of interest to *Quantum* readers, let us know about it! Help us fill Happenings and the Bulletin Board with short news items, firsthand reports, and announcements of upcoming events.

What's on your mind?

Write to us! We want to know what you think of *Quantum*. What do you like the most about it? What would you like to see more of? And, yes—what *don't* you like about *Quantum*? We want to make it even better, but we need your help.

You can contact us via e-mail at quantum@nsta.org, leave a message in our guestbook at http://www.nsta.org/quantum, or send us a letter at

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QUANTUM

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Planetary building blocks

Blueprints for creating terra firma

by V. Meshcheryakov

IT WAS LONG, LONG AGO. Judging by the fact that Mankind does not remember the face of the Creator, who gave birth to the World, we can reasonably suppose that people did not participate in this historical event. What can we know about it? All hopes are focused on physics, which helps us look into the past, learn the laws of the Universe, and realize where we should go in the future.

It is believed that the Universe is composed of atoms and that these atoms are made of electrons and nuclei. This seems quite probable since there are gobs of electronic devices around, and thinking linguistically, they should be made of electrons. However, the Universe is also composed of stars and planets. Remember, we ourselves live on a planet. It seems like the existence of planets and electrons should be connected in one way or another. But how?

The answer to this question cannot be simple, short, and exhaustive. Therefore, if you want to look at the Universe as a whole, let's get to work and let the visage of our Creator—at first as undetermined as Nature herself—accompany us.

The Creator sat at His workbench and ran His fingers over the atoms. They consisted of positively charged nuclei surrounded by dense clouds of negatively charged electrons. Drawing two atoms nearer to each other, He could see the distortions of the electron clouds resulting from a complex combination of electromagnetic interactions.

At small distances the dominating phenomenon was the attraction of the electrons of one atom to the nucleus of the other. As a result, some electrons became common to both nuclei (so-called "collective electrons"), and the total energy of the system composed of two atoms with their electron clouds stuck together turned out to be less than the sum of the energy of the same atoms when separated by a large distance. However, if we try to press the atoms still closer together, forces of repulsion arise, forces caused mainly by the interaction of the inner shells of the electron clouds.

It is easy to imagine that the strange construction composed of a few dozen atoms (called an atomic cluster) has a cell structure and is similar in many respects to an individual atom. The energy of a single cell of this cluster can be evaluated in the following way.

Suppose an atomic cell of radius R consists of a pointlike nucleus, the inner electronic shells (known as the ionic core, occupying a volume of radius R_c), and the outermost electrons, which fill most of the cell's volume. Let's determine the dependence of the cell's energy on R, say, for a univalent atom with nuclear charge Ze, where Z is the number of charges (equal to the number of electrons in the neutral atom) and $e = 1.6 \cdot 10^{-19}$ C is the elementary charge. This energy is approximately the sum of the potential energy of the coulomb attraction between the outer electron and the nucleus

$$E_1 = \frac{-Ze^2}{4\pi\varepsilon_0 R}$$

(here ε_0 is the permittivity of free space), the potential energy due to the coulomb repulsion

$$E_2 = \frac{(Z-1)e^2}{4\pi\varepsilon_0 R}$$

and the non-coulomb repulsion E_3 of the outer electron and the ionic core as well as the kinetic energy of the outer electron E_4.

The energy E_3 arises from the nonpointlike nature of the ionic core. If we assume that the electrons of the ionic core and the outer electron are distributed homogeneously in their respective volumes $V_1 = 4\pi R_c^3/3$ and $\Omega = 4\pi R^3/3$, then E_3 will be directly proportional to the surface area of the core $4\pi R_c^2$ and inversely proportional to the cell's volume Ω. Thus,

$$E_3 = \frac{3e^2R_c^2}{4\pi\varepsilon_0 R^3}.$$

The energy E_4 can be evaluated using de Broglie's formula for the momentum of an electron: $p = 2\pi\hbar/\lambda$, where λ is the electron's wavelength and $\hbar = 1.0 \cdot 10^{-34}$ J · s is Planck's constant. This formula is based on the fundamental experimental fact that in many respects electrons behave like waves. For this reason even a single free electron should be considered as a cloud similar to electrons confined in the cluster cell. Assuming λ to be equal to the length $2\pi R$ of the outermost orbit and describing the kinetic energy as $m_e v^2/2$, where $v = p/m_e$ is the electron's speed and $m_e = 9.1 \cdot 10^{-31}$ kg its mass, we obtain

$$E_4 = \frac{\hbar^2}{2m_eR^2}.$$

Therefore, the total energy is

$$E(R) \cong -\frac{e^2}{4\pi\varepsilon_0 R} + \frac{3R_c^2e^2}{4\pi\varepsilon_0 R^3} + \frac{\hbar^2}{2m_eR^2}. \quad (1)$$

Note that there is no parameter Z in this formula. This means that our estimate can be applied to atoms with an arbitrary number of electrons.

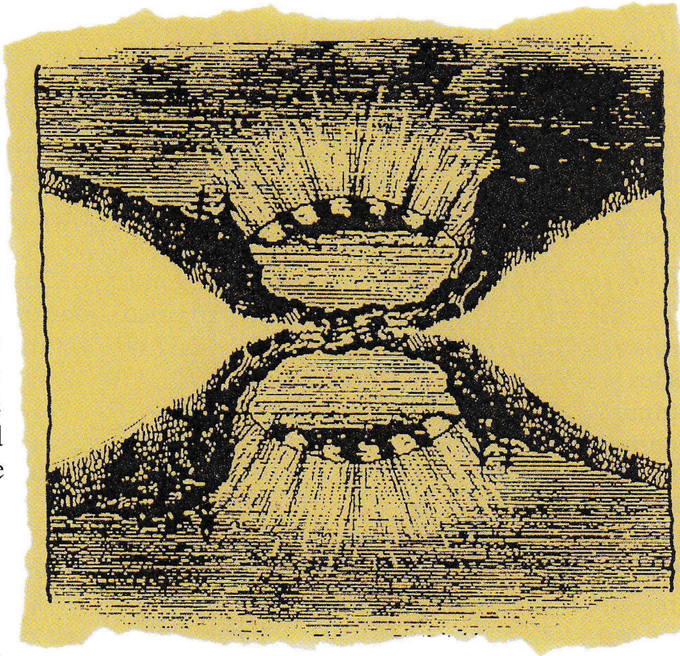
The different signs in formula (1) mean that the coulomb energy tends to compress a cell (and the entire cluster) but collapse is prevented by the non-coulomb energy of the inner shells' repulsion and the kinetic energy of the outer-shell electrons. So the function $E(R)$ has a minimum at $R = R_m$ that can be found by taking the derivative $dE(R)/dR$ and setting it equal to zero. The result $R_m \sim 3.5\, R_c$ corresponds to the minimal value of the energy E_m.

In condensed matter, the sizes of the inner electron shells of most elements do not differ appreciably from each other and on average are close to the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 0.53 \cdot 10^{-10}\ \text{m}^3.$$

This means that the ionic cores of radius $R_c = a_0$ occupy a very small part of the cell's volume

$$\frac{R_c^3}{R_m^3} \cdot 100\% \cong 10\%,$$



and so they do not prevent the outer cloud from uniting the atoms into a strong and elastic cluster whose properties depend upon the ratio R_c/R_m. The volume of a cell in such a cluster is on the order of

$$\Omega = \frac{4\pi}{3}R_m^3 \cong 10^2 a_0^3$$
$$\cong 10^2\left(\frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}\right)^3 \cong 10^{-29}\ \text{m}^3. \quad (2)$$

It is noteworthy that most of the substances in the Universe are composed of clusters with atomic volumes that differ from (2) by no more than one order of magnitude. This fact makes it possible, in a rough approximation, to subdivide the Universe into two universes: (1) the microscopic universe with electrons and nuclei that sets the value of Ω, and (2) the macroscopic universe with planets, mountains, and stones, the size of which is determined by this value of Ω.

Note the difference in the poetic sound of the words "electron" and "planet" on the one hand, and such a prosaic, plain word as "stone" on the other. This results from the nearness and familiarity of the stones. On the contrary, the distant and unknown look more attractive. But are you sure you know a Stone? It seems as if this term is appearing for the first time on the noble pages of our esteemed magazine. Next we shall discuss the phenomenon of Stone and its brethren.

The energy E_m can also be considered as the work W performed by the external force f that would need to be applied to an atom to remove it from a cluster to a distance larger than the cell's size. In other words this distance should correspond to the breaking of the interatomic bonds. In this case $E_m = W = fR \sim f\Omega^{1/3}$.

To characterize the rigidity of the interatomic bond, it is useful to consider the value $B = E_m/\Omega$, known as the bulk modulus. This constant characterizes the energy density in a cell and can be calculated from formulas (1) and (2):

$$B \cong 10^{-2}\frac{e^2}{4\pi\varepsilon_0 a_0^4}$$
$$= \frac{10^{-2}m_e^4e^{10}}{(4\pi\varepsilon_0)^5\hbar^8} \cong 10^{11}\ \text{N/m}^2. \quad (3)$$

This value is of the same order of magnitude as the experimental values of the bulk modulus of solid bodies.

Collages by Vera Khlebnikova

Thus in the first approximation the complicated picture of electromagnetic interaction inside the atomic cell can be described by two parameters: B and Ω, which make it possible to evaluate the macroscopic characteristics of the cluster.

Step by step, the Creator proceeded to construct the cluster, now joining hundreds of atoms to it, then thousands. He wondered when the gravitational effects would start to show themselves. Of course, He knew that as early as the third century A.D. the great Aristotle would begin to develop the concept of spherically symmetric gravitation:

> *Its shape must necessarily be spherical. For every portion of earth has weight until it reaches the center, and the jostling of parts greater and smaller would bring about not a waved surface, but rather compression and convergence of part and part until the center is reached* (Aristotle, *On the Heavens,* II:14).

One can't help admiring this neglecting of the heterogeneities on Earth's surface, such as mountains, which are 3 to 4 orders of magnitude larger than the average size of a human being. This was not a trivial step on the long road to understanding gravity. However, in spite of the fact that Aristotle was a first-rate mathematician as well as an outstanding physicist, he did not finish his study of gravitation, and it took an additional 2,000 years to find the mathematical description of this physical idea. Still later another important problem was solved—the description of electromagnetic phenomena in a medium, and it was found, surprisingly, that gravitational forces are far weaker than electromagnetic forces. The calculations made previously can illustrate this feature.

The elastic constant B can be determined in another way, via the pressure of the critical force $f \cong B\Omega^{2/3}$ applied to the cell's surface. Larger forces break the cell. Let's compare the critical force f with the gravitational force f_{gr} acting between two atoms. According to Newton's law of universal gravitation, $f_{gr} = Gm^2/(2R_m)^2$, where $G = 6.67 \cdot 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$ is the gravitational constant. Assuming the typical density of solid bodies to be $\rho \cong 5{,}000\ \text{kg/m}^3$, we get

$$\frac{f_{gr}}{f} \cong \frac{Gm^2}{B\Omega^{4/3}} \cong 10^{-35}.$$

These estimates convinced the Creator that a cluster could be composed of a very large number of atoms. However, He was uneasy about the moment when the gravitational force F compressing one of the cells of the macroscopic universe would surpass the elastic force that determined the pressure in a cell of the microscopic universe, which would mean the collapse of the entire construction. What would follow? A nuclear catastrophe? The full responsibility rested with the Creator, so He proceeded with the numerical estimation.

Let's express the force F in terms of the number of atoms N contained in a spherical volume $V = N\Omega$ of mass $M = Nm$ and radius

$$R = \left(\frac{3N\Omega}{4\pi}\right)^{1/3}.$$

The weight of a surface atom of such a cluster is mg, where g is the acceleration due to gravity, $g(R) = GM/R^2$. Assuming the cluster's density to be a constant $\rho = M/V$, we get

$$g(R) = \frac{4\pi}{3}G\rho R. \qquad (4)$$

This relationship says that the acceleration due to gravity at the surface of a spherical cluster grows proportionally to its radius. Thus, atoms farther from the center have larger weights. In this reasoning the cluster's surface just marks the distance to the probe atom. However, considering an atom at a distance r from the center, we do not find any changes in its weight for arbitrary variations of the cluster's radius $R \geq r$. Of course, the conditions around the atom will vary. Let's see how.

We begin to drill a well in the cluster along its radius. The cross section of the well will be only one atom. In so doing we'll weigh every atom we meet. Clearly, the result of the measurements will be the function $F_k = mg(r_k)$, where r_k is the distance from the center to the k-th atom. For example, this function will give 0 for the weight of the central atom, because $g(r_0) = 0$. This is in accordance with Aristotle's hypothesis about the disappearance of gravity at the center of a sphere.

Having finished drilling to the center, we start to fill the well with atoms. After the first atom goes to the bottom of the well, the central cell is affected by this atom's weight. The next atom increases this force by its own weight, which, however, differs from the weight of the first atom because it is farther from the center. The third atom then adds its weight to the sum, and so on.

Thus, to find the force affecting one of the central cluster's cells, we should add all the forces F_k acting along the entire length of the well:

$$F = \sum_{k=1}^{n} F_k = \frac{4\pi}{3}G\rho m(r_1 + r_2 + \ldots + r_n).$$

As the neighboring atoms are separated by the distance $2R_m$, so $r_1 = 2R_m, r_2 = 4R_m, \ldots, r_k = 2kR_m$. The last term of the sum must be equal to the force acting on the atom located at the surface of the cluster of radius R. So $r_n = R = N^{1/3}R_m$. Expressing F in terms of R_m we get

$$F = \frac{4\pi}{3}G\rho m \times 2R_m\left(1 + 2 + \ldots + k + \ldots + \frac{N^{1/3}}{2}\right). \qquad (5)$$

When the last term of the bracketed arithmetic sum $n = N^{1/3}/2$ is large, the sum is approximately equal to

$$\frac{n^2}{2} = \frac{N^{2/3}}{8}.$$

Inserting this value and using the formulas $R_m = (3\Omega/4\pi)^{1/3}$ and $\rho = m/\Omega$ yields

$$F \cong G\rho^2\Omega^{4/3}N^{2/3}. \qquad (6)$$

Comparing F with $f = B\Omega^{2/3}$, we obtain the number of atoms in a cluster:

$$N \cong \frac{1}{m\rho^2}\left(\frac{B}{G}\right)^{3/2}, \qquad (7)$$

which corresponds to its "elastic"-gravitational stability. The quotation marks stress the fact that in addition to the given atomic mass, the parameters B and $\rho = m/\Omega$ in this formula are defined by the microscopic mechanisms of interaction in the atomic cell.

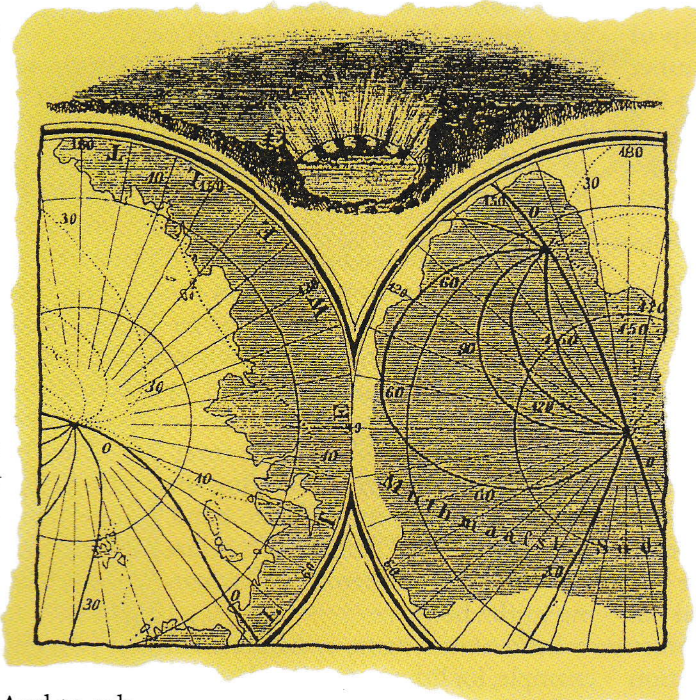
Let's stop and think about our analysis. Why is the force F determined by the pressure of only one atomic column? Why did we weigh the atoms in such a narrow well? Isn't it clear that the electromagnetic interaction of an atom with the wall of our well will prevent the realization of such an experiment? Why shouldn't we use a wider well, and why didn't we consider the relationships between the forces f_{gr} and F_k? Would you like to be the Creator for a moment and answer these questions on your own? And to ask some new ones? Equation (7) was obtained in 1905 by the English physicist Sir James Jeans. It was the first formula discovered of a number of relationships that later determined the gravitational stability of different systems.

Now let's evaluate the order of magnitude of N using the atomic volume (2), the bulk modulus (3), and the previously given value of the characteristic density of solid matter. According to (7) we have $N \cong 10^{49}$. "Mein Gott!" exclaimed the Creator, "What might the other parameters of the cluster be?" It is easy to check that the so-called Jeans mass and Jeans radius are $M \cong 10^{24}$ kg and $R \cong 10^7$ m.

This is how Earth was created. According to formulas (3), (4), and (5), the acceleration due to gravity at Earth's surface was related neither to the planet's size nor to the total mass of its atoms; it was entirely determined by the set of fundamental physical constants:

$$g \cong (BG)^{1/2} \cong \frac{10^{-1}G^{1/2}m_e^2e^5}{(4\pi\varepsilon_0)^{5/2}\hbar^4},$$

which resulted in about 10 m/s^2. The planet made by the Creator met the requirements of "elastic"-gravitational stability and since



"earth in motion, whether in a mass or in fragments, necessarily continues to move until it occupies the center equally every way, the less being forced to equalize itself by the greater owing to the forward drive of the weight impulse.... It must have been formed in this way, and so clearly its generation was spherical" (Aristotle, *On the Heavens*, II:14).

Alas, the newborn planet was naked. "What to do next?" pondered the Creator. "If I settle human beings on such a planet, they will need building materials! So He proceed with His work. He recalled the strange atomic formations, the clusters He dealt with from the very beginning, and decided to use them. But first He was to discover a physical mechanism that would determine the stable formation of small-size bodies at the planet's surface. He had a feeling that such a mechanism existed because the planet He had created met the requirements of stability, having its gravitational field determined only by fundamental constants. By intuition He felt that this mechanism should exist in the gravitational field of a stable planet and not in an arbitrarily constructed one. The idea came unexpectedly and consisted of a gravitationally stable cluster (this will prevent breakage of interatomic bonds) that is placed on the surface of such a planet and is affected by its own weight.

He embodied this idea as follows. If a gravitating planet of mass M and an atomic cluster of mass $M_0 << M$ are set in physical contact with each other, their static equilibrium (determined by the force of attraction $F_0 = GMM_0/R^2$ and by the oppositely directed supporting force) will be achieved by the breaking of the interatomic bonds at their interface, followed by a redistribution of atoms resulting in the formation of n supporting atoms:

$$n = \frac{F_0}{f}, \qquad (8)$$

where, we recall, f is the force required to tear an atom away from the cluster.

As a fundamental building unit, the cluster with both the maximum number of constituent atoms and the minimal number of supporting atoms was used. Clearly, a cluster of larger mass would better protect the interface against a load of short duration. A similar situation occurs in the circus trick in which a heavy sledgehammer strikes a massive slab resting on a man. The conservation of momentum in the hammer-slab system makes the slab virtually motionless. We invite the reader to prove this (without, of course, resorting to the use of a sledgehammer).

The condition of the minimal number of supporting atoms $n = 3$ is necessary because incorporating a fourth atom is only possible at the expense of breaking interatomic bonds, which would cause the collapse of the entire cluster. The case $n < 3$ is excluded because it doesn't provide the stability of a cluster on a plane.

The three-support cluster was called a Grain. With the help of Jeans's formula for the total number of atoms in a planet, equation (7), and by inserting $f = B\Omega^{2/3}$ and $n = 3$ into equation (8), we can evaluate such an important building parameter of the Grain as its mass:

$$M_0 \cong \Omega^{2/3}\left(\frac{B}{G}\right)^{1/2}. \qquad (9)$$

This formula shows that the mass of a gravitationally stable cluster placed on the planet's surface doesn't depend on atomic mass, and thus should be determined only by the character of the interatomic bonds. However, in that case the variety of forms and numerical parameters of natural interatomic bonds would result in the formation of clusters quite different with respect to volume per atom, number of atoms per cluster, and mass. Does this happen in nature? At present we know only one established fact: Atomic volumes in condensed matter differ no more than by one order of magnitude. Indeed, when expressing Ω and B in terms of the Bohr radius, we see that the experimental data taken from the periodic table are in the following range:

$$\Omega \cong \left(10 - 10^2\right)a_0^2,$$
$$B \cong \frac{\left(10^{-5} - 10^{-3}\right)e^2}{4\pi\varepsilon_0 a_0^4}, \qquad (10)$$

which is quite narrow.

Clearly, the quasi-constant nature of Ω and B is at odds with the existence of the vast variety of interatomic bonds. However, inserting (10) into (9) yields

$$M_0 = \frac{ze}{4\pi\varepsilon_0 G^{1/2}}, \qquad (11)$$

where z is a constant coefficient of about 1. This result is really surprising: The mass of the critical cluster not only does not depend on the mass of the constituent atoms but does not depend on the Bohr radius either. This is quite a fundamental fact! Perhaps this is the place where the Creator built the narrow pathway connecting the microscopic universe with the macroscopic universe! Indeed, the absence of the Bohr radius from the formula for M_0 says that the Grain's mass doesn't depend on the particular features of the interatomic interactions and is possibly determined by some kind of averaged properties included in the coefficient z. The nature of this parameter can be seen with the help of formula (11): Being multiplied by the elementary charge, it results in the product ze, which can be interpreted as the charge underlying the formation of the interatomic bond. In such a case the parameter z is just the number of electrons participating in the bond.

Let's obtain some estimates. For example, for lithium $\Omega = 2.1 \cdot 10^{-17}$ m^3 and $B(78\text{K}) = 0.11 \cdot 10^{11}$ N/m^2, so $M_0 \cong 1.0 \cdot 10^{-9}$ kg and $z \cong 0.53$. For beryllium $\Omega = 0.81 \cdot 10^{-17}$ m^3 and $B(0) = 1.7 \cdot 10^{11}$ N/m^2, and thus $M_0 \cong 2.0 \cdot 10^{-9}$ kg and $z \cong 1.1$. By the way, can the number of electrons participating in the bond formation really be a noninteger? Yes, it can. Remember, at the beginning of the article we spoke about electron clouds that occupy some space and thus can participate in binding not as a whole, but with part of the cloud.

Let's assume that to an order of magnitude $M_0 \cong 10^{-9}$ kg. Then for a typical density of solid matter $\rho \cong 5{,}000$ kg/m^3, we get the total number of atoms in the Grain,

$$N_0 = \frac{M_0}{m} = \frac{M_0}{\rho\Omega} \cong 10^{16},$$

as well as its characteristic size,

$$R_0 = (N_0\Omega)^{1/3} \cong 10^{-4}\ \text{m}.$$

Jeans's formula for the number of atoms in a planet and relationship (9) result in a formula that connects the Jeans mass with that of the gravitationally stable cluster:

$$M_0 \cong \left(Mm^2\right)^{1/3}. \qquad (12)$$

By inserting $M_0 = N_0 m$ and $M = Nm$, we get

$$N_0 \cong N^{1/3} \qquad (13)$$

This result shows that, to within one order of magnitude, the number of atoms in a Grain is equal to the number of atoms located along the planet's radius. Of course, this is a manifestation of the fact that both the radial column of atoms and a Grain (which is simply a crumpled radial column) produce with their weights the maximum load corresponding to the breaking of individual interatomic bonds.

The Creator made the Grain very quickly, and the result was excellent. This construction of 10^{16} atoms firmly stood on its three supports, wherever it was placed on the planet's surface.

"I wonder," He thought, "If the people will ever guess that My World rests on three supports or if they will always imagine something primitive like round and unsupported structures?" A troubled shadow came over His face.

The remnants from the production of the Grains can still be encountered in space as particles of stellar dust. However, there is not very much of it, which testifies to the high effectiveness and ecological purity of the Creator's work.

According to the Creator's design, these Grains were destined to form a grainy structure of far greater bodies. So He started to erect a new construction.

A cluster of mass m_1 of the next

structural level must be composed of clusters with critical mass M_0. Then among the masses m_1 it is possible to find an M_1 such that the number n_1 of support clusters M_0 is equal to 3: $m_1(n_1 = 3) = M_1$. The clusters of mass $m_1 < M_1$ prevent breakage of interatomic bonds between a pair of clusters with critical masses M_0, and in this sense they are also gravitationally stable in the gravitation field of the Jeans mass.

The number of supporting atoms n_0 of cluster M_1 cannot surpass the approximate number of the surface atoms of cluster M_0, so $n_0 < N_0^{2/3}$. Therefore, the force of gravitational attraction of cluster M_1 should not be greater than $F_1 = N_0^{2/3}f$. As the field at the surface of the Jeans mass is $g = GM/R^2$, and since the gravitational force affecting cluster M_1 is $F_1 = M_1g$, then $M_1g = N_0^{2/3}f$. As the number of atoms of the new cluster is $N_1 = M_1/m$, we get

$$N_1 = N_0^{2/3}\frac{f}{mg} = N_0^{5/3} = N^{5/9}, \quad (14)$$

where $f/mg = N_0$. Using this formula, we find the parameters of the first structural level:

$$N_1 \cong 10^{27},$$
$$M_1 \cong 50 \text{ kg},$$
$$R_1 \cong 0.3 \text{ m}.$$

This was how Stones and Boulders emerged from the Grains. Look at them—they are so beautiful! Their grainy structure can be seen with the naked eye.

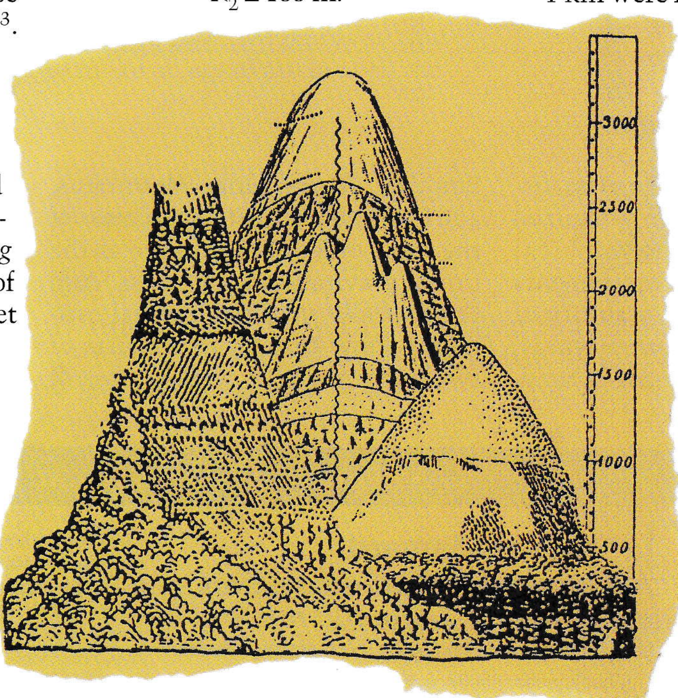
"Surely," the Creator thought, "In a Boulder (or in a Stone) that has lain on the planet's surface for thousands of years, the upper Grains will have fewer supports and thus fewer interatomic bonds than the lower Grains. So in due time the Boulders and Stones will degrade and become Sand. But, first, people will need Sand, and second, the next structural level will be made of Boulders and Stones."

In a similar way the cluster M_2 can be constructed, duly accounting for equilibrium between the weight $F_2 = M_2g$ and the supporting force acting from the side of the Jeans mass, providing it rests on three M_1-type clusters. The result is

$$N_2 = N_1^{2/3}\frac{f}{mg} = N_0^{19/9} = N^{19/27}, \quad (15)$$

This formula yields the following estimate for the cluster's parameters in the second structural level:

$$N_2 \cong 10^{34}$$
$$M_2 \cong 5 \cdot 10^8 \text{ kg},$$
$$R_2 \cong 100 \text{ m}.$$



That's good! This building material can be used to make Rocks and Cliffs, and their disintegration will produce Stones and Boulders!

Inspired by the invented succession, the Creator decided to generalize formulas (14) and (15). He saw that the number of atoms in the n-th structural level was

$$N_n = N_0^{\sum_0^n (2/3)^k} = N^{\frac{1}{3}\sum_0^n (2/3)^k},$$

where $k = 0, 1, 2, \dots$.

The numerical convergent series

$$\sum_{k=0}^{k=n}\left(\frac{2}{3}\right)^k = 1 + \frac{2}{3} + \frac{4}{9} + \dots + \left(\frac{2}{3}\right)^n$$

has the n-th partial sum

$$S_n = 3 - \frac{2^{n+1}}{3^n},$$

so

$$N_n = N_0^{3-3(2/3)^{n+1}} = N^{1-(2/3)^{n+1}}. \quad (16)$$

He checked and saw that this formula yielded equation (13) at $n = 0$. Now He resumed the numerical estimation and compiled a table of structural levels in solid matter. At $n = 3$ He obtained $N_3 = N^{65/81}$. The corresponding Hills with a height of 1 km were rather stable, large, and beautiful. Still He wanted to crown His work with something really wonderful. So He took an extra step and created the fourth level. The results based upon the relationship $N_4 = N^{211/243}$ are given in table 1.

This is how the Mountains were created, which had heights of up to 10 km. Of course, with the passage of time, they also broke down, because the upper slabs had fewer interacting bonds than the lower ones, which were pressed by a too-heavy load. But who could invent a better construction? And is it worth doing? How could people live without mountains, which provide them with a great number of useful minerals?

The atoms produced Grains, which made Boulders and Stones, and in their turn composed Rocks and Cliffs, and finally Hills and Mountains appeared. Is it the End? Not at all! Now it is time for Slabs, which will support the Continents. The Continents need Crust, then Mantle. Does this process go to infinity?

At this juncture He saw that when $n \to \infty$, formula (16) yielded $N_\infty = N$. That is, the number of atoms corresponding to the Jeans

Level number	Level name	Number of atoms	mass (g)	Characteristic size (cm)
0	Grain	10^{16}	10^{-6}	10^{-2}
1	Boulders and Stones	10^{27}	10^{4}	10^{1}
2	Rocks and Cliffs	10^{34}	10^{11}	10^{4}
3	Hills	10^{39}	10^{16}	10^{5}
4	Mountains	10^{42}	10^{19}	10^{6}
∞	The Earth	10^{49}	10^{27}	10^{9}

Table 1
Structural levels of solid matter.

mass. This means that the mass distribution of solid matter organized by a three-support hierarchical mechanism meets the requirements of elastic-gravitational stability of the limiting body. In other words, the next construction made of Sand, Boulders, Rocks, Hills, Mountains, and Continents must have a mass of the same order of magnitude as the initial gravitating Jeans mass. And this is really so! Experimental estimates show that Earth consists of two parts: the core with a mass of about $2 \cdot 10^{24}$ kg and the mantle with an approximate mass of $4 \cdot 10^{24}$ kg.

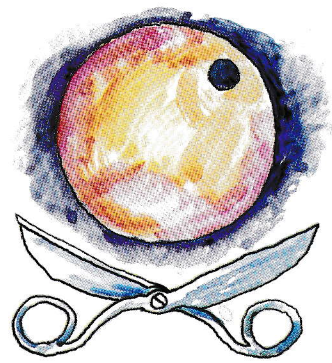
"I like my work," thought the Creator. "First, people will easily understand this mechanism of formation of macroscopic structural levels in solid matter. Let them know what a simple and pleasant World they live in. And second, now I know the answer to the puzzle that tormented me: At what structural level will people live? Of course, at the planetary level, because a planet recreates itself."

"Really, I like my work," repeated the Creator. "On such a planet people should live and raise themselves. Let them give the name for this planet, each people in its native language. And let it have a thousand names, all of which will be correct!"

Quantum articles about planet construction:

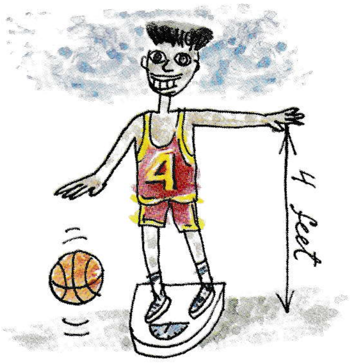
Bruk, Y. and A. Stasenko. "Hardcore heavenly bodies." Jul/Aug 1993, pp. 34–38.

Just for the fun of it!



B236

Circular reasoning. Cut the circle in the figure into two pieces so that it is possible to put the pieces together to make an equal circle with the hole in the center.



B237

Distinguishing traits. Of 20 children in a class, 14 have brown eyes, 15 have dark hair, 17 weigh more than 80 lbs., and 18 are more than 4 feet tall. Show that at least four of the children must have all four characteristics.

B238

Ahoy, matey! A raft and a motorboat left town A simultaneously and traveled downstream to town B. (The raft always moves at the same speed as the current, which is constant.) The motorboat arrived at town B, immediately turned back, and encountered the raft two hours after they had set out from A. How much time did it take the motorboat to go from A to B? (Assume that it travels at a constant rate of speed.)

B239

Goldbricking. Three bars of gold alloys of different percentages of gold have masses of 1 kg, 2 kg, and 3 kg. Can you find a way to divide these pieces in order to make another three bars with masses of 1 kg, 2 kg, and 3 kg but with equal percentages of gold? (Your method must work no matter what the original percentages of gold may be.)

B240

Sparkling snow. Why is snow often described as "sparkling"? (A. Panov)

ANSWERS, HINTS & SOLUTIONS ON PAGE 52

Art by Pavel Chernusky

Van der Waerden's pathological function

Examining a "miserable sore"

by B. Martynov

STUDENTS OF CALCULUS KNOW THAT IF A function is differentiable at a given point, then it is also continuous at that point. They also know that the converse is not true.

For example, the function $y = |x|$ is continuous when $x = 0$ (and at all other points), but it has no derivative at the point $x = 0$. Generally speaking, any function whose graph has a "corner" at some point is not differentiable at that point. Clearly, it is not difficult to construct a continuous function with an infinite number of "corners" in a segment. (Figure 1 represents a part of a graph of such a function.) However, all functions of this type, and all other continuous functions that you might know, are differentiable at the majority of their points.

For a long time after the invention of differential calculus, mathematicians thought that continuous functions were "usually" differentiable. In fact, a great commotion arose in the world of mathematics in the 1880s when the German mathematician Karl Weierstrass published an example of a continuous function that wasn't differentiable at any point. (This type of function was first suggested by the Czech mathematician Bernhard Bolzano some 40 years earlier, but his work was not widely publicized.)

"How on Earth could intuition play such a trick on us?" asked the French mathematician Henri Poincaré. Even more emphatic was his countryman Charles Hermite, who stated that he "turns away with horror and disgust from this miserable sore—a continuous function that has no derivative anywhere."

Weierstrass's construction was very difficult, but a much simpler example was proposed during the twentieth century by the Dutch mathematician B. L. van der Waerden. It begins with the function ϕ_0, whose graph appears in figure 2. The function ϕ_0 is continuous at every point on the number line, periodic (with a period of 1), and bounded, since $0 \le \phi_0(x) \le 1/2$ for all $x \in \mathbf{R}$. Also, the graph of this function is symmetric with respect to every line of the form $x = k/2$ ($k \in \mathbf{Z}$). The function ϕ_0 is not differentiable at all points $x = k/2$. The function

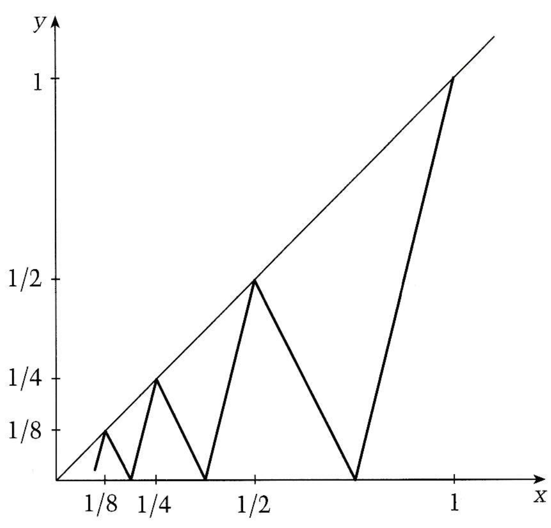


Figure 1

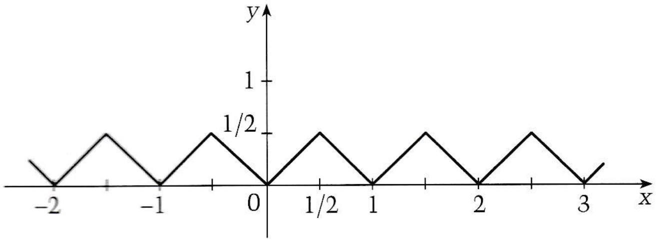


Figure 2

Art by Vasily Vlasov

1
2
3
6
7
8
9
A
B
C
-1
0
1
2
B.B.97

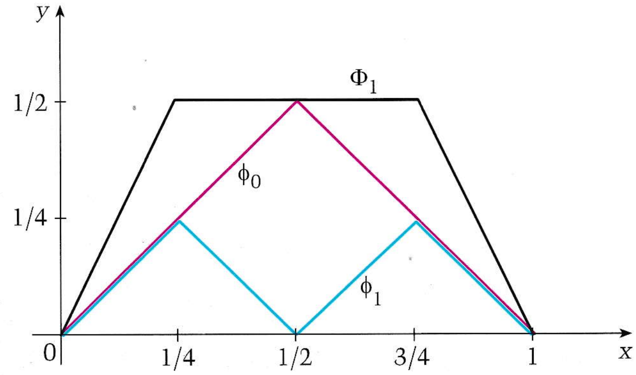


Figure 3

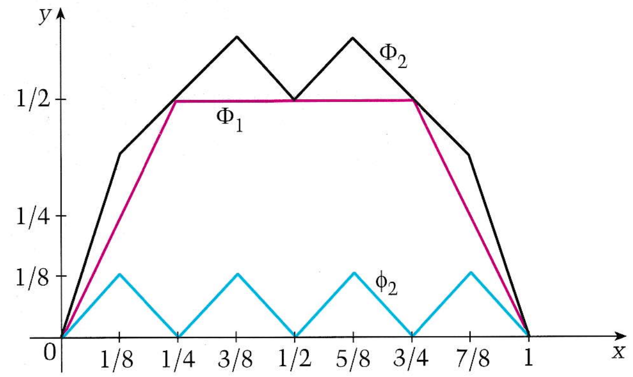


Figure 4

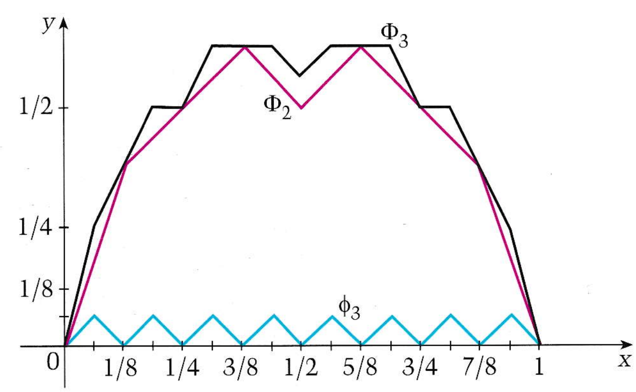


Figure 5

$\phi_1(x) = (1/2)\phi_0(2x)$, whose graph is the blue line in figure 3, does not have a derivative at the points $x = k/4$ ($k \in \mathbf{Z}$), and the function $\phi_2(x) = (1/2^2)\phi_0(2^2x)$, which is graphed in blue in figure 4, at the points $x = k/8$ ($k \in \mathbf{Z}$).

For all $n \in \mathbf{N}$, let $\phi_n(x) = (1/2^n)\phi_0(2^nx)$. (The function ϕ_3 is graphed in blue in figure 5.) The function ϕ_n is continuous at all points on the number line and has no derivative when $x = k/2^n$ ($k \in \mathbf{Z}$). And $0 \le \phi_0(x) \le 1/2^{n+1}$ for all $x \in \mathbf{R}$.

We see that the number of points at which the function ϕ_n has no derivative increases as n grows.

What if we add up all the functions ϕ_n? There is every reason to hope that the sum will be continuous at all points and nondifferentiable at any point of the form $k/2^n$ ($k \in \mathbf{Z}$, $n \in \mathbf{N}$). Unfortunately, the meaning of the sum of an infinite set of terms is not something we want to clear up right now. However, we can consider the sequence of functions

$$\Phi_n(x) = \phi_0(x) + \phi_1(x) + \ldots + \phi_n(x)$$

(thus, $\Phi_{n+1}(x) = \Phi_n(x) + \phi_{n+1}(x)$, see figs. 3–5) and prove that for every x the limit

$$\lim_{n\to\infty} \Phi_n(x)$$

exists. To prove it, we'll have to use Weierstrass's theorem, which states that every monotonic bounded sequence of numbers has a limit.

From the inequality $\phi_{n+1}(x) \ge 0$ we get $\Phi_{n+1}(x) \ge \Phi_n(x)$. Therefore, the sequence $(\Phi_n(x))$ is monotonic. Since

$$\Phi_n(x) = \phi_0(x) + \phi_1(x) + \ldots + \phi_n(x)$$
$$\le 1/2 + 1/4 + \ldots\ 1/2^{n+1} = 1 - 1/2^{n+1} < 1,$$

we see that the sequence $(\Phi_n(x))$ is bounded for every x. Therefore, the limit

$$\lim_{n\to\infty} \Phi_n(x)$$

must exist. Let's denote it by $\Phi(x)$. Thus, we've defined a function $\Phi(x)$ for all real x. It is clearly periodic (with a period of 1), and $0 \le \Phi(x) \le 1$ for all $x \in \mathbf{R}$, and the graph of the function $\Phi(x)$ is symmetric with respect to every line $x = k/2$ (where $k \in \mathbf{Z}$). All these statements can easily be derived from the properties of ϕ_0 and the properties of the limit of sequences. It is impossible to draw the graph of $\Phi(x)$. We can tell from figures 3–5 that the number of points where the functions $\Phi_n(x)$ have "corners," and where no derivative exists, increases as n grows.

Our function is continuous

The function $\Phi(x)$ is continuous at every point in the number line. As far as our intuition is concerned, this statement is clear enough: If the function $\Phi(x)$ had a "gap" at some point x_0, then, for sufficiently large $n \in \mathbf{N}$, the function $\Phi_n(x)$ would have a similar gap at the same point x_0, yet it would be continuous everywhere else in the number line. The following proof clarifies these considerations.

Let's start with a brief study of the influence of the difference $r_n(x) = \Phi(x) - \Phi_n(x)$, where we think of x as a constant and n as a variable:

$$\Phi_{n+m}(x) - \Phi_n(x) = \phi_{n+1}(x) + \phi_{n+2}(x) + \ldots + \phi_{n+m}(x).$$

Clearly,

$$0 \le \Phi_{n+m}(x) - \Phi_n(x) \le \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \ldots + \frac{1}{2^{n+m+1}}$$

$$= \frac{1}{2^{n+2}} \cdot \frac{1 - \dfrac{1}{2^m}}{1 - \dfrac{1}{2}} < \frac{1}{2^{n+1}}.$$

(We've used here the formula for the sum of a geometrical progression.)

Since

$$\lim_{m\to\infty} \Phi_{n+m}(x) = \Phi(x)$$

(this is just a peculiar way of writing the definition of $\Phi(x)$) and

$$\lim_{m\to\infty} \Phi_n(x) = \Phi_n(x),$$

(the variable m in fact does not affect the values of $\Phi_n(x)$, so this sequence is constant), we have

$$0 \le r_n(x) \le \frac{1}{2^{n+1}}.$$

Now we can prove that the function Φ is continuous at an arbitrary point $x_0 \in \mathbf{R}$. Let's take any $\varepsilon > 0$ and look at the absolute value of the difference

$$\begin{aligned}&\left|\Phi(x_0+h)-\Phi(x_0)\right|\\ &\quad=\left|\Phi_n(x_0+h)+r_n(x_0+h)-\Phi_n(x_0)-r_n(x_0)\right|\\ &\quad\le\left|\Phi_n(x_0+h)-\Phi_n(x_0)\right|+\left|r_n(x_0+h)\right|+\left|r_n(x_0)\right|.\end{aligned}$$

However,

$$\left|r_n(x_0)\right| \le \frac{1}{2^{n+1}}$$

and

$$\left|r_n(x_0+h)\right| \le \frac{1}{2^{n+1}}.$$

Therefore,

$$\left|\Phi(x_0+h)-\Phi(x_0)\right| \le \left|\Phi_n(x_0+h)-\Phi_n(x_0)\right|+\frac{1}{2^n}.$$

When n is sufficiently large,

$$\frac{1}{2^n} < \frac{\varepsilon}{2}.$$

The function Φ_n is continuous at x_0, as a sum of a finite number of continuous functions. Thus, taking δ small enough, we can be sure that as soon as $|h| < \delta$,

$$\left|\Phi_n(x_0+h)-\Phi_n(x_0)\right| < \frac{\varepsilon}{2}.$$

So, choosing δ in this way, we see that if $|h| < \delta$,

$$\left|\Phi(x_0+h)-\Phi(x_0)\right| < \varepsilon,$$

which proves that the function Φ is continuous at the point x_0.

The function we've constructed is not differentiable

The function Φ is not differentiable at any point on the number line.

At first this statement might seem obvious: The number of points at which the function Φ_n has no derivative increases as the index n increases, so it's natural to expect that at the limit (that is, for the function Φ), it will fill the entire line with nondifferentiable points. However, this reasoning is naïve, as is clear from the following example:

$$G(x) = \lim_{n\to\infty} G_n(x).$$

While the number of points where $G(x)$ is not differentiable increases with x, it is clear the function $G(x) = 1 - x^2$ is differentiable everywhere.

We will prove that Φ is nowhere differentiable by contradiction. Let's suppose that there is an $x_0 \in \mathbf{R}$, such that $\Phi'(x_0)$ exist.

We "squeeze" x_0 by sequences of deficient and excessive binary approximations:

$$\frac{s_k}{2^{k+1}} \le x_0 < \frac{(s_k+1)}{2^{k+1}} \quad (k = 0, 1, 2, \ldots; s_k \in \mathbf{Z}).$$

Set

$$\alpha_k = \frac{s_k}{2^{k+1}}, \beta_k = \frac{s_k+1}{2^{k+1}}.$$

Then

$$\alpha_k \le x_0 < \beta_k \quad (k = 0, 1, 2, \ldots) \qquad (1)$$

and

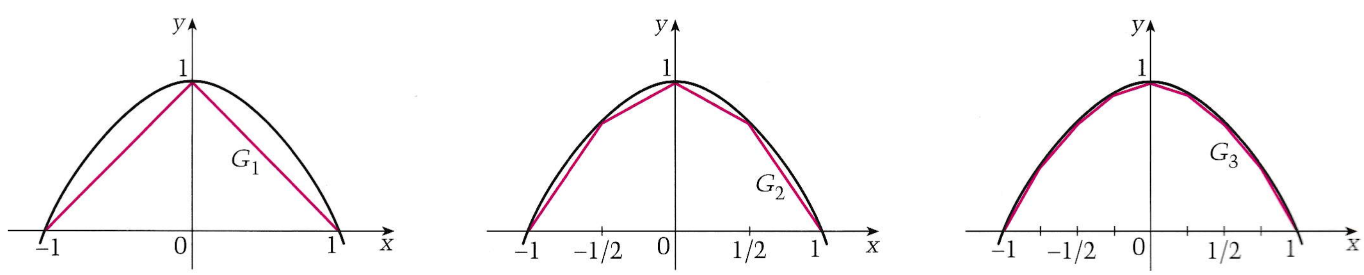


Figure 6

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = x_0. \tag{2}$$

It follows from (1) that

$$0 < \frac{\beta_k - x_0}{\beta_k - \alpha_k} \le 1,\ 0 < \frac{x_0 - \alpha_k}{\beta_k - \alpha_k} \le 1.$$

Now, from these inequalities, from equality (2), and from the easily checked identity

$$\frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k} - \Phi'(x_0)$$
$$= \frac{\beta_k - x_0}{\beta_k - \alpha_k}\left[\frac{\Phi(\beta_k) - \Phi(x_0)}{\beta_k - x_0} - \Phi'(x_0)\right]$$
$$+ \frac{x_0 - \alpha_k}{\beta_k - \alpha_k}\left[\frac{\Phi(x_0) - \Phi(\alpha_k)}{x_0 - \alpha_k} - \Phi'(x_0)\right],$$

we conclude that

$$\lim_{k\to\infty} \frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k} = \Phi'(x_0).$$

Now let's demonstrate that, as a matter of fact,

$$\lim_{k\to\infty} \frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k}$$

does not exist, which will create the desired contradiction.

From the definition of the function Φ_n, we derive

$$\frac{\Phi_n(\beta_k) - \Phi_n(\alpha_k)}{\beta_k - \alpha_k} = \sum_{i=0}^{n} \frac{1}{2^i} \cdot \frac{\phi_0(2^i\beta_k) - \phi_0(2^i\alpha_k)}{\beta_k - \alpha_k}.$$

When $i > k$, it is true that 2^{i-k-1}, and therefore,

$$\phi_0(2^i\alpha_k) = \phi_0(2^{i-k-1}s_k) = 0,$$
$$\phi_0(2^i\beta_k) = \phi_0\left[2^{i-k-1}(s_k + 1)\right] = 0.$$

for such i. Therefore,

$$\frac{\Phi_n(\beta_k) - \Phi_n(\alpha_k)}{\beta_k - \alpha_k} = \sum_{i=0}^{k} \frac{1}{2^i} \cdot \frac{\phi_0(2^i\beta_k) - \phi_0(2^i\alpha_k)}{\beta_k - \alpha_k}.$$

Thus, the ratio

$$\frac{\Phi_n(\beta_k) - \Phi_n(\alpha_k)}{\beta_k - \alpha_k}$$

does not depend on n. Passing to the limit as n approaches infinity, we obtain

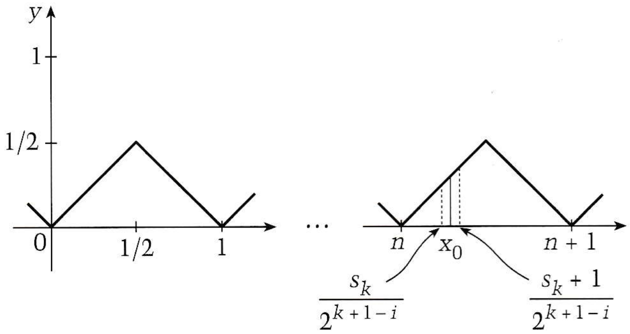


Figure 7

$$\frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k} = \sum_{i=0}^{k} \frac{1}{2^i} \cdot \frac{\phi_0(2^i\beta_k) - \phi_0(2^i\alpha_k)}{\beta_k - \alpha_k}.$$

When $i < k$,

$$\frac{1}{2^i} \cdot \frac{\phi_0(2^i\beta_k) - \phi_0(2^i\alpha_k)}{\beta_k - \alpha_k} = \frac{\phi_0\left(\frac{s_k + 1}{2^{k+1-i}}\right) - \phi_0\left(\frac{s_k}{2^{k+1-i}}\right)}{\frac{1}{2^{k+1-i}}}.$$

But the function ϕ_0 is linear on the segment

$$\left[\frac{s_k}{2^{k+1-i}}, \frac{s_k + 1}{2^{k+1-i}}\right]$$

(see fig. 7). Thus the ratio on the right is simply the slope of the corresponding line. That is, it is either +1 or –1. Therefore,

$$\frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k} = p \cdot 1 + q \cdot (-1),$$

where $p + q = k + 1$.

The parity of the number $p \cdot 1 + q \cdot (-1)$ coincides with the parity of the number

$$p \cdot 1 + q \cdot (-1 + 2) = p \cdot 1 + q \cdot 1 = (p + q) \cdot 1 = (k + 1) \cdot 1 = k + 1.$$

Thus the ratio

$$\frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k}$$

is even when k is odd, and it is odd when k is even. Therefore,

$$\lim_{k\to\infty} \frac{\Phi(\beta_k) - \Phi(\alpha_k)}{\beta_k - \alpha_k}$$

does not exist.

Looking for maxima

Let's prove that the greatest value of

$$M = \max_{\mathbf{R}} \Phi(x) = \max_{[0,1]} \Phi(x)$$

equals 2/3 and determine the set of points where Φ attains this value. We'll slightly violate the strict definitions given in some textbooks by referring to these as the maximum points, and to the greatest value as the maximum of the function Φ. We cannot employ the standard methods here. We cannot look for the points where the derivative vanishes, since the function is not differentiable. But we can write

$$\begin{aligned}\Phi_n(2x) &= \phi_0(2x) + \phi_1(2x) + \ldots + \phi_n(2x)\\ &= \phi_0(2x) + \frac{1}{2}\phi_0(2^2 x) + \ldots + \frac{1}{2^n}\phi_0(2^{n+1}x)\\ &= 2\left[\frac{1}{2}\phi_0(2x) + \frac{1}{2^2}\phi_0(2^2 x) + \ldots + \frac{1}{2^{n+1}}\phi_0(2^{n+1}x)\right]\\ &= 2[\phi_1(x) + \phi_2(x) + \ldots + \phi_{n+1}(x)] = 2[\Phi_{n+1}(x) - \phi_0(x)]\\ &= 2\Phi_{n+1}(x) - 2\phi_0(x).\end{aligned}$$

Passing to the limit when $n \to \infty$, we obtain

$$\Phi(2x) = 2\Phi(x) - 2\phi_0(x). \tag{3}$$

It's worth remarking that there is only one function with nonnegative values that satisfies equation (3). We invite the reader to prove this.

Since 1 is a period of the function Φ, we see that $\tilde{x}$ is a maximum point of Φ if and only if its fractional part $\{\tilde{x}\} \in [0, 1]$ is a maximum point, too.

Let's now consider the set E, consisting of the points of maximum lying on the segment [0, 1]. Using the symmetry of the graph of the function Φ with respect to the line $x = 1/2$ and equation (3), we find $\Phi(1/3)$

$$\Phi\left(\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) = 2\Phi\left(\frac{1}{3}\right) - 2\phi_0\left(\frac{1}{3}\right),$$
$$\Phi\left(\frac{1}{3}\right) = 2\phi_0\left(\frac{1}{3}\right) = \frac{2}{3}.$$

Let's prove that

$$E \subset \left[\frac{1}{3}, \frac{2}{3}\right].$$

Indeed, let $\tilde{x} \in E$ and $\tilde{x} \le 1/2$. Then we have:

$$\Phi(\tilde{x}) = M$$
$$\Phi(2\tilde{x}) \le M$$
$$\frac{2}{3} = \Phi\left(\frac{1}{3}\right) \le M,$$

and from equation (3) we obtain

$$\Phi(2\tilde{x}) = 2\Phi(\tilde{x}) - 2\phi_0(\tilde{x}) = 2\Phi(\tilde{x}) - 2\tilde{x} = 2M - 2\tilde{x} \le M.$$

Thus

$$\tilde{x} \le M/2 \le 1/3.$$

Since the graph of Φ is symmetric with respect to the line $x = 1/2$, we conclude that there are no points of maximum in the interval [2/3, 1], either.

Now let us prove that $M = 2/3$. Once again, we use equation (3):

$$\begin{aligned}\Phi(4x) &= 2\Phi(2x) - 2\phi_0(2x) = 2[2\Phi(x) - 2\phi_0(x)] - 2\phi_0(2x)\\ &= 4\Phi(x) - 4\phi_0(x) - 2\phi_0(2x),\end{aligned}$$
$$\Phi(x) = \phi_0(x) + \frac{1}{2}\phi_0(2x) + \frac{1}{4}\Phi(4x) = \Phi_1(x) + \frac{1}{4}\Phi(4x).$$

The function Φ_1 is constant and equal to 1/2 at the segment [1/4, 3/4], as we can see in figure 3. Therefore, for all x on this segment,

$$\Phi(x) = \frac{1}{2} + \frac{1}{4}\Phi(4x). \tag{4}$$

Since

$$\tilde{x} \in \left[\frac{1}{3}, \frac{2}{3}\right] \subset \left[\frac{1}{4}, \frac{3}{4}\right],$$

we can substitute its value into equation (4):

$$M = \Phi(\tilde{x}) = \frac{1}{2} + \frac{1}{4}\Phi(4\tilde{x}) \le \frac{1}{2} + \frac{1}{4}M.$$

$$M \le \frac{2}{3}.$$

Recalling that

$$M \ge \Phi\left(\frac{1}{3}\right) = \frac{2}{3},$$

we conclude that

$$\Phi\left(\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) = \frac{2}{3}.$$

Besides this, $\Phi(1/3) = \Phi(2/3) = 2/3$.

To continue, we need the following statement: If $\tilde{x}$ is a maximum point, then $4\tilde{x}$ is also a maximum point, and if $4\tilde{x}$ is a maximum point and

$$\{\tilde{x}\} \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

then x is also a maximum point. To prove this, let's note that equation (4) is true for all x such that

$$\{x\} \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

(We invite the reader to check this.) Let $\tilde{x}$ be a maximum point. Then

$$\{\tilde{x}\} \in \left[\frac{1}{3}, \frac{2}{3}\right] \subset \left[\frac{1}{4}, \frac{3}{4}\right].$$

Therefore,

$$\frac{2}{3} = M = \Phi(\tilde{x}) = \frac{1}{2} + \frac{1}{4}\Phi(4\tilde{x}),$$
$$\Phi(4\tilde{x}) = \frac{2}{3} = M.$$

Now suppose that $4\tilde{x}$ is a maximum point and

$$\{\tilde{x}\} \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Then

$$\Phi(4\tilde{x}) = M = \frac{2}{3},$$

and

$$\Phi(\tilde{x}) = \frac{1}{2} + \frac{1}{4}\Phi(4\tilde{x}) = \frac{2}{3} = M.$$

The set of maximum points

Let's now investigate the structure of the set E.

It seems very likely that if Hermite had understood just how complex van der Waerden's function was, he would have turned away with even greater disgust. On the other hand, it seems possible that the subtle and beautiful structure of this set might have inspired Hermite to study nondifferentiable functions. Who's to say?

We already know two points of the set E: 1/3 and 2/3. We have also demonstrated that $E \subset [1/3, 2/3]$. Further, we can show that the set E is *closed.* This means that if a sequence of points of E converges to a limit, then this limit is also in E. To prove it, suppose that $x_n \in E(n = 1, 2, 3 \ldots)$. This means that $\Phi(x_n) = M$. Let

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Then, since Φ is continuous at $\bar{x}$, it follows that

$$\Phi(\bar{x}) = \Phi\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \Phi(x_n) = M.$$

This means that $\bar{x} \in$ E, so E is closed.

From the statements proved at the end of the previous section, and the fact that 1 is a period of Φ, it follows that if x is a point of E, then both

$$1/4 + x/4 = (1 + x)/4$$

and

$$2/4 + x/4 = (2 + x)/4$$

also belong to E. Thus, starting from the points 1/3 and 2/3, we can construct many other points in the set E:

$$E: \frac{1}{4} + \frac{1}{4}\cdot\frac{1}{3},\ \frac{2}{4} + \frac{1}{4}\cdot\frac{1}{3},\ \frac{1}{4} + \frac{1}{4}\cdot\frac{2}{3},$$

and so on.

It's easy to see that all the numbers of the form

$$x = \frac{\alpha_1}{4} + \frac{\alpha_2}{4^2} + \ldots + \frac{\alpha_n}{4^n} + \frac{1}{4^n}\cdot\frac{\alpha_{n+1}}{3},$$

where every $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$ is either one or two, are elements of the set E. Since

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}\ldots,$$

we can represent such numbers x as

$$x = \frac{\alpha_1}{4} + \frac{\alpha_2}{4^2} + \ldots + \frac{\alpha_n}{4^n} + \frac{\alpha_{n+1}}{4^{n+1}} + \frac{\alpha_{n+1}}{4^{n+2}} + \ldots,$$

or, drawing an analogy with the usual decimal fractions, in the form of an infinite periodic "quaternary" fraction with a one-digit-long period:

$$x = 0.\alpha_1\alpha_2 \ldots \alpha_n\alpha_{n+1}\alpha_{n+1} \ldots = 0..\alpha_1\alpha_2 \ldots \alpha_n(a_{n+1}).$$

Since the set E is closed, any number of the form

	$\frac{1}{12}$	$= 0, 0\,1\,1\,1\,1\,1\,1 \ldots$
	$\frac{1}{6} = \frac{2}{12}$	$= 0, 0\,2\,2\,2\,2\,2\,2 \ldots$
	$\frac{1}{4} = \frac{3}{12}$	$= 0, 1\,0\,0\,0\,0\,0\,0 \ldots$
V	$\frac{1}{3} = \frac{4}{12}$	$= 0, 1\,1\,1\,1\,1\,1\,1 \ldots$
V	$\frac{5}{12}$	$= 0, 1\,2\,2\,2\,2\,2\,2 \ldots$
	$\frac{1}{2} = \frac{6}{12}$	$= 0, 2\,0\,0\,0\,0\,0\,0 \ldots$
V	$\frac{7}{12}$	$= 0, 2\,1\,1\,1\,1\,1\,1 \ldots$
V	$\frac{2}{3} = \frac{8}{12}$	$= 0, 2\,2\,2\,2\,2\,2\,2 \ldots$
	$\frac{3}{4} = \frac{9}{12}$	$= 0, 3\,0\,0\,0\,0\,0\,0 \ldots$
	$\frac{5}{6} = \frac{10}{12}$	$= 0, 3\,1\,1\,1\,1\,1\,1 \ldots$
	$\frac{11}{12}$	$= 0, 3\,2\,2\,2\,2\,2\,2 \ldots$

Figure 8

$$x = \frac{\alpha_1}{4} + \frac{\alpha_2}{4^2} + \ldots + \frac{\alpha_n}{4^n} + \frac{\alpha_{n+1}}{4^{n+1}} + \frac{\alpha_{n+2}}{4^{n+2}} + \ldots,$$

where each α_i is either one or two belongs to E. In fact, for every x of this sort, we can find a sequence (x_n) of points from E converging to x; for instance, if

$$x_n = \frac{\alpha_1}{4} + \frac{\alpha_2}{4^2} + \ldots + \frac{\alpha_n}{4^n} + \frac{1}{4^n} \cdot \frac{1}{3},$$

then

$$\lim_{n\to\infty} x_n = x \left(\text{for } 0 \le x - x_n \le \frac{1}{4^{n+1}} \right).$$

Thus, the set E consists of all numbers that can be represented as the infinite quaternary fraction $0.\alpha_1\alpha_2 \ldots \alpha_n \ldots$, where every α_1 is either one or two (in fig. 8, we see the quaternary decompositions of the number 1/12, 2/12, ... , 11/12).

Now let's prove that E contains no other numbers. Suppose that the quaternary notation of a number $x \in E$ contains 0 somewhere after the decimal (or quaternary) point; that is, $x = 0.\alpha_1\alpha_2 \ldots \alpha_n 0\alpha_{n+2} \ldots$; then

$$\{4^n x\} = 0.0\alpha_{n+2} \ldots \in E.$$

But

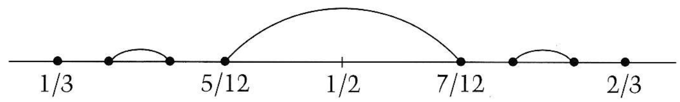


Figure 9

$$0.0\alpha_{n+2}\alpha_{n+3} \ldots < 0.033\ldots = 3\left(\frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) = \frac{1}{4},$$

and thus $\{4^n x\}$ lies outside the segment [1/3, 2/3] that contains E.

On the other hand, if there is a numeral 3 in the quaternary notation of a number $x \in$ E (that is, $x = 0.\alpha_1\alpha_2 \ldots \alpha_n 3\alpha_{n+2} \ldots$,) then

$$\{4^n x\} = 0.3\alpha_{n+2} \ldots \in E,$$

which is impossible because $0.3\alpha_{n+2} \ldots > 0.3 = 3/4$.

We can give a geometric construction of the set E as follows. Divide the segment [1/3, 2/3] into four equal parts and delete the interior points of the two parts in the middle and the midpoint of the segment (fig. 9). Then, again divide each of the remaining segments into four equal parts and delete the interior points of the two parts in the middle and the midpoints, and so on. The set remaining after all such deletions consists precisely of all the points in whose quaternary notation we find only the numerals 1 and 2 (try to prove this yourself), in other words, the set E.

Hydroparadoxes

When fluids forsake model behavior

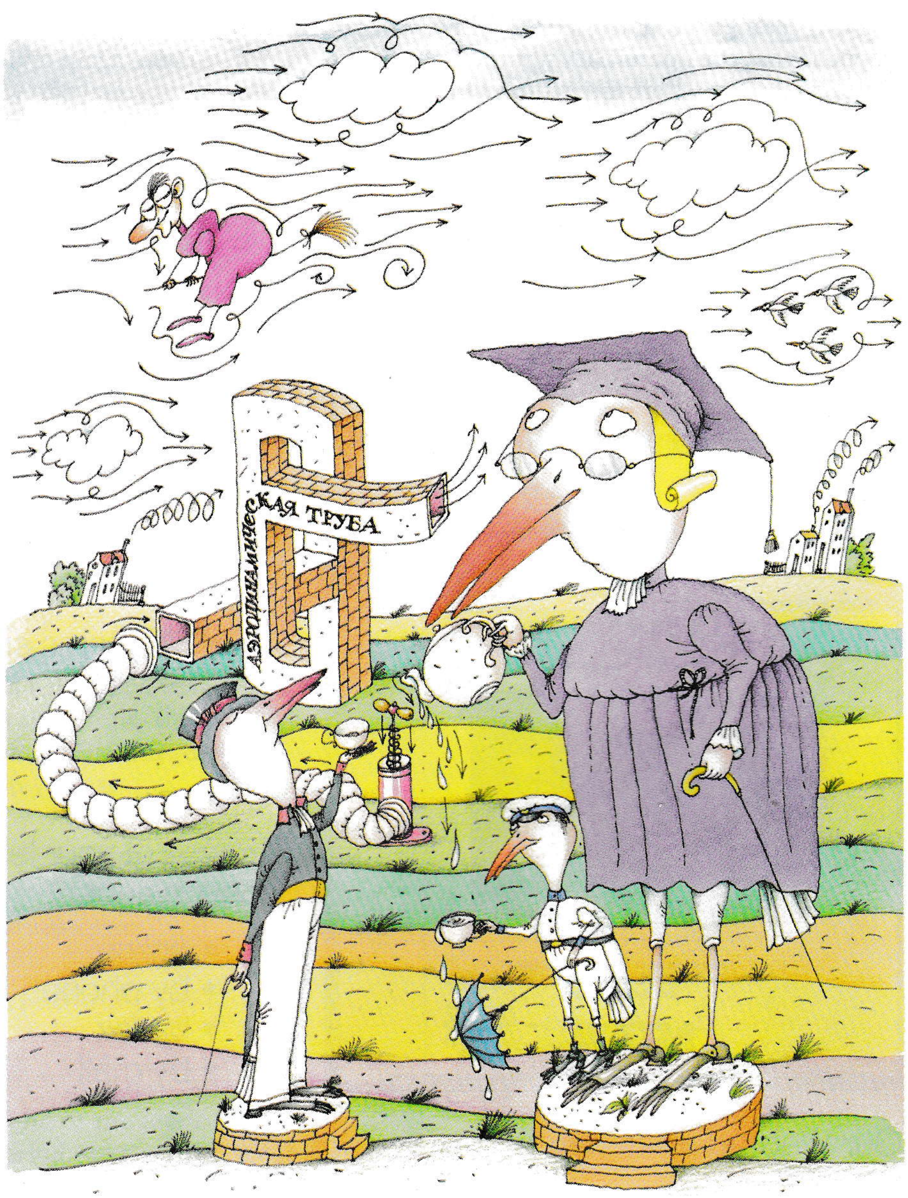
by S. Betyaev

PARADOXES ARE SURPRISING statements that drastically contradict common sense. The practical importance of paradoxes, "the motors of progress," consists of opening up new vistas of an old theory or paving the way for a more perfect theory (and sometimes even for a new branch of science).

The theory of relativity incorporated the paradox of a velocity limit for information transmission into modern science, and quantum mechanics did the same with the paradox of signal discontinuity in a microcosm. Paradoxes gave rise to the fields of elementary particle physics and cosmology and stimulated the development of modern mathematics. The most fundamental paradoxes, cornerstones of science, are formulated and interpreted by geniuses. This was evident to the giant of Russian poetry, Alexander Pushkin:

Oh, the amazing discoveries
That come from the spirit of learning
And experience, son of errors,
And genius, friend of paradoxes,
And chance, the inventor-god.

Science clearly distinguishes between experimental facts and theoretical generalizations based on mathematical simulations. Scien-



Art by V. Ivanyuk

tific paradoxes can be subdivided into three types.

The first type is a contradiction between a generally accepted theoretical concept and a new theoretical inference. These simplest of paradoxes (the "theory-to-theory" type) result from an improvement of a mathematical model or calculational method.

The second type, contradictions between general experience and a new, experimentally based notion ("test-to-test" paradoxes) deserve more detailed consideration, which we'll give now, putting aside for a while the definition and analysis of the third type of paradox.

The paradoxes of symmetry

Do symmetrical causes always result in symmetrical effects? This is not true on the microscopic scale (see Richard Feynman's book *Theory of Fundamental Processes*). However, it isn't always true on the macroscopic scale either. For example, the streamlined flow of a fluid around a symmetrical body is frequently asymmetrical. This is the hydrodynamic symmetry paradox.

Figure 1 shows the symmetrical water flow around a circular cylinder. The trajectories of fluid particles are made visible with the help of aluminum powder. In the figure, the water flows from left to right. The upper and lower halves of the figure are symmetrical: One is the mirror reflection of the other. Moreover, the flow around the front and rear of the cylinder is also almost symmetrical.

Figure 2 shows the flow around the same cylinder under different conditions. The vertical symmetry still exists, but the symmetry between the left and right sides of the flow is broken: There are two closed zones with counter-rotating fluid particles behind the cylinder.

Finally, figure 3 shows the flow around the cylinder when both types of symmetries are disturbed. In this figure, visualization was achieved with air bubbles dispersed in the water.

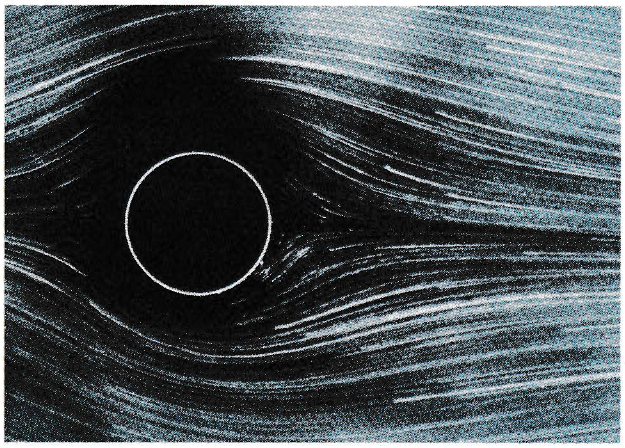


Figure 1

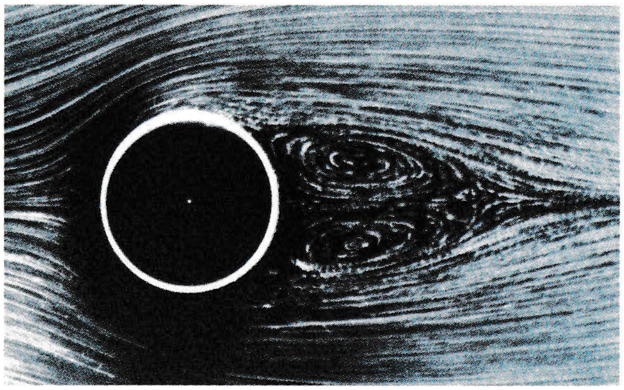


Figure 2

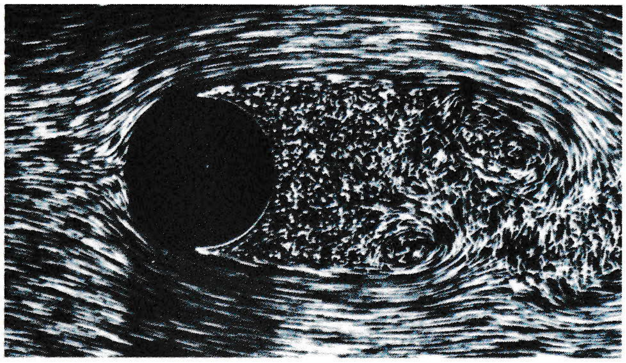


Figure 3

Why does the flow lose symmetry? Right now we can't give a complete answer to this question. Let's try to deal with simpler ones first. For example, what are the differences in flow around the cylinder in the three cases? Each case has different values of the ratio of the forces affecting the fluid particles: the ram force and the viscous drag force. This ratio is characterized by the dimensionless Reynolds (*Re*) number. At small *Re* values the viscous forces are large, and a body moves in a fluid like a pellet in honey (the *Re* numbers for the cases in figs. 1 and 2 are 1.5 and 26). At large *Re* numbers the viscous forces are small, and the flow becomes unstable and even turbulent (*Re* = 2000 in fig. 3).

The change of symmetry type and its abrupt destruction is a fundamental feature of modern hydrodynamics. In real-life conditions, absolute symmetry is impossible, and there is always some asymmetry in a flow. Therefore, even if symmetrical causes result in symmetrical effects, near-symmetrical causes lead to quite asymmetrical consequences. This is one explanation of the symmetry paradoxes in hydrodynamics.

The Eiffel paradox

Another paradox, which is physically close to paradoxes of symmetry, was discovered in 1912 by the French engineer Alexandre Gustave Eiffel (1832–1923). In his later years, he was interested in hydrodynamics, especially the effects of wind loads on architectural structures. When experimenting with balls inside a self-made wind tunnel, he discovered the paradox that was later named for him: When the flow around a ball reached a "critical" *Re* number of about 150,000, the air resistance dropped drastically (by 4 to 5 times) when the velocity increased. This observation runs counter to common physical experience and intuition.

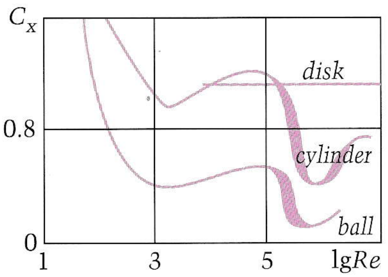


Figure 4

Let's write the aerodynamic drag force as

$$F = C_d(Re)\frac{\rho u_\infty^2}{2}\frac{\pi l^2}{4},$$

where ρ is density, u_∞ is the velocity of the undisturbed free stream, and l is a characteristic size of the streamlined body. The proportionality $F \propto \rho u_\infty^2 l^2$ can be easily obtained by dimensional analysis.[1] The numerical coefficients 1/2, $\pi/4$, and C_d are written for the sake of convenience. The dimensionless drag coefficient C_d can be determined experimentally. Usually it decreases with Re—that is, with a decrease in viscous friction.

The Eiffel paradox does not occur only in flows around spherical bodies. Figure 4 shows the experimentally obtained function $C_d(Re)$ for a ball, cylinder, and disk of the same diameter l. For the ball and cylinder, there is a large scatter of experimental data at the region of drastic change of C_d, shown by the wide parts of the curves. On the other hand, the drag coefficient for the disk is virtually constant! This result can be generalized: The Eiffel paradox does not hold for bodies with sharp edges.

The Eiffel paradox is explained by a transition near the critical Re number from smooth (laminar) flow to turbulent (stochastic) flow. Thus, a small change in the Re number results in a drastic rearrangement of the flow.

Small variations in parameters that lead to radical changes in flow are typical in hydrodynamics. This effect is responsible for many instances of surprisingly large scatter in experimental data obtained at seemingly identical conditions. Therefore, in simulating the flows around bodies in wind tunnels, one should take into account effects of the tunnel's walls, the apparatus supporting the body, the heterogeneity of the free stream, and the physical and chemical properties of the body's surface (such as roughness, wettability, and thermal conductivity). In practice, it is very difficult (if not impossible) to do all this.

The Dubois paradox

One of the founders of experimental hydrodynamics was the French military engineer P. Dubois (1734–1809). Dubois showed that within a certain range of Re numbers, the resistance force affecting a body resting in a tunnel with water running at a certain speed is less than the resistance force affecting a body moving with the same speed in motionless water. According to Galileo's relativity principle, the result of this experiment should not depend on whether a body moves in resting fluid or whether fluid flows around a resting body.

How can we explain the Dubois paradox? Using the factors we've just discussed, of course. The flow in an experimental basin or aerodynamic tunnel is less uniform than in a tranquil sea or atmosphere, so the transition to turbulent flow takes place at smaller (subcritical) Re numbers. This transition manifests itself in a narrowing of the flow "tail" and by a decrease in the resistance.

The Dubois paradox is still unsolved. The difference between experimental data obtained in a tunnel and data obtained in real flight is the number one problem in hydrodynamics and aerodynamics.

Look at a helicopter resting on the ground: Its blades curve downward by almost 1 meter. However, they are straight during flight. Similarly, the wings of an airplane change shape because of aerodynamic forces in real flight. This change is not very large, but it is large enough to nullify the results of scrupulous (and expensive!) experiments. Therefore, to explain the discrepancies between wind tunnel and full-scale experiments, we need to take into account the elastic properties of materials subjected to the action of hydrodynamic forces.

The Euler-d'Alembert paradox

Now we're ready to learn about the third type of paradox. In addition to the "theory-theory" and "test-test" paradoxes, there are "test-theory" paradoxes. These are characterized by a drastic contradiction between theoretical results and our experience, intuition, or common sense.

The most famous hydrodynamic paradox of this type is the Euler-d'Alembert paradox. In 1742 Euler calculated the drag force affecting a cylinder moving in a frictionless fluid, and he obtained a paradoxical result: There was no resistance at all! Seven years later, the great French mechanician Jean d'Alembert calculated the flow around an arbitrary body of a finite volume, and came to the same striking result of zero resistance.

This calculation directly challenges common sense. As everyone knows from experience, it is necessary to supply force to keep a body moving in a fluid. This is why aircraft, ships, and submarines have motors and sails. D'Alembert could not explain the paradox, and he bitterly remarked that zero resistance was the only paradox left for future geometers to solve.

The future geometers (hydrodynamicists and mathematicians) had inherited a hard nut to crack. Before cracking it, let's explain the geometric character of the paradox. The flow studied by Euler and d'Alembert is symmetrical—that is, its right part is a mirror reflec-

[1]Y. Bruk and A. Stasenko, "The Power of Dimensional Thinking," May/June 1992, pp. 34–39.

tion of its left part (similar to fig. 1). Thus, the projection of the momentum of the circumflow jet onto the free stream direction is constant. It is the same at cross-sections on the left and right of the body. In accordance with conservation of momentum, no drag forces affect either the jet or the streamlined body. The point is that the mathematical models used by Euler and d'Alembert were oversimplified. The real flow is not symmetrical and looks similar to the flows in figures 2 and 3.

As you may have guessed, viscous friction disturbs the symmetry of a flow. This friction is responsible for the tail formation behind a moving body. So do we understand the Euler-d'Alembert enigma now? Not quite. The complete explanation is far more complicated. Look again at figure 4. Even at the highest attainable *Re* numbers, when the viscous forces are negligible, the drag coefficient is not zero. Therefore, asymmetry and fluid resistance can arise even in nonviscous fluids.

Such a fluid was "constructed" by the German physicist Hermann Helmholtz (1821–1894), who closed the book on the Euler-d'Alembert paradox. The flow around a cylinder according to the Helmholtz model is illustrated in figure 5, which shows a region of stationary fluid behind the cylinder. Thus, the true mathematical model should take into account both friction and separation of the flow from a body.

Many other paradoxes besides the Euler-d'Alembert paradox have originated from oversimplified mathematical models. For example, continuous flow around the sharp edge of a plate (fig. 6a) results in the "infinity paradox": The velocity of the flow increases to infinity near the edge. Additionally, some kind of centripetal force is needed to turn the flow through 180°. According to Newton's third law, the plate will be affected by the "leading edge" force, which is equal in value to the centripetal one. Where is this force applied? To the very edge of the plate—a point without size! In reality, the flow around an edge separates, and it is characterized by the line of separation of the tangent velocity (shown in red in fig. 6b). Thus, the velocity at an edge is actually finite.

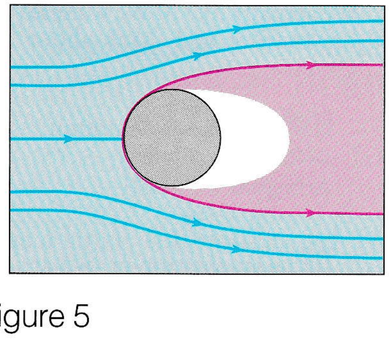


Figure 5

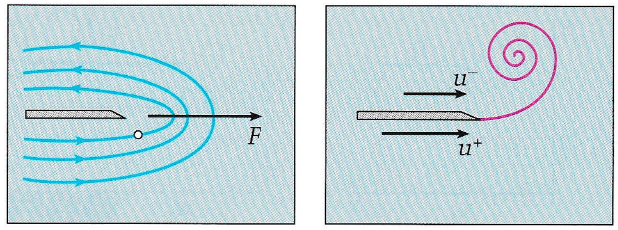


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Figure 6

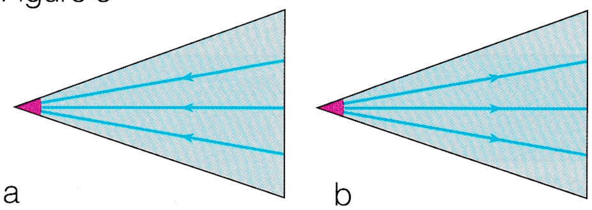

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Figure 7

Correctness of mathematical models

The development of a consistent mathematical model which adequately describes a physical process is a very complicated matter. In most applications such a model is just a dream. The mathematician D. Birkhoff humorously proposed subdividing hydrodynamicists into experimentalists, who watch phenomena that cannot be described, and theoreticians, who describe events that can't be seen.

To avoid being trapped in a paradox, a mathematical model shouldn't be oversimplified, and it should take into account the factors that result in a paradox when they are neglected. From a physical viewpoint, this is a natural requirement. However, mathematicians consider problems on a much stricter basis—a problem should also be formulated *correctly.* Correctness implies three requirements for a mathematical model: the existence of a solution, uniqueness, and stability.

Clearly, lack of a solution is a consequence of model oversimplification. For example, a solution describing a radial flow converging in the vertex of an angle (fig. 7a) exists at any *Re* number. On the contrary, a similar solution for a radial flow diverging from the vertex of an angle (fig. 7b) exists only at small *Re* numbers, which are less than a certain critical value Re^*. When $Re > Re^*$, a solution doesn't exist. However, experimentally we can see nonstationary, separated flow at sufficiently large *Re* numbers. We've come upon the paradox of the lack of a solution in describing the flow radiating from the vertex of an angle.

Another problem arises when there are several solutions. For example, there are two roots of a quadratic equation. Which should be used? Let's consider the possible variants from an experimental point of view. If no possible root is realized in an experiment, it simply means that the chosen mathematical model and corresponding quadratic equation are not correct. If only one root describes the experimental event, it usually means that this root is stable, while the other is not. Finally, both roots can be realized in an experiment.

When at certain values of the parameter ($x = x_0$) one solution

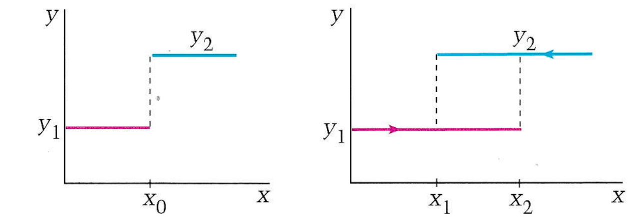

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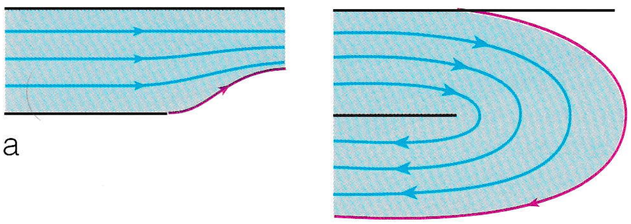
Figure 8



a b

Figure 9

$(y = y_1)$ is substituted by another $(y = y_2)$, *bifurcation* of the solution takes place (fig. 8a). When both solutions exist in some range of the parameter $(x_1 \le x \le x_2)$, this is referred to as *hysteresis:* One solution $(y = y_1)$ is realized when parameter (x) is increased from some value $x < x_1$ (the direct pass), while another solution $(y = y_2)$ takes place when the parameter is decreased from $x > x_2$ (the reverse pass). Thus, the choice of the branch of the hysteretic loop depends on the history of the process. In aerodynamics, the hysteretic modes of flow are observed, for example, near the value of the attack angle corresponding to the maximum value of the lift force.

The paradox of multiple solutions was solved by scientists at the dawn of aviation. In 1910 at an aeronautical show near Paris, Henri-Marie Coanda, a young Romanian engineer, launched an airplane that was a prototype of modern jet planes. It had jets of fire emerging from lateral thrust nozzles. After a successful flight, and disembarking with only minor bruises, the young aviation designer and pilot received enthusiastic congratulations. "Mon cher, you advanced our epoch by 30, no, by 50 years," exclaimed Eiffel. But the pilot was distracted by something else. During takeoff he had observed strange behavior of the jet plume. Instead of being reflected from the special metal plates protecting the plywood fuselage, the jet plume followed their surfaces and even turned backward.

However, at that time this phenomenon, which became known as the Coanda effect[2], did not attract scientific attention. In the next 25 years, Coanda, now a famous aircraft designer, conducted experiments searching for possible practical applications of this phenomenon. Now the Coanda effect is used in designing hovercrafts and hydrofoils, increasing the propulsive force of jet nozzles, braking aircraft upon landing, and muffling jet engine noise.

We encounter the Coanda effect in everyday life—for instance, in the stream of water that "sticks" to the tea kettle's spout and doesn't make it into the cup. This turning of the stream and its following a solid surface is jokingly called the "Kettle effect." Figure 9a shows a stream outflow without turning, and figure 9b shows the outflow with the turning of the stream. These are both mathematical solutions. So, has the Coanda enigma been solved? Unfortunately, no. We don't know the conditions with respect to either mode of flow.

We mentioned another criterion for the correctness of mathematical models—the stability of their solutions. Stochastic, unstable, and relatively small perturbations cannot be analyzed using classic mathematical tools. We cannot define an individual stochastic trajectory, just as we cannot say whether it will be raining a month from now. At best, we can hope to obtain a general description. There is a rhymed illustration to this remark by the Russian poet and philosopher Vladimir Solovyov:

Nature does not allow
The veil to be drawn, her beauty shown;
With instruments you will not find
What your soul had not already
guessed at on its own.

The paradox of instability arises when the flow around a body kept under constant external conditions nevertheless varies with time. Figure 3 shows an example of a nonstationary flow. A flow becomes nonstationary when the *Re* number surpasses a certain critical value. Although it is known for sure that instability is caused by the nonstationary character of flow separation from a body, there is still a long way to the final solution of the instability paradox.

[2]Jet Raskin, "Foiled by the Coanda Effect," January/February 1994, pp. 5–11.

Quantum articles on fluid mechanics:

I. Vorobyov, "Canopies and Bottom-flowing Streams," July/August 1995, pp. 45–47.

H. D. Schreiber, "A Viscous River Runs Through It," November/December 1995, pp. 43–46.

A. Mitrofanov, "Against the Current," May/June 1996, pp. 22–29.

A. Stasenko, "Whirlwinds over the Runway," July/August 1997, pp. 36–39.

HOW DO YOU FIGURE?

Challenges

Math

M236

Is that right? Consider two circles that intersect at points A and B. Let a line through B meet the circles at points K and M (see fig. 1). Let E and F be the midpoints of arcs AK and AM, respectively (the arcs that don't contain B), and let L be the midpoint of segment KM. Prove that $\angle ELF$ is a right angle.

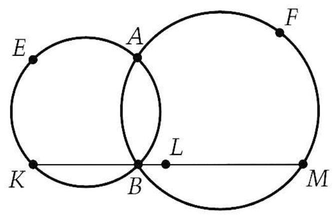


Figure 1

M237

Complementary rhombus angles. Consider the rhombus $ABCD$. Find the locus of points M such that $\angle AMB + \angle CMD = 180°$.

M238

It's only natural. For what natural number n is the expression $\log_{2n-1}(n^2 + 2)$ rational?

M239

Systematic curve. Let the numbers a and b be such that the system

$$\begin{cases} y = x^2 + a \\ x = y^2 + b \end{cases}$$

has a single solution (x_0, y_0). Draw the curve consisting of all possible positions of the point (x_0, y_0).

M240

Run for the border. A math student is lost in a vast forest whose border is a line. (Imagine that the forest covers a half-plane.) The student knows that she is no more than 2 miles from the border. Propose a route for her such that she would come out of the forest having walked no more than 13 miles. (Of course, the student doesn't know where the border lies, and no matter how close to it she passes, she cannot see it. We say that the student comes out of the forest when she reaches the border.)

Physics

P236

Speed of lunar rover. Estimate the maximum speed of a camera-equipped, self-propelled vehicle moving on the Moon's surface and controlled from Earth. (V. Shelest)

P237

Galaxy mass. According to scientist's visual estimates, a mass of $M_1 = 1.5 \cdot 10^{11} M_0$ is concentrated within the limit $R = 3 \cdot 10^9 R_0$ from the Galaxy's center, where M_0 is the mass of the Sun and R_0 is the radius of Earth's orbit. However, the period of revolution of the stars located at this distance from the Galaxy's center is $3.75 \cdot 10^8$ years, which corresponds to a larger mass.

Find the "hidden mass" of the Galaxy—that is, the mass of invisible objects inside the sphere of radius R. When calculating the stellar motion, assume the mass M_1 to be concentrated at the center of the Galaxy. (V. Belonuchkin)

P238

Refrigerator in a room. A refrigerator maintains an interior temperature of −12°C. If the temperature of the room is 25°C, the regrigerator's motor turns on every 8 minutes. After $\tau_1 = 5$ minutes, the motor turns off. Considering the refrigerator to be an ideal heat engine working in reverse, find how often and how long the refrigerator's motor would turn on at a room temperature of 15°C. At what maximum room temperature could the refrigerator maintain the given internal temperature? (A. Zilberman)

P239

Railroad resistance. Long, bare conducting rods made of copper are randomly strewn on the rails of a toy railroad. Find the resulting resistance between the rails if the width of the railroad is $l = 5$ cm, the diameter of the rods is $d = 0.2$ mm, the length of a rod is $h = 30$ cm, the number of rods is $N = 100$, and the resistivity of copper is $\rho = 1.7 \cdot 10^{-8}\ \Omega \cdot \text{m}$. (A. Zilberman).

P240

Speeding UFO. A UFO flies horizontally above Earth at a very high speed v. What speed will a ground observer measure when the direction to the UFO makes an angle ϕ with the vertical (fig. 2)? (S. Krotov)

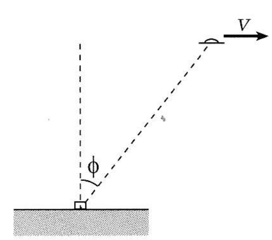


Figure 2

ANSWERS, HINTS & SOLUTIONS ON PAGE 50

The legacy of al-Khwarizmi

by Z. D. Usmanov and I. Hodjiev

THE WORDS *ALGEBRA* AND *algorithm* are familiar to readers of *Quantum*, but do you know their origins? In fact, both these words are associated with one scientist: Muhammad ibn Musa al-Khwarizmi, the outstanding Arabian mathematician and astronomer born about A.D. 780 in Hiva (modern Uzbekistan).

Al-Khwarizmi's most productive period was around the year 825, when he worked in Baghdad. The reigning caliph al-Ma'mun patronized astronomy and mathematics. He built the "House of Wisdom" in Baghdad, complete with its own library and observatory. This improvised academy of sciences concentrated nearly all the best Arabian scientists of the time.

Muhammad ibn Musa al-Khwarizmi was one of the scientists commissioned by the caliph to translate the treatises of Greek mathematicians, calculate the length of a meridian, and do other research. Al-Khwarizmi wrote many books on mathematics and astronomy.

In his work on arithmetic, al-Khwarizmi explains the Indian system of notation for numbers and details the rules of written calculations in the digital system of notation. The original Arabic version of this book is lost, but a Latin translation made in the twelfth century survives. This book was one of the main sources that brought the digital system of notation to western Europe. The title of this translation is *Algoritmi de numero Indorum* ("Al-Khwarizmi Concerning the Hindu Art of Reckoning"). Thus the term *algorithm*, the latinized form of al-Khwarizmi's name, was introduced to the mathematical lexicon.

At first *algorithm* meant just the positional decimal numeration. Later this name was given to all works that spread the Indian numerical system in Europe, and, finally, it came to mean the system itself. Today the word *algorithm* denotes a finite set of rules that allows one to solve a problem in a purely mechanical way, as in a computer program.

In his *Algoritmi de numero Indorum*, al-Khwarizmi explains how to write numbers and perform the four basic arithmetic operations with integers and simple fractions. Still, he considers the doubling of a number and division by two to be separate operations. All the reasoning in the book is carried out only in words; there is not a single formula in it, and all the examples are explained by numbers denoted by words or Roman numerals rather than by the usual decimal figures. Al-Khwarizmi does not explain how to carry out subtraction when a figure in the subtrahend is greater than the corresponding figure in the minuend.

Another famous book of al-Khwarizmi is *Kitab al-Jabr wa al-muqabalah*, which means "The Book of Integration and Equation." The Latin translation of this book became popular in western Europe, and thus the word *algebra* (from *al-Jabr* in the title) was used to name a whole branch of mathematics—the branch concerned, till the middle of the nineteenth century, almost exclusively with equations.

In effect, the word *al-Jabr*, according to al-Khwarizmi, means the operation that allows one to move the terms from one side of an equality to the other so that at the end both parts contain only positive terms. The word *al-muqabalah* means the following operation of collecting similar terms so that only one positive term of each degree remains on one side. Thus, for instance, using the first operation, one turns the equation

$$3x^2 - 5x + 6 = x^2 + 7x + 2$$

into

$$3x^2 + 6 = x^2 + 12x + 2,$$

and then, through the second operation, the equation becomes

$$2x^2 + 4 = 12x.$$

So, the whole science of equations ("algebra"), the symbols developed for this purpose, and the whole theory of abstract operations that grew from these investigations bear the name of the operation *al-Jabr*. Unlike ancient Greeks and Arabs and their successors in Europe, we don't demand today that both parts of an equation contain only positive terms.

In his book *Kitab al-Jabr wa al-muqabalah*, al-Khwarizmi considers linear and quadratic equations, but he doesn't use any algebraic formulas. Everything is explained with words. Thus, he calls the variable of an equation the *root*, and he calls its square simply the *square*. Six kinds of equations are considered:

- "squares equal to roots,"
- "squares equal to a number,"
- "roots equal to a number,"
- "squares and roots equal to a number,"
- "squares and a number equal to roots," and
- "roots and a number equal to squares."

The translation of this set into modern algebraic language can be written as

$$ax^2 = bx,$$
$$ax^2 = b,$$
$$ax = b,$$
$$ax^2 + bx = c,$$
$$ax^2 + b = cx,$$
$$ax + b = cx^2,$$

where $a, b, c > 0$.

Al-Khwarizmi investigates the last three types of equations using the following examples:

$$x^2 + 10x = 39,$$
$$x^2 + 21 = 10x,$$
$$3x + 4 = x^2$$

(later these equations appeared in many books on algebra). He describes the equation $x^2 + 10x = 39$ as "a square and ten of its roots make thirty-nine dirhams, ... and this means that if one adds to a square something equal to ten of its roots, one gets thirty-nine" (a dirham is a silver coin; here it is used to denote the constant term). And the procedure that allows one to calculate the root of this equation, the formula

$$x = \sqrt{\left(\frac{10}{2}\right)^2 + 39} - \frac{10}{2} = 3,$$

al-Khwarizmi explains as follows: "Take one-half of the number of roots in this problem, the result is five; multiply it by its equal, you get twenty-five. Add it to thirty-nine, you get sixty-four. Extract the root from this number, you get eight; subtract one half of the number of roots, which is five, and three will remain. This is the root of the square that you sought, and the square is nine."

Al-Khwarizmi proposes two geometric ways of solving this equation (figs. 1, 2). In figure 1 four rectangles with a side equal to 10/4 are drawn on the sides of the square with side x, after which the corners of the figure obtained are filled with squares with sides 10/4. The area of the square with side $x + 10/2$ that appears in this way will be equal to a known number: $39 + (10/2)^2$. And in figure 2 we see the square with the same area 64, composed of the squares with areas x^2 and 25, which corresponds to another possible way of rewriting the equation $x^2 + 10x = 39$, which is

$$x^2 + 2 \cdot 5x + 25 = 64.$$

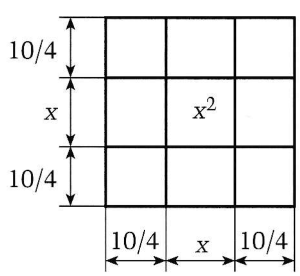


Figure 1

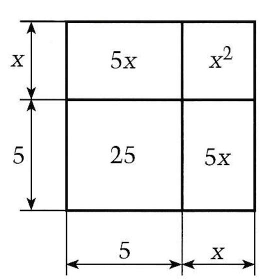


Figure 2

Although the equation $x^2 + 21 = 10x$ has two positive roots ($x_1 = 3$ and $x_2 = 7$), al-Khwarizmi proposes a geometrical solution only for $x_1 = 3$. However, he points out that $x_2 = 7$. The proposed solution is shown in figure 3. First we draw a rectangle *ABCD* with sides x and 10, then the square *EFCH* with side 5. After this, we can calculate the area of the square *EMLO* in two different ways. On one hand it is equal to $(5 - x)^2$; on the other hand,

$$S_{EMLO} = S_{EFCH} - S_{MFCHOLM}$$
$$= 25 - S_{GBCT} = 25 - (10x - x^2)$$
$$= 25 - 21 = 4.$$

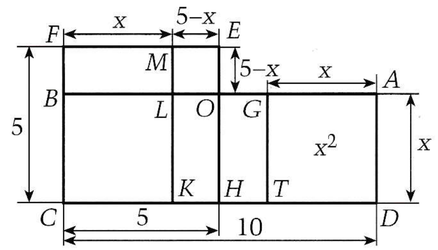


Figure 3

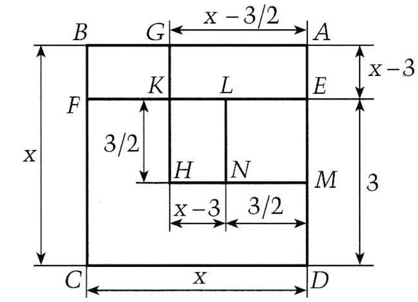


Figure 4

Thus, $(5 - x)^2 = 4$, and therefore $x_1 = 3$.

Figure 4 represents the constructions al-Khwarizmi used to solve the equation $3x + 4 = x^2$. The square *ABCD* with side x is divided into two rectangles *ABFE* and *EFCD*, whose areas are 4 and $3x$, respectively. Further, we draw the square *AGHM* with side $x - 3/2$. It is not difficult to calculate its area: It is 25/4, since $S_{ELNM} = 9/4$ and $S_{AGHLNE} = S_{ABFE} = 4$. But then we have $(x - 3/2)^2 = 25/4$, $x = 4$.

Al-Khwarizmi concludes his treatise with the "Book about Legacies," in which we find numerous applications of equations to the questions of everyday life, for instance, to the hereditary laws that existed in the Arab world at that time.

Al-Khwarizmi's works played an important role in the history of mathematics. They were the main source from which western Europe learned of Indian numbers and Arabic algebra. But al-Khwarizmi's activity was not limited to mathematics. He wrote a geographical treatise that started the development of geographical studies in the medieval East. He organized scientific expeditions to Byzantia, Hazaria (the region around the lower part of the Volga river), and Afghanistan. He also directed the work that allowed the calculation of the length of one degree of arc along a meridian with good precision.

PHYSICS CONTEST

Doppler beats

"The most persistent sound which reverberates through man's history is the beating of war drums." —Arthur Koestler

by Larry D. Kirkpatrick and Arthur Eisenkraft

AS A POLICE CRUISER DRIVES by with its siren sounding, you notice that the pitch of the siren decreases. The same thing happens at the Indianapolis 500 as a race car passes you. The pitch of the engine is steady as the car approaches, decreases as the car passes by, and is steady (but lower) as the car recedes into the distance.

These are two examples of the Doppler shift. The motion of the source shifts the frequency of the sound you hear. If you are at rest relative to the air, the frequency f you hear is given by

$$f = f_0\left(\frac{v}{v \mp v_s}\right),$$

where f_0 is the frequency heard by the observer at rest, v_s is the speed of the source, and v is the speed of sound. The minus sign is used when the source moves toward the observer and the plus sign is used when the source moves away from the observer.

Doppler worked out this mathematical relationship in 1842. He pointed out that the motion of the source toward the observer causes the sound waves to reach the ears at shorter time intervals—therefore, the higher frequency. The reverse is true when the source moves away from the observer.

Doppler's formula was put to an experimental test a few years later. For two days trumpet players rode on a flat car that was pulled at different speeds. Musicians who had perfect pitch stood on the ground and recorded the notes that they heard as the train approached and receded. Their observations were in agreement with Doppler's formula.

The motion of the observer also changes the frequency. When you ride in a train, the bell at the crossing has a higher (but steady) pitch as you approach the crossing and a lower pitch as you leave the crossing behind. This effect is described by

$$f = f_0\left(\frac{v \pm v_0}{v}\right),$$

where v_0 is the speed of the observer. The plus sign is used when the observer moves toward the source and the minus sign when the observer moves away from the source.

These two effects can be combined into a single relationship

$$f = f_0\left(\frac{v \pm v_0}{v \mp v_s}\right),$$

where the upper signs refer to the motion of one toward the other and the lower signs refer to motion of one away from the other.

Another interesting sound effect occurs when two sirens produce sound waves with approximately the same pitch. The two sound waves produce a sound with a pitch halfway between the two pitches, but with an intensity that varies periodically from no sound to a sound with four times the loudness of either source. The period of this *beat* frequency is just the difference of the two frequencies.

Piano tuners use beats to tune the wires corresponding to the same note on a piano. After one string is tuned to the correct frequency, it is struck at the same time as another wire. If the two wires have the same frequency, there is no variation in loudness, that is, the beat frequency is zero. However, if the second wire has a higher or lower pitch, the loudness of the sound will vary with a frequency equal to the difference of the two frequencies produced by the wires. The piano tuner then adjusts the tension in the second wire until the beating disappears.

These two sound effects were combined in an interesting way on the second exam used to select the members of the U.S. Physics Team that will compete in the International Physics Olympiad in Reykjavik, Iceland, this July.

Two sirens located on the x-axis are separated by a distance D. As

Art by Tomas Bunk


SAVING BANK
NEW
Doppler
RELIABLE
NOW WITH
IMPROVED
SOUND WAVES
POLICE
POLICE
1
2
DOPPLER
EXPRESS
TOMAS
BUNK

heard by an observer at rest relative to the sirens, the left-hand siren has a frequency f_L and the right-hand siren has a frequency f_R. Assume that you are moving with a constant speed v_0 along the x-axis and record the following observations:

1. When you are on the right-hand side of both sirens, you hear a beat frequency of 1.01 Hz.

2. When you are on the left-hand side of both sirens, you hear a beat frequency of 0.99 Hz.

3. When you are between the two sirens, the beat frequency is zero.

A. In which direction are you moving along the x-axis?
B. What is your speed as a fraction of the speed of sound?
C. Which frequency is greater?
D. What are the numerical values of the two frequencies?

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington, VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space.

Local fields

We asked our readers in the January/February issue of *Quantum* to calculate the local fields on an idealized spherical Earth.

1. At the North Pole, the local field is due only to the gravitational force, since the angular velocity there is zero.

$$\sum F = \frac{Gm_1m_2}{R^2} = mg_N,$$

$$g_N = \frac{GM_E}{R_E^2} = 9.804 \text{ m/s}^2,$$

where

$G = 6.6726 \cdot 10^{-11}$ N $\cdot$ kg^2/m^2,
$M_E = 5.977 \cdot 10^{24}$ kg,
$R_E = 6.378 \cdot 10^6$ m.

At the Equator, part of the gravitational force is needed to provide the centripetal force on the rotating Earth. The local field is reduced by this amount.

$$\sum F = \frac{GM_Em}{R_E^2} - m\omega^2R_E = mg_E,$$

$$g_E = \frac{GM_E}{R_E^2} - \omega^2R_E$$

$$= (9.804 - 0.034) \text{ m/s}^2 = 9.770 \text{ m/s}^2.$$

At 40° latitude, a component of the gravitational force is needed to provide the centripetal force on the rotating Earth. The local field is once again reduced by this amount. In this case, we resolve the gravitational force into components parallel and perpendicular to Earth's axis of rotation and reduce the perpendicular component by the centripetal force.

In the perpendicular direction:

$$\sum F_{perp} = \frac{GM_Em}{R_E^2}\cos\theta - m\omega^2R_E\cos\theta$$

$$= mg_{perp},$$

where $R_E \cos\theta$ is the radius of the circle that objects at 40° latitude rotate. Therefore,

$$g_{perp} = \left(\frac{GM_E}{R_E^2} - \omega^2R_E\right)\cos\theta$$

$$= 7.484 \text{ m/s}^2.$$

In the parallel direction:

$$\sum F_{par} = mg_{par},$$

$$g_{par} = \frac{GM_E}{R_E^2}\sin\theta = 6.302 \text{ m/s}^2.$$

The vector sum of the two components is therefore

$$g_{40} = \sqrt{7.484^2 + 6.302^2} \text{ m/s}^2$$

$$= 9.784 \text{ m/s}^2,$$

$$\theta = \tan^{-1}\frac{6.302}{7.484} = 40.10°.$$

2. The angular deviation between the local field at 40° latitude and the radial line toward the center of Earth is 0.10°.

3. The local field is along the radial line at the equator (both the gravitational force and centripetal force are along the same line) and along the radial line at the North Pole (no centripetal force). There must be a latitude for which the deviation is greatest. Finding this latitude and its corresponding deviation requires us to use the equations derived in part 1:

$$g_{perp} = \left(\frac{GM_E}{R_E^2} - \omega^2R_E\right)\cos\theta,$$

$$g_{par} = \frac{GM_E}{R_E^2}\sin\theta,$$

$$\theta' = \tan^{-1}\frac{\frac{GM_E}{R_E^2}\sin\theta}{\left(\frac{GM_E}{R_E^2} - \omega^2R_E\right)\cos\theta}.$$

To find where the deviation of this angle from θ is a maximum, we can plot the equation $(\theta' - \theta)$ versus θ and find the maximum. Alternatively, we can find the maximum on a spreadsheet or take the derivative and set it equal to zero:

$$\theta' - \theta = \tan^{-1}\frac{\frac{GM_E}{R_E^2}\sin\theta}{\left(\frac{GM_E}{R_E^2} - \omega^2R_E\right)\cos\theta} - \theta$$

$$= \tan^{-1}K\tan\theta - \theta.$$

where

$$K = \frac{\frac{GM_E}{R_E^2}}{\left(\frac{GM_E}{R_E^2} - \omega^2R_E\right)},$$

$$\frac{d(\theta' - \theta)}{d\theta} = \frac{K\sec^2\theta}{1 + K^2\tan^2\theta} - 1 = 0,$$

$$\sin\theta = \sqrt{\frac{1}{K+1}}.$$

On the rotating Earth, K is very close to 1 (that is, 9.804/9.770 = 1.0035), and the maximum deviation occurs at an angle slightly less than 45°. On objects where the rotational speed is much greater, we find that the maximum deviation occurs at even smaller latitudes.

Weightlessness in a car?

by Sergei Pikin

TO DRIVE A CAR WITHOUT getting into an accident, you should not only know the rules of the road, but also the laws of mechanics. We can see this by considering the following problem. How fast should a car travel at the top of a convex bridge with a radius of 40 m to momentarily put the driver in a state of weightlessness?

Let's consider a frame of reference attached to the ground. The driver is affected by two forces: gravity $m\mathbf{g}$ and the supporting force **N.** Since at the top of the bridge the driver is weightless, $N = 0$. In projections onto the y-axis, Newton's second law says (fig. 1) $mg = ma$, where $a = v^2/R$.

This yields

$$v = \sqrt{gR} = 20 \text{ m/s} = 72 \text{ km/h}.$$

At first everything looks O.K. This speed is below the speed limit. But let's think further and ask what will happen to the car (and the driver) after passing the top of the bridge, and what happened to them before the top? You'll see from the next calculations that the situation described in this problem is actually impossible! Find, for example, the normal force N before the car gets to the top, provided its speed is a con-

stant $v = \sqrt{gR}$ (fig. 2). In projections onto the y-axis, the equation of motion looks like this:

$$mg \cos \alpha - N = ma,$$

where $a = v^2/R$ and $v = \sqrt{gR}$, from which we get

$$N = mg(\cos \alpha - 1).$$

So, if at the top of the bridge $N = 0$, N is negative ($N < 0$) everywhere else! The passengers would need to fasten their seatbelts to prevent being slammed into the ceiling! But the car can't be "fastened" to the road, so it will take off from the bridge and after a spectacular flight will land back on the road. The most probable result of such weightlessness would be a damaged car. In other words, when trying to ride on the convex bridge at the speed $v = \sqrt{gR}$, not only can't you become weightless, but you may well be injured.

Well then, *is* there a constant speed at which one can ride on the convex bridge to experience a moment of weightlessness? Assume that the bridge has a radius R and subtends the angle 2α (fig. 3).

The formula for N shows that the normal force is minimal at the entrance of the bridge. So if the car doesn't take off when it first hits the bridge, it won't do so later. Thus,

$$mg \cos \alpha - N = ma,$$

where $N \ge 0$ and $a = v^2/R$, from which we get

$$v \le \sqrt{gR \cos \alpha}.$$

This is the maximum speed at which to drive a car on a convex bridge. The state of weightlessness will be experienced twice: at the entrance and at the exit of the bridge.

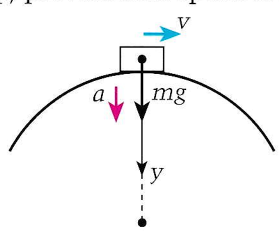


Figure 1

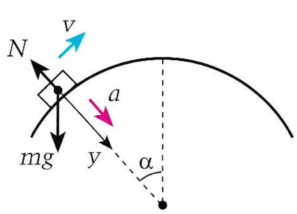


Figure 2

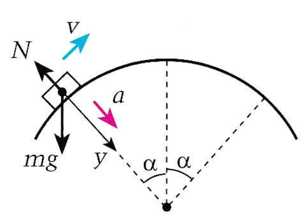


Figure 3

Art by Sergey Ivanov

Triangles with th

SEVERAL INTERESTING ELementary theorems concern a right triangle with the altitude drawn to the hypotenuse. This seemingly ordinary situation can give rise to many interesting and nontrivial results. We can build these up from simple properties studied in most geometry classes.

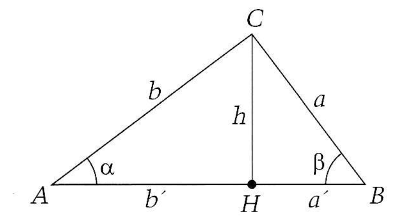


Figure 1

Let ABC be a right triangle and let H be the foot of the altitude to hypotenuse AB (fig. 1). As usual, we let $AB = c$, $BC = a$, $AC = b$, $AH = b'$, $BH = a'$, $CH = h$, $\angle A = \alpha$, and $\angle B = \beta$. The following statements and properties are easy to prove.

1. Triangles BCH, ACH, and ABC are similar to each other.

2. $a^2 = a'c$, $b^2 = b'c$. (These relations imply the Pythagorean theorem: $a^2 + b^2 = a'c + b'c = c(a' + b') = c^2$.)

3. $h^2 = a'b'$, $h = ab/c$.

4. Let R be the center of the circle inscribed in ABC and let r be its radius (fig. 2). Then $r = (a + b - c)/2$.

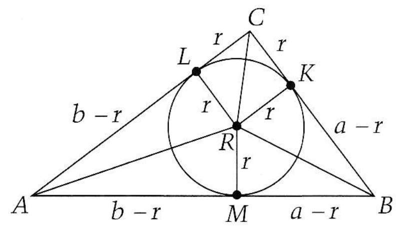


Figure 2

Proof. We note that the quadrilateral $RKCL$ is a square and use the fact that tangents drawn to a circle from one point are equal. Then $c = (a - r) + (b - r)$, and thus $r = (1/2)(a + b - c)$.

5. $\angle ARB = 180° - (1/2)\angle A - (1/2)\angle B = 135°$.

Things get more interesting if we inscribe circles in triangles ACH and BCH (fig. 3). Let their radii be r_1 and r_2 respectively, and let their centers be at R and S. Then:

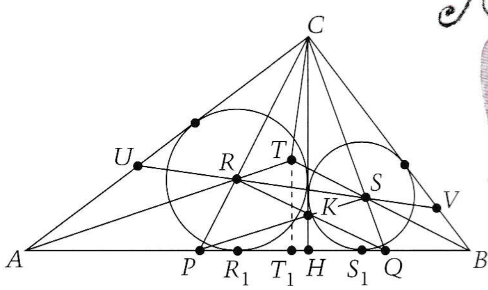


Figure 3

6. $r + r_1 + r_2 = h$.
Outline of proof. We have

$$r = \frac{a + b - c}{2},$$

$$r_1 = \frac{h + b' - b}{2},$$

and

$$r_2 = \frac{h + a' - a}{2}.$$

Adding these formulas, we obtain the relation

7. $r^2 = r_1{}^2 + r_2{}^2$.
Proof. Since triangles ACH and ABC are similar, we conclude that

$$\frac{r_1}{r} = \frac{b}{c} = \cos\alpha.$$

Similarly

$$\frac{r_2}{r} = \frac{a}{c} = \cos\beta.$$

Thus,

$$r_1 = r\cos\alpha,\ r_2 = r\sin\alpha,$$

and therefore,

$$r^2 = r_1{}^2 + r_2{}^2.$$

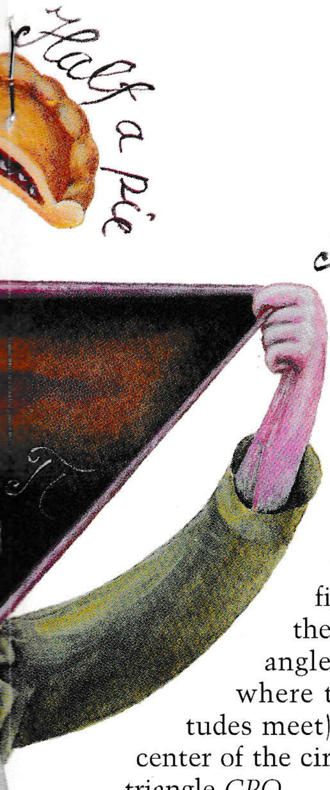
8. If line CR intersects AB at P, and line CS intersects AB at Q (fig. 3), then $AC = AQ$ and $BC = BP$.

Outline of proof. The first equality follows from the following sequence of equalities:

$$\begin{aligned}\angle CQA &= \angle BCQ + \angle B \\ &= (1/2)\alpha + \beta = \angle ACH + \angle HCQ \\ &= \beta + (1/2)\alpha = \angle ACQ.\end{aligned}$$

Art by Jose Garcia

the right stuff



The second equality is proved similarly.

9. Suppose point T is the center of the circle inscribed in triangle ABC (see fig. 3). Then T is also the orthocenter of triangle CRS (the point where the triangle's altitudes meet), and T is also the center of the circle circumscribing triangle CPQ.

Outline of proof. The lines BS and AT are the bisectors of $\angle B$ and $\angle A$, respectively. Since triangles ACQ and BCP are isosceles, this means that the lines BS and AT are perpendicular to CP and CQ and that they divide these segments into halves.

10. The lines T_1R and T_1S are parallel to BC and AC, respectively.

Outline of proof. Let T_1 be the foot of the perpendicular from T to AB. Draw the perpendicular T_1R_2 from T_1 to AC (see fig. 4). Let R^* be the point where T_1R_2 meets AT.

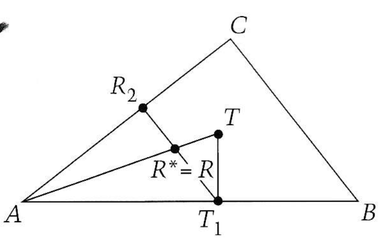


Figure 4

We'll show that $R^* = R$.

Triangles ATT_1 and AR^*R_2 are similar. Therefore,

$$\frac{R^*R_2}{TT_1} = \frac{AR_2}{AT_1}.$$

From right triangle AR_2T, we see that $AR_2/AT_1 = \cos\alpha$. Thus

$$R^*R_2 = TT_1 \cos\alpha = r\cos\alpha.$$

But triangles ABC and ACH are similar, and the ratio of any two corresponding parts is equal to $AC/AB = \cos\alpha$. Thus $r_1/r = \cos\alpha$, and $RR_2r \cos\alpha = r_1$.

Exercises. Prove the following statements. Selected proofs are outlined.

11. The line PS is parallel to AT, and the line RQ is parallel to BT.

12. Triangles ATB, BSC, and ARC are similar.

13. Points H and T_1 lie on the circle with diameter RS.

14. $ST_1 = RT_1$.

Outline of proof. Since triangle T_1RR_1 is a right triangle and since $\angle RT_1R_1 = \beta$, we conclude that $T_1R \sin\beta = r_1 = r\sin\beta$, and thus $T_1R = r$. We can similarly show that $T_1S = r$.

15. $CU = CV$.

Outline of proof. Triangle ST_1R is isosceles (see statement 14) and right (see statement 10). Therefore, $\angle RST_1 = 45°$. But $T_1S \parallel AC$; thus $\angle VUC = \angle RST_1 = 45°$, and triangle CUV is isosceles.

16. Points Q, S, T, R, and P lie on a circle with center at T_1 and radius r.

Outline of proof. Points S and R lie on this circle (see statement 14). Let's show, for instance, that the point Q also lies on it.

Triangles T_1SQ and CQA are similar ($T_1S \parallel AC$), and thus

$$\frac{ST_1}{AC} = \frac{T_1Q}{QA}.$$

But $AC = QA$ (see statement 8). Therefore, $ST_1 = T_1Q = r$.

17. Point K is the orthocenter of triangle CPQ.

18. The lines PS, RQ, and CH meet at K.

19. $RT = KS = SQ$ and $RP = RK = TS$.

20. $RS = CT$.

21. Points A, R, S, and B lie on one circle; points A, P, T, and C lie on another circle; and points B, Q, T, and C lie on one circle, too.

22. $AT_1 \cdot BT_1 = S_{ABC}$ (S_{ABC} denotes the area of triangle ABC).

Outline of proof. Let $AT_1 = u$ and $BT_1 = v$. Since $S_{ABC} = rs$ (here s is the semiperimeter of ABC), we have

$$S_{ABC} = \frac{1}{2}r(a+b+c) = r(r+u+v).$$

But $(u+r)^2 + (v+r)^2 = c^2 = (u+v)^2$, and thus $r^2 + r(u+v) = uv$. The left part of this identity is the area of triangle ABC.

23. $S_{CPQ} = (abr)/c$. *Hint.* The height of triangle CPQ drawn from the vertex C is $h = ab/c$, and the length of PQ is $2r$.

24. Triangles HSR and ABC are similar.

25. Triangles RR_1T_1 and SS_1T_1 are congruent and are similar to triangle ABC.

25. Triangles AQR and ARC are congruent.

26. The circles circumscribed about triangles ARC and CSB touch each other at C, and CT is their common tangent.

—L. D. Kurlyandchik

So, what's wrong?

Debunking problematic solutions

by I. F. Sharygin

MANY PEOPLE, EVEN SOME who consider themselves intellectuals, have only a vague understanding of mathematics and often feel misgivings about the simplest of mathematical statements. For example, here is a problem dealing with a subject everyone learns in school: percentage.

Problem 1. A farmer harvested 10 tons of watermelons and sent them by river to the nearest town. It is well known that a watermelon, as reflected in its name, is made almost entirely of water. When the barge left, the content of the watermelons was 99% water by weight. On the way to the town, the watermelons dried out somewhat, and their water content decreased by 1% (to 98%). What was the weight of the watermelons when they arrived at the town?

Many people won't believe the answer, even if they find it themselves. We invite the reader to solve the problem independently. (Explanations for the problems in this article can be found beginning on page 53.)

Many people are ready to believe the silliest reasoning, especially if it is presented in public in a convincing manner. Consider, for example, this old problem.

Problem 2. A retired general decided to sell his old boots. He sent his butler to the market with the pair of boots and instructions to sell them for \$15. The butler met two one-legged veterans at the market and sold them each one boot for \$7.50. When the butler told his master about it, the general said that military veterans should be charged less. So, he gave the butler \$5 and had him return it to the buyers. On his way to the market, the servant squandered \$3 on drink and returned \$1 to each of the veterans. Now let's count the money: each veteran paid \$6.50. Multiplying \$6.50 by 2, we get \$13. And \$3 dollars was squandered by the servant: \$13 + \$3 = \$16. Where does the extra dollar come from?

(This sort of reasoning can be found, for instance, in the promises of many politicians.)

This example is rather simple, but it illustrates the way in which many mathematical paradoxes are obtained. The reader is pressed to believe plausible but erroneous reasoning, the outcome of which is a statement contradicting some obvious or well-known mathematical fact. (Sometimes this error is very slight and not easy to find.) To demonstrate this, we present a geometrical "theorem."

Problem 3. The following "theorem" is an additional test for the congruence of triangles. If in triangles ABC and $A_1B_1C_1$ the equalities $AB = A_1B_1$, $AC = A_1C_1$, and $\angle ABC = \angle A_1B_1C_1$ hold, then these triangles are congruent. That is, the criterion $SSA = SSA$ guarantees congruence for any two triangles.

"Proof." Construct triangle AB_2C as it is shown in figure 1. In this triangle $\angle CAB_2 = \angle C_1A_1B_1$ and $AB_2 = A_1B_1$. Triangles $A_1B_1C_1$ and AB_2C are congruent by SAS (since it is given that $AC = A_1C_1$). Thus, $\angle ABC = \angle AB_2C$ and $AB = AB_2$. Now draw segment BB_2. Triangle BAB_2 is isosceles. Therefore, $\angle ABB_2 = \angle AB_2B$. We also see that $\angle CBB_2 = \angle CB_2B$. Thus triangle CBB_2 is also isosceles, and $CB = CB_2$. Finally, we conclude that triangle ACB_2 is congruent to triangle ACB because three of their sides are equal, and therefore triangles ABC and $A_1B_1C_1$ are also congruent.

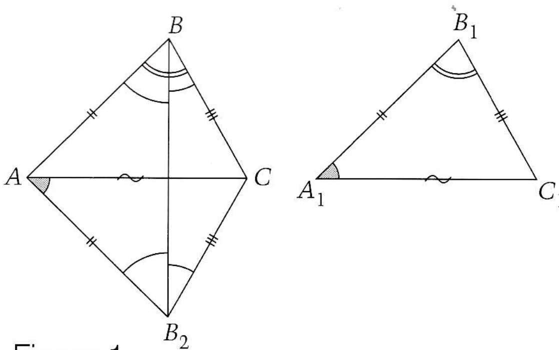
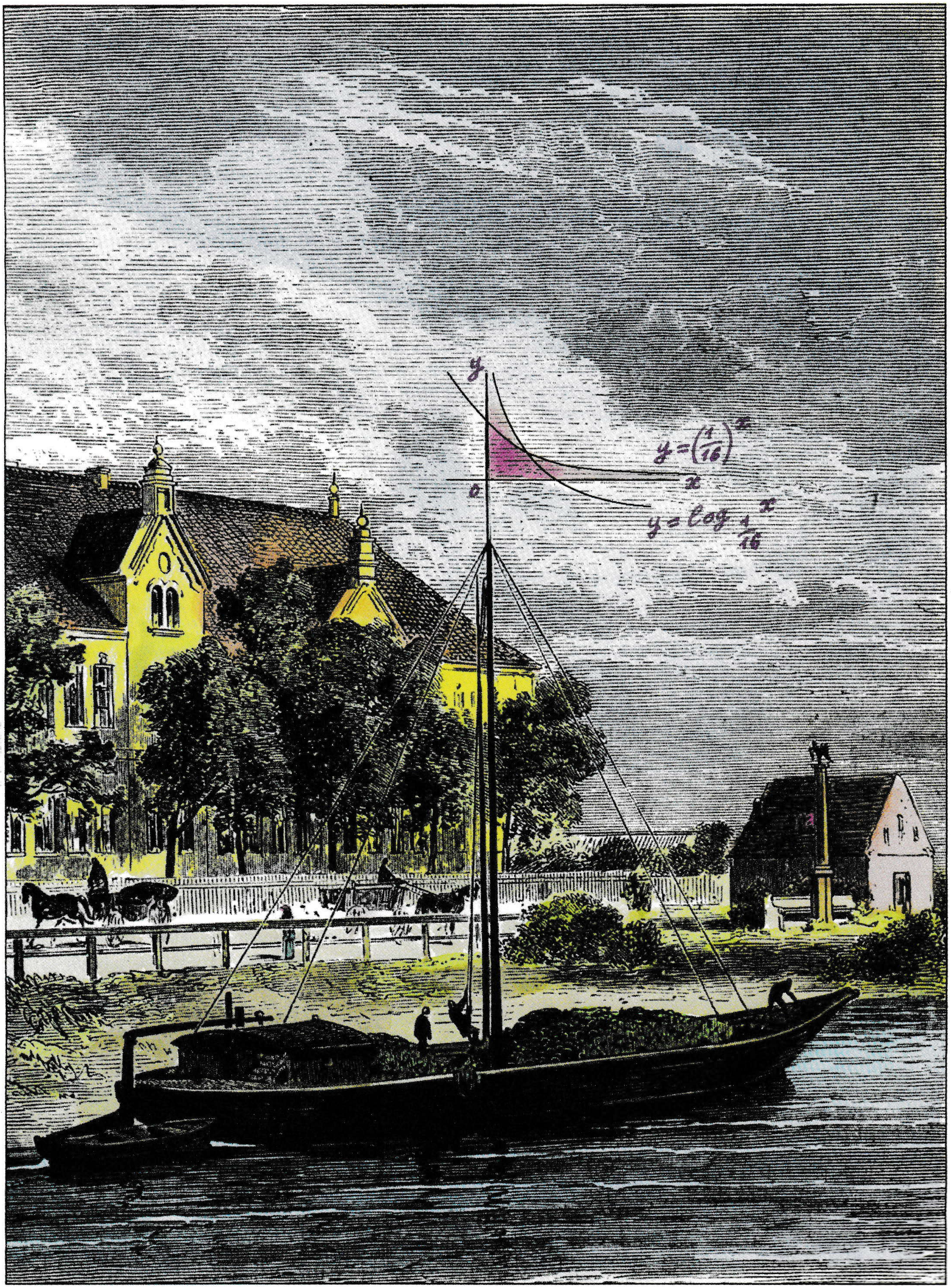


Figure 1

Collage by Vera Khlebnikova


y
$y=\left(\frac{1}{16}\right)^x$
o
x
$y=\log_{\frac{1}{16}} x$

Is the "theorem" proved?

We do not ask you to refute the conclusion of the "theorem." It is not difficult to see that it is wrong. But where is the error?

It is not always easy to understand whether or not a mathematical statement is true. The ability to find a mistake in reasoning is one of the most important skills a professional mathematician can possess. The history of mathematics is replete with instances when mathematicians found mistakes in proofs that had been considered flawless for decades.

We'll consider a few more scholarly examples. Each of the following problems will be supplied with a "solution." The solutions will contain an error for you to find.

Problem 4. A parallelogram $ABCD$ is given in which $\angle ABD = 40°$. The centers of the circles circumscribed about triangles ABC and CAD lie on BD. What kind of parallelogram is $ABCD$?

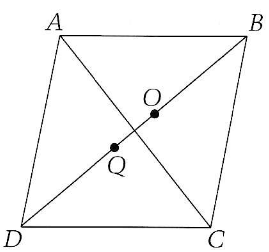


Figure 2

"Solution." Let O and Q be the centers of the circles circumscribed about ABC and CAD (fig. 2). Since the perpendiculars drawn to AC from these centers bisect AC, we conclude that the line OQ is perpendicular to the diagonal AC. It follows that the diagonals of the parallelogram are perpendicular to each other. Thus, it is a rhombus.

Do you like this solution?

Sometimes the trick is in the statement of the problem, and not the solution.

Problem 5. The numbers p and q satisfy the equation $x^2 + px + q = 0$. Find p and q.

"Solution." Using facts about the sum and product of the roots of a quadratic equation, we have the following system:

$$\begin{cases} p+q=-p, \\ pq=q. \end{cases}$$

Solving it, we obtain two pairs:

$$p = q = 0,$$

and

$$p = 1,\ q = -2.$$

Do you have any doubts concerning the solution?

Problem 6. Solve the equation

$$\tan(x + \pi/4) = 3\cot x - 1.$$

"Solution." Transform the right side of this equation by using the formula for the tangent of a sum, and introduce the new variable y tan x. We find that

$$\frac{y+1}{1-y} = \frac{3}{y} - 1.$$

This leads to $y = 3/5$, thus,

$$x = \arctan(3/5) + \pi k.$$

Is that all?

Problem 7. How many solutions does the equation $\log_{1/16} x = (1/16)^x$ have?

"Solution." The functions that appear on the left- and right-hand sides of the equation are inverses of each other. If we draw their

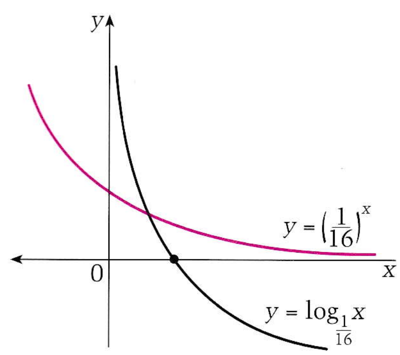


Figure 3

graphs, we'll "see" that they intersect in only one point on the bisector of the first quadrant. Therefore, the equation has only one solution. Any objections?

The next two problems lie somewhat outside the focus of our article. The situations they describe seem to be impossible. This is what attracts our attention.

Problem 8. A section of greatest possible area was drawn through the vertex of a right circular cone. It turned out that its area is twice the area of an axial section. Find the angle at the axial section.

The conditions of the problem seem to be impossible, since the axial section of a cone has the greatest area.

Problem 9. The center of a sphere lies on another sphere. It is known that the part of the second sphere lying within the first one has an area five times smaller than the surface area of the first sphere. Find the ratio of the spheres' radii.

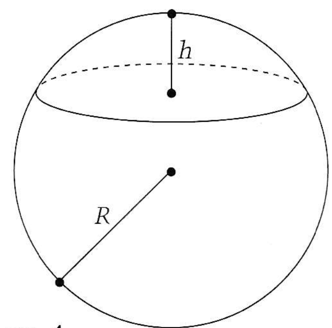


Figure 4

To solve the problem we'll need the formula for the area of a spherical segment: $S = 2\pi hR$, where R is the radius of the sphere and h is the height of the segment.

"Solution." Let r and R be the radii of the first and the second spheres, respectively. Draw a planar section through the centers of the spheres (fig. 5). We have $OA = OB = R$ and $AB = r$. Drop the perpendicular BC from B to OA. Now AC is the height of the spherical segment that is the part of the second sphere lying within the first one. If we let $AC = h$, then using the Pythagorean theorem in triangles ABC and OBC, we eventually find that the equation

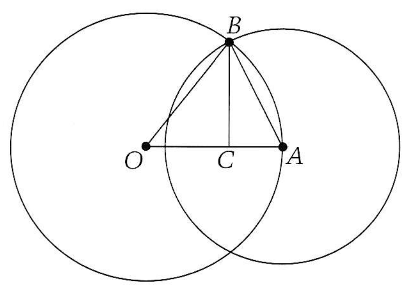


Figure 5

$$r^2 - h^2 = R^2 - (R - h)^2,$$

from which we get $h = r^2/2R$.

Now, the formula for the area of a spherical segment gives $S = \pi r^2$. But the whole surface of the first sphere is $4\pi r^2$. Thus the surface of the part of the second sphere within the first one is always 4 times smaller than the surface of the first. But the statement of the problem says that it is 5 times smaller, and thus we have a contradiction. So, does the problem have no solution?

Problem 10. A convex quadrilateral with two sides of length 10 and two other sides of length 6 forms the base of a quadrilateral pyramid. The altitude of this pyramid is 7. All the angles between its lateral faces and its base are 60°. Find the volume of the pyramid.

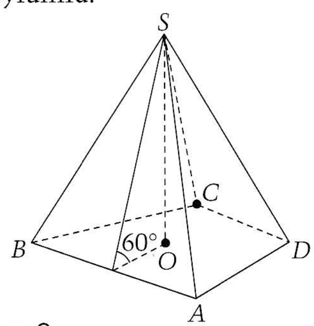


Figure 6

"Solution." Since all the lateral faces form equal angles with the base, we conclude that the projection of the vertex S of the pyramid $SABCD$ coincides with O—the center of the circle inscribed in $ABCD$ (fig. 6). The radius of the circle is

$$7 \cot 60° = 7/\sqrt{3}.$$

The area of quadrilateral $ABCD$ equals the sum of the areas of triangles ABO, BCO, CDO, and DAO. All these areas are easy to find. Finally, we see that the area of the base is

$$(10 + 6) \cdot 7/\sqrt{3} = 112/\sqrt{3}.$$

And the volume of the pyramid equals $784/3\sqrt{3}$. Do you agree with this answer?

ANSWERS, HINTS & SOLUTIONS ON PAGE 53

Rivers, typhoons, and molecules

by Albert Stasenko

WHAT DO RIVERS, TYPHOONS, and molecules have in common? Only that they are all composed of atoms? Not so. They are all affected by a phenomenon known as the *Coriolis force*, which is caused by motion relative to a rotating frame of reference.

Do we feel this force when we run on Earth's surface? After all, we are moving in the rotating system of our planet. No, we don't. Still, it is the Coriolis force that makes one bank of a river steep and the other flattened, that spins huge air masses into typhoons, and even intrudes into the private life of molecules. So, is this force negligible or not?

Let's consider two adjoining circular bands on Earth's surface located at latitudes θ_1 and θ_2. In figure 1 these bands are different colors. Clearly, the higher the latitude θ, the smaller the linear (circumferential) velocity: $v_2 < v_1$. For example, on the North Pole ($\theta = 90°$), the linear velocity is zero.

Imagine a river in the Northern Hemisphere flowing from south to north along a meridian (perpendicular to the parallels of latitude—see fig. 1). When flowing from latitude θ_1 to latitude θ_2, the water particles "try" to keep their velocity v_1 (which is directed to the east), and if Earth's surface were smooth and slippery, they would be deflected to the right (to the east) when reaching latitude θ_2. (This path is indicated by the dashed line in fig. 1). An observer on Earth's surface would say that the water particles experience a force perpendicular to their velocity. This is the Coriolis force, discovered in 1835 by the nineteenth-century French scientist Gustave-Gaspard Coriolis.

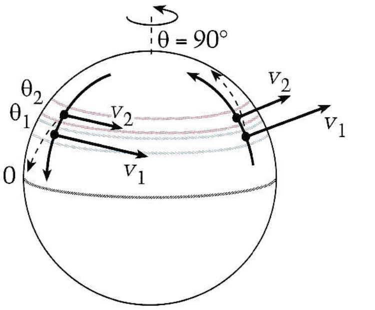
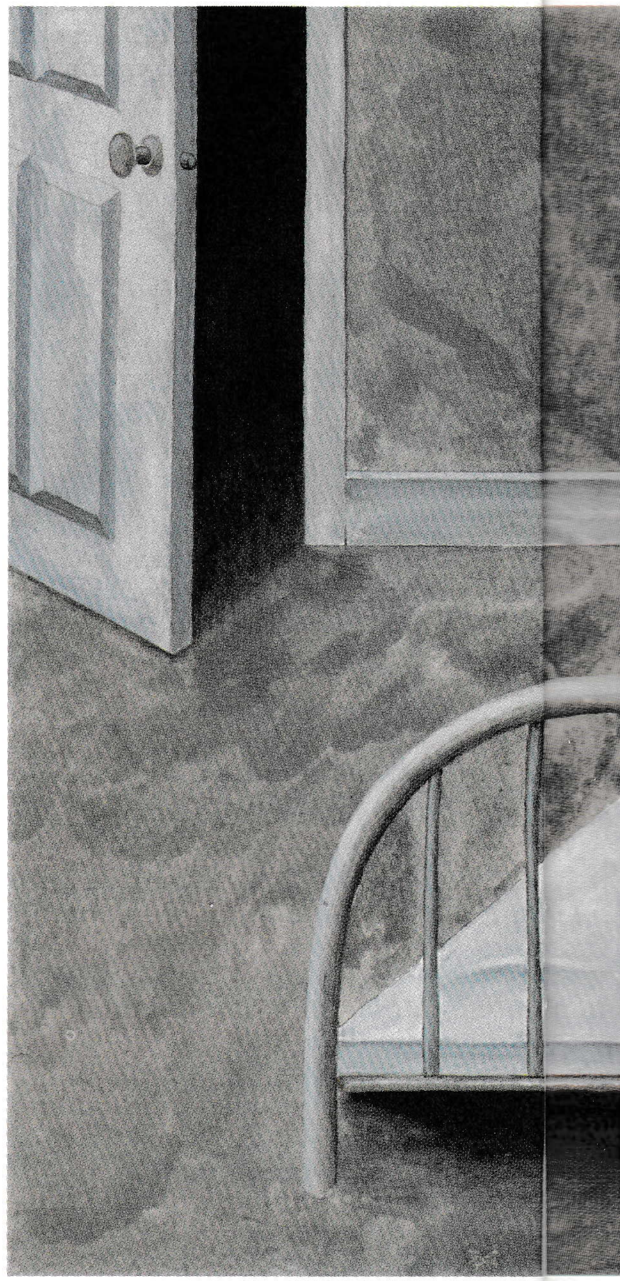


Figure 1

Thus, the Coriolis force tries to push flowing water aside. However, if a river is confined to its bed, the water particles will hit the right bank because they move eastward with velocity $v_2 < v_1$, and thus they will gradually destroy the right bank of the river.

If we imagine another Northern Hemisphere river flowing from north to south along a meridian (fig.1), we realize that it would try to turn to the west, again to the right of the direction of its motion. Now it is clear why all meridional rivers in the Northern Hemisphere have steep right banks and flattened left banks. In addition, the water level at the right bank is always higher than that at the left bank.

Clearly, in the Southern Hemisphere meridional rivers must wash out their left banks. This geographi-



cal phenomenon was first discovered in 1857 by the outstanding naturalist Carl Maximovich Barr, who analyzed his own observations and earlier reports (beginning in 1826) of Russian travelers. In addition, he gave the correct explanation of this phenomenon as being caused by Earth's rotation.

The effects of Coriolis forces are manifested most spectacularly in the motion of water and air masses. Is there anyone who doesn't know that the most famous oceanic current, the Gulf Stream (directed to the north in the Northern Hemisphere), deviates to the right, depriving Canada of warmth and heating Europe instead! It is a kind of river, only without banks.

And how are typhoons created, these formidable atmospheric phenomena with characteristic diameters on the order of a thousand miles, which inflict colossal destruction? First a region of decreased atmospheric pressure forms somewhere due to nonuniform heating of Earth's surface by the Sun. The air masses from the adjacent regions of higher atmospheric pressure rush toward the depression along radial directions. As we already know, all moving masses tend to deviate to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. Therefore, a colossal vortex arises, which rotates counterclockwise in the Northern Hemisphere or clockwise in the Southern Hemisphere (fig. 2).

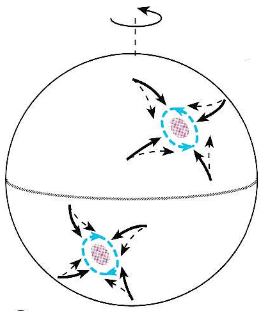


Figure 2

Now let's consider gas molecules. They are known not only to move

Art by Jose Garcia

stochastically in any direction but also to rotate very quickly in such a way that the energy of their rotational motion will be of the same order of magnitude as the energy of their translational motion. In addition, under certain conditions some molecular fragments (individual atoms or atomic groups and radicals in very complicated molecules) can vibrate relative to the center of molecular mass. Again, the energy of this oscillation will be of the same order of magnitude as the energy of either the translational or rotational motion. By the way, in physics this fact is referred to as the principle of equipartition of energy per degree of freedom.

Now consider a simple model of a triatomic molecule that has two identical atoms attached by elastic, weightless springs to a central atom (figs. 3 and 4). This model simulates, for example, the carbon dioxide molecule CO_2, which is an extremely important agent in powerful infrared lasers. When nothing disturbs this molecule, its center of mass moves along a straight line. Considering the time axis to point to the right as usual, we consider the motion of the molecule's atoms in the reference frame that rotates about the center of mass with the same angular velocity as the molecule itself. We have already used such a rotating frame of reference in considering the flow of rivers and the motion of oceanic and atmospheric streams on the surface of the rotating Earth.

There are two basic modes of oscillation of the system: (1) the peripheral atoms simultaneously move toward and away from the center of mass (both springs are simultaneously compressed or elongated) and (2) the peripheral atoms simultaneously move in the same direction, which means that one spring contracts while the other stretches out.

It is easy to see that in the first case (fig. 3), the molecule's rotation is either accelerated or decelerated. For example, when both atoms move toward the center, they are affected by the Coriolis force, which deflects them to the right relative to their centripetal motion. Therefore, in this case the molecule's rotation is accelerated. On the contrary, when the peripheral atoms move away from the center, the Coriolis force again deflects them to the right, and this time their rotation is decelerated. We can observe the same phenomenon when a figure skater spins faster by drawing his or her hands close to the body. This can also be explained in the inertial frame of reference by conservation of angular momentum.

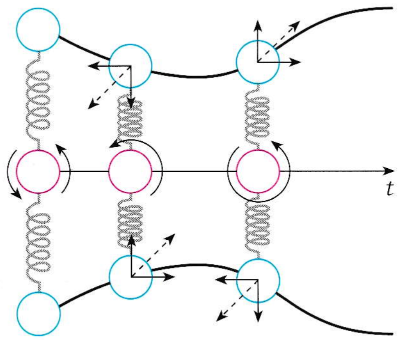


Figure 3

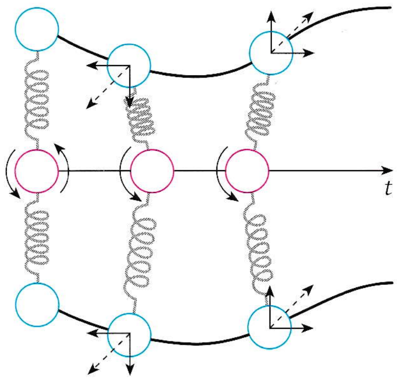


Figure 4

A new and much more interesting phenomenon arises when the molecule oscillates in the second mode. Indeed, when the peripheral atoms move in the same direction, the Coriolis force again shifts them to the right. However, while the rotation of one atom is accelerated, the rotation of the other is decelerated. As a result, the molecule will be bent. In a quarter of a period bending will occur again, but this time in the opposite direction. Therefore, the oscillation of atoms in a rotating molecule leads to additional types of vibration, called *flexural vibration*.

However, since the energies and velocities of vibration and rotation in a gas are of the same order of magnitude, their frequencies can be similar to each other, so the phenomenon of resonance can occur. As the molecules usually radiate electromagnetic waves, this resonance will be manifested in the infrared spectrum of carbon dioxide. Indeed, spectroscopists have confirmed this.

We note that in the second case, the oscillation of the peripheral atoms and the flexing of the springs will periodically shift the central atom from the position of the center of mass, but this will not affect our qualitative inferences.

Thus we see that seemingly quite different objects—rivers, typhoons, and gas molecules—have something in common. Seek and you will find!

Quantum articles about rotation and the Coriolis force:

V. Surdin, "A Venusian Mystery," July/August 1996, pp. 4–8.

M. Emelyanov et al., "In Foucault's Footsteps," November/December 1996, pp. 26–27.

Symmetry in algebra, part III

by Mark Saul and Titu Andreescu

LET'S GO BACK TO BASICS. Suppose we wanted to factor $x^3 - 5x^2 + 5x - 1$. We can note that if $x = 1$, the value of the given polynomial is 0. It follows from the factor theorem that $(x - 1)$ is a factor of the polynomial, and we can obtain the other factor by division. Indeed,

$$x^3 - 5x^2 + 5x - 1 = (x - 1)(x^2 - 4x + 1).$$

Remember the factor theorem?

Factor Theorem: For any number a, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

Problem 1. Factor

$$150x^2 - 77x - 73.$$

Problem 2. Factor

$$x^3 + 15x^2 + 15x + 1.$$

If the value of a polynomial is 0 when $x = k$, we sometimes say that the polynomial *vanishes* when $x = k$.

Problem 3. Factor $x^3 - 1$.

Solution: The given polynomial vanishes when $x = 1$. This leads to the factorization

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

Some readers may have encountered this factorization already. Both $x^3 - 1$ and $x^3 + 1$ can be factored, and it may be difficult to remember how each factored form looks. But if we recall the factor theorem, it is easy to see that $x - 1$ must be a factor of $x^3 - 1$ and that $x + 1$ must be a factor of $x^3 + 1$.

Problem 4. Factor

$$x^3 - 7x^2 + 7x - 1.$$

Answer: $(x - 1)(x^2 - 6x + 1)$.

Problem 5. Factor

$$x^3 - 137x^2 + 137x - 1.$$

Answer: $(x - 1)(x^2 - 136x + 1)$.

What's going on? Problems 4 and 5 are not very interesting.

What's interesting is the pattern that they indicate.

Problem 6. Factor

$$x^3 - ax^2 + ax - 1.$$

Solution: Once more it is clear that one factor of this polynomial is $x - 1$. We can obtain the other factor easily, for example by division: It is $x^2 + (1 - a)x + 1$. Thus we have the complicated looking, but really not so difficult identity

$$x^3 - ax^2 + ax - 1 = (x - 1)(x^2 + x - ax + 1),$$

which can be checked by multiplication. The reader is invited to look back at problems 4 and 5 to see that the answers are indeed of this form.

Problem 7. Factor

$$x^3 + ax^2 + ax + 1.$$

Problem 8. Factor

$$x^3 - ax^2 + 2ax - 2a^2.$$

Hint: What happens if $x = a$?

Problem 9. Factor

$$x^4 - 6x^3y + 4xy^3 + y^4.$$

Problem 10. Factor

$$ab(a - b) + bc(b - c) + ca(c - a).$$

Solution: Let us first consider this expression as a polynomial in a, and think of b and c as "constants." The polynomial vanishes when $a = b$ and when $a = c$, so it has factors $(a - b)$ and $(a - c)$. Now let us consider the expression as a polynomial in b. We already know that it vanishes when $b = a$, but it also vanishes when $b = c$. Thus it has another factor of $(b - c)$.

Therefore, we can write

$$ab(a - b) + bc(b - c) + ca(c - a) = (a - b)(a - c)(b - c)M,$$

where M is some polynomial in a, b, and c. Let us think again of these two expressions (whose identity is being asserted) as polynomials in a. Then the left-hand polynomial is quadratic in a, so the right side must also be quadratic in a, and M cannot contain any positive powers of a. But the same is true for b and c, so M must be a constant. We can find the value of the constant, for instance, by plugging in numerical values for a, b, and c. We quickly find that $M = 1$.

Problem 11. Factor

$$(a - b)^3 + (b - c)^3 + (c - a)^3.$$

Problem 12. Factor

$$(a + b + c)^3 - (a^3 + b^3 + c^3).$$

Hint: What happens if $a = -b$?

Problem 13. For all real numbers a, b, c, x, prove that

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

Hint: Consider the problem as an equation in x. What is its degree? How many roots can you find by inspection? What kind of equation has more roots than its degree?

Problem 14. For all real numbers a, b, c, x, prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$$

Problem 15. Let m and n be two odd integers. Show that

$$\frac{1}{a^m+b^m+c^m}=\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m}$$

if and only if

$$\frac{1}{a^n+b^n+c^n}=\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}.$$

Hint: One approach is to construct a cubic equation for which a^m, b^m, and c^m are the roots. Then guess at one of the roots of the equation.

Problem 16. Factor

$$x^3+y^3+z^3-3xyz.$$

Hint 1: Try letting $y+z=-x$. Hint 2: Alternatively, and if you know something about determinants, note that the given polynomial is equal to the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Problem 17. Let the symbol $\overline{abc}$ denote the decimal numeral with a in the hundreds place, b in the tens place, and c in the units place. Prove that if the numbers $\overline{abc}$, $\overline{bca}$, and $\overline{cab}$ are all divisible by some integer n, then $a^3+b^3+c^3-3abc$ is also divisible by n. (Note: The solution we give depends on properties of determinants, and is related to the second solution to problem 16.)

ANSWERS, HINTS & SOLUTIONS ON PAGE 54

AT THE BLACKBOARD III

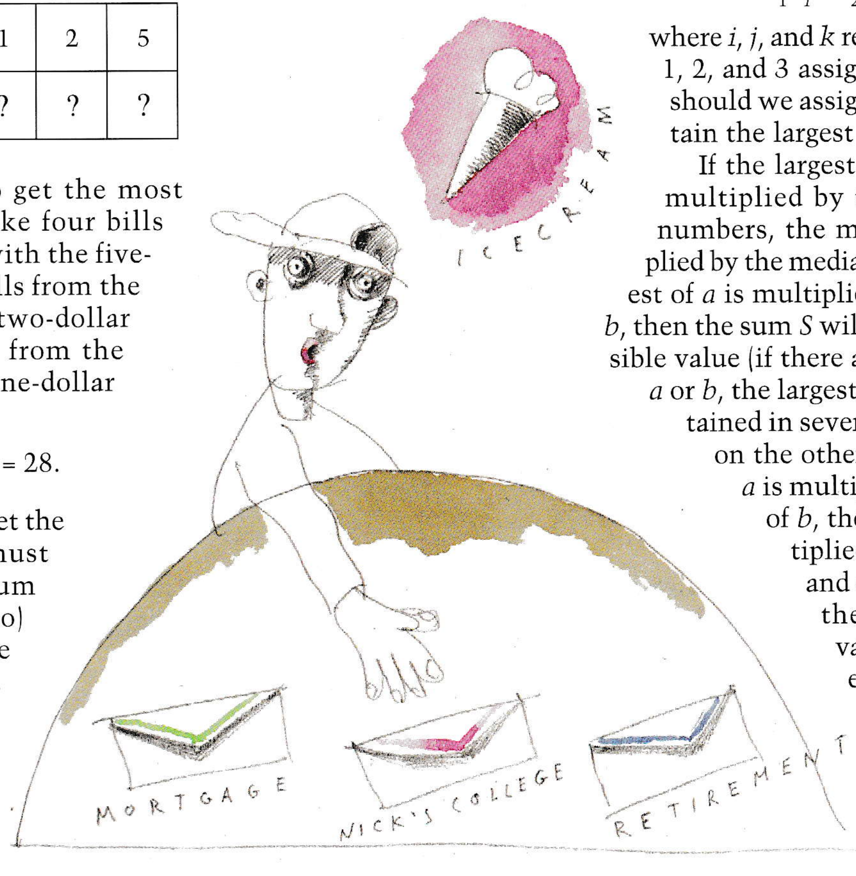
Ordered sets

by L. Pinter and I. Khegedysh

NICK'S PARENTS KEEP PAPER MONEY IN ENVElopes. One day they take three envelopes and place them before Nick. One of the envelopes holds one-dollar bills, another holds two-dollar bills, and the third holds five-dollar bills. They ask Nick to take two bills from one of the envelopes, three bills from another, and four bills from the third envelope.

How should Nick choose envelopes to get the most money? How should he choose envelopes to get the least? To calculate the sum for a random selection of envelopes, place the numbers 2, 3, and 4 into the lower row of the table below in the corresponding order and then add the products of the numbers in the columns.

1	2	5
?	?	?



Art by Virginia Smith

If Nick wants to get the most money, he must take four bills from the envelope with the five-dollar bills, three bills from the envelope with the two-dollar bills, and two bills from the envelope with the one-dollar bills:

$$5 \cdot 4 + 2 \cdot 3 + 1 \cdot 2 = 28.$$

If Nick wants to get the least money, he must take the minimum number of bills (two) from the envelope with five-dollar bills, three bills from the envelope with two-dollar bills, and four bills from the envelope with one-dollar bills:

$$5 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 = 20.$$

Ordered triplets

The above problem can also be approached in the following way. Suppose we have two triplets of positive numbers:

$$a_1, a_2, a_3,$$
$$b_1, b_2, b_3.$$

Consider the sum

$$S = a_1 b_i + a_2 b_j + a_3 b_k,$$

where i, j, and k represent the numbers 1, 2, and 3 assigned arbitrarily. How should we assign the numbers to obtain the largest (or smallest) sum S?

If the largest of the a numbers is multiplied by the largest of the b numbers, the median of a is multiplied by the median of b, and the smallest of a is multiplied by the smallest of b, then the sum S will have its largest possible value (if there are equal numbers in a or b, the largest value of S can be obtained in several different ways). If, on the other hand, the largest of a is multiplied by the smallest of b, the median of a is multiplied by the median of b, and so on, we will obtain the smallest possible value of S. This method enables us to solve the following problem.

Problem 1. Prove that for any positive numbers a, b, and c the following inequalities hold:

$$a+b+c \le \frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{c^2+a^2}{2b} \le \frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^2}{ab}. \quad (1)$$

Let us first prove the first inequality. Write two triplets of numbers

$$a^2, b^2, c^2,$$
$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$$

Since the numbers a, b, and c are positive, the largest number of the first triplet is greater than the smallest number of the second triplet, and the smallest number of the first triplet is greater than the largest number of the second triplet. Because of this, the sum

$$a^2\frac{1}{a}+b^2\frac{1}{b}+c^2\frac{1}{c}$$

is the smallest sum for the given triplets, and consequently,

$$a^2\frac{1}{a}+b^2\frac{1}{b}+c^2\frac{1}{c} \le a^2\frac{1}{b}+b^2\frac{1}{c}+c^2\frac{1}{a} \quad (2)$$

and

$$a^2\frac{1}{a}+b^2\frac{1}{b}+c^2\frac{1}{c} \le a^2\frac{1}{c}+b^2\frac{1}{a}+c^2\frac{1}{b}. \quad (3)$$

Combining inequalities (2) and (3), we obtain

$$2(a+b+c) \le a^2\frac{1}{b}+b^2\frac{1}{c}+c^2\frac{1}{a}+a^2\frac{1}{c}+b^2\frac{1}{a}+c^2\frac{1}{b}$$
$$=\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}.$$

To obtain the second of inequalities (1), let's consider the following triplets:

$$a^3, b^3, c^3,$$
$$\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}.$$

The largest (and respectively, smallest) numbers of the second triplet are written under the largest (and smallest) numbers of the first triplet. Consequently,

$$a^3\frac{a}{abc}+b^3\frac{b}{abc}+c^3\frac{c}{abc}$$
$$\ge a^3\frac{b}{abc}+b^3\frac{c}{abc}+c^3\frac{a}{abc}=\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}, \quad (4)$$

$$a^3\frac{a}{abc}+b^3\frac{b}{abc}+c^3\frac{c}{abc}$$
$$\ge a^3\frac{c}{abc}+b^3\frac{a}{abc}+c^3\frac{b}{abc}=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}. \quad (5)$$

Combining inequalities (4) and (5), we obtain

$$2\left(\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}\right) \ge \frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$
$$=\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}.$$

Exercises.

Prove the following inequalities:

1. $a^4 + b^4 \ge a^3b + ab^3$.
2. $a^3b + b^3c + c^3a \ge a^2bc + b^2ca + c^2ab$.
3. $\frac{a^3b}{c}+\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{b^3c}{a}+\frac{c^3a}{b}+\frac{c^3b}{a} \ge 6abc$

(here $a > 0$, $b > 0$, and $c > 0$).

4. If $a_1 \ge a_2 \ge a_3$ and $b_1 \ge b_2 \ge b_3$, then

$$3(a_1b_1 + a_2b_2 + a_3b_3) \ge (a_1 + a_2 + a_3)(b_1 + b_2 + b_3).$$

5. If $a > 0$, $b > 0$, and $c > 0$, then

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \ge \frac{3}{2}.$$

Some generalizations

Now let there be two sets of n numbers:

$$a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n,$$
$$b_1 \ge b_2 \ge b_3 \ge \dots \ge b_n. \quad (A)$$

Consider all possible sums of the form

$$\sigma = a_1b_{i_1} + a_2b_{i_2} + \dots + a_nb_{i_n}$$

where $i_1, i_2, \dots, i_n$ is some permutation of the numbers $1, 2, \dots, n$.

There is a finite number of such sums, so there must be a maximum S and a minimum s. It is easy to see that

$$S = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

and

$$s = a_1b_n + a_2b_{n-1} + \dots + a_nb_1.$$

Let's prove this fact. Note that for any four numbers a, b, c, and d such that $a \ge b$ and $c \ge d$, the following inequality holds:

$$ac + bd \ge ad + bc,$$

(since it is equivalent to the obvious inequality

$$(a-b)(c-d) \ge 0).$$

Now our assertion can easily be proved. Indeed, if some sum σ involves the terms $a_e b_q$ and $a_k b_p$ for which $e < k$ and $q > p$ (that is, $b_p \geq b_q$), then we can obtain a sum σ' that is not less than σ with the terms $a_e b_p$ and $a_k b_q$, by exchanging the numbers b_p and b_q.

Performing a series of such permutations, we can obtain the sum S such that $S \geq \sigma$. Similarly, if $b_q \geq b_p$, we can obtain a sum not greater than σ by exchanging these numbers. Since the sum s can be obtained as a result of a series of such permutations, we have $\sigma \geq s$. Thus, for any sum σ,

$$s \leq \sigma \leq S, \qquad \text{(B)}$$

and the equality in (B) is possible only if one of the sets $a_1, \ldots, a_n$ or $b_1, \ldots, b_n$ contains equal numbers.

Summing up, we can formulate a general method for proving inequalities: If the sets of numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are ordered identically—that is, if $a_k \geq a_l$ implies $b_k \geq b_l$, then

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \geq a_1b_{i_1} + a_2b_{i_2} + \ldots + a_nb_{i_n},$$

where $i_1, i_2, \ldots, i_n$ is an arbitrary permutation of the numbers $1, 2, \ldots, n$.

Some remarkable inequalities

1. Cauchy inequality:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n} = G.$$

Proof. We can assume that $a_1 \geq a_2 \geq \ldots \geq a_n$. Consider the sets of numbers

$$\frac{a_1}{G}; \frac{a_1a_2}{G^2}, \ldots, \frac{a_1a_2 \ldots a_n}{G^n} = 1,$$
$$\frac{G}{a_1}; \frac{G^2}{a_1a_2}, \ldots, \frac{G^n}{a_1 \ldots a_n} = 1.$$

These two sets are listed in opposite orders. Therefore,

$$n = \frac{a_1}{G} \cdot \frac{G}{a_1} + \frac{a_1a_2}{G^2} \frac{G^2}{a_1a_2} + \ldots + \frac{a_1 \ldots a_n}{G^n} \frac{G^n}{a_1 \ldots a_n}$$
$$\leq \frac{a_1}{G} \cdot 1 + \frac{a_1a_2}{G_2} \frac{G}{a_1} + \ldots + \frac{a_1a_2 \ldots a_n}{G^n} \frac{G^{n-1}}{a_1a_2 \ldots a_{n-1}}$$
$$= \frac{a_1 + a_2 + \ldots + a_n}{G}.$$

2. Chebyshev inequality. If

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$$

and

$$b_1 \geq b_2 \geq \ldots \geq b_n \geq 0,$$

then

$$n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n).$$

Proof. Combining n inequalities

$$a_1b_1 + \ldots + a_nb_n \geq a_1b_1 + a_2b_2 + \ldots + a_nb_n,$$

$$a_1b_1 + \ldots + a_nb_n \geq a_1b_2 + a_2b_3 + \ldots + a_nb_1,$$

$$a_1b_1 + \ldots + a_nb_n \geq a_1b_n + a_2b_1 + \ldots + a_nb_n,$$

we obtain what is desired.

Remark. Similarly, we can prove that if $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$, then

$$n(a_1b_i + a_2b_2 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n).$$

3. Mean-square inequality:

$$\sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

Proof. We can assume that $a_1 \geq a_2 \geq \ldots \geq a_n$. Using the Chebyshev inequality for the case of $a_i = b_i$ for all $i = 1, 2, \ldots, n$, we obtain

$$n\left(a_1^2 + a_2^2 + \ldots + a_n^2\right) \geq (a_1 + a_2 + \ldots + a_n)^2,$$

and the desired inequality can be easily obtained.

Exercises.

6. If $a_1, a_2, \ldots, a_n$ are the lengths of the sides of a convex polygon (where n is the number of sides, and $n \geq 3$), then

$$\frac{a_1}{p - 2a_1} + \frac{a_2}{p - 2a_2} + \ldots + \frac{a_n}{p - 2a_n} \geq \frac{n}{n-2},$$

where $p = a_1 + a_2 + \ldots + a_n$ is the perimeter of the polygon.

7. If $a_1, a_2, \ldots, a_n$ are nonnegative, then

$$n\left(a_1^{k+m} + a_2^{k+m} + \ldots + a_n^{k+m}\right) \geq \left(a_1^k + a_2^k + \ldots + a_n^k\right)\left(a_1^m + a_2^m + \ldots + a_n^m\right).$$

8. If a, b, and c are positive, then

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

9. (Generalization of problem 1.) If $a_1, a_2, \ldots, a_n$ are positive, then

$$(n-1)\left(a_1^m + a_2^m + \ldots + a_n^m\right) \leq \frac{a_2^k + a_3^k + \ldots + a_n^k}{a_1^{k-m}} + \frac{a_1^k + a_3^k + \ldots + a_n^k}{a_2^{k-m}} + \ldots + \frac{a_1^k + a_2^k + \ldots + a_{n-1}^k}{a_n^{k-m}},$$

where $k > m > 0$.

Suds studies

by P. Kanaev

EXPERIMENTS WITH SOAP films and bubbles can be done easily at home without sophisticated equipment. However, you will need to carefully wash the glassware used and skillfully prepare a good soap solution. The best solution is made from shampoo dissolved in water, with small amounts of pure glycerin and a strong solution of ammonium hydroxide added.

An empty ballpoint pen with its tip cut off can be a thin enough tube for blowing soap bubbles. It's better to blow small bubbles. Try not to leave drops of solution at the bottoms of the bubbles. A bubble can be easily released from the tube with a quick upward motion of your hand.

After you have prepared everything needed for the experiments and have practiced blowing soap bubbles, begin the experiments described below.

1. Blow a soap bubble, release it from the end of the tube, and immediately move away quickly—first backward, then to the left and the right. The bubble will follow you!

The bubble's behavior is explained by the creation of low air pressure zones during your quick movements—these are where the bubble goes.

2. Take a wide glass tube with a diameter of 2 cm or greater. Take a piece of foil and cut out a circle with a diameter a little larger than that of the tube. Wet the foil with soap solution and press it to the top of the tube. Sink the opposite end of the tube into a deep container filled with water. Soon you will see the foil cap open slightly—this is due to the soap bubble created under the cap by the compressed air in the tube. The deeper the tube sinks into the water, the larger the foil's angle of inclination will be. At some depth this angle becomes 90° (fig. 1).

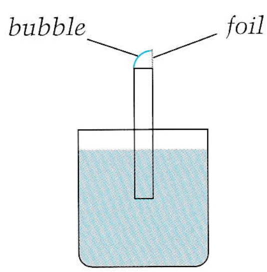


Figure 1

3. Carefully place a soap bubble on a flat, wet surface, such as water, paper, or glass. Because the surface is wet, the bottom part of the bubble will spread over the surface and the bubble will become a hemisphere.

Now make a flat soap film on a wire ring. Make a soap bubble with about the same diameter as the ring and place it on the film. Both the film and the bubble will change shape, and you will get a very thin, double-convex lens that is symmetrical with respect to the plane of the wire ring.

4. Dip the empty frame of a pair of glasses into the soap solution—the frame will be covered with two plane films. Look through these soap glasses, and you'll see things as they naturally appear—neither magnified nor attenuated. Visibility will also be normal.

Now modify the experiment. Holding the frames horizontally this time, dip them into the soap solution—you'll get glasses with double-convex lenses. Surprisingly, objects observed through these soap spectacles are not distorted either. Why?

There is air between the thin curved films, including these double-convex lenses. The light beams passing through such lenses are refracted very little.

5. Can you obtain a layer of soap film inside a test tube? That is, not at the top, but deep inside the tube? Here we show two ways of forming such a film.

(a) Fill part of the test tube with water. Take a strip of paper, wet it in the soap solution, and then drag it over the top of the test tube—the opening will be covered by a soap film. To sink the film down into the tube, tilt the test tube and pour out some of the water. The amount the film sinks is determined by the amount of water you pour out.

(b) Pour a little water into the test tube and heat it to boiling using a candle. Remove the test tube from the heat and cover the top of it with a soap film. After a while you'll see the film gradually sink down into the tube.

Water vapor condenses under the soap film during cooling and produces lower pressure in the tube, so the film moves. To accelerate this process, the tube can be cooled with running water.

6. Blow a bubble at the end of a thin tube. Using modeling clay, attach the free end of the tube to a horizontal beam secured at about 20 cm above a table. Using a syringe (or other thin tube), you can pump air in and out of the bubble. Will this distort the shape of the bubble? Certainly not. This follows directly from Pascal's law.

7. Find a glass flask or bottle about 6 cm tall and fill it two-thirds full of soap solution. Make two electrodes from copper wire about 8 cm long and insert them into a rubber stopper or cork. Insert the stopper into the neck of the bottle (fig. 2). The lower ends of the electrodes should be near the bottom of the jar, but they must not contact it. When the electrodes are connected to a flashlight battery, small bubbles of a gas will be released at the cathode (negative terminal) in the solution. The bubbles will rise and create a foam at the surface of the solution. What is the gas? Will bubbles be produced if alcohol, glycerin, or kerosene is poured into the jar instead of the water-soap solution?

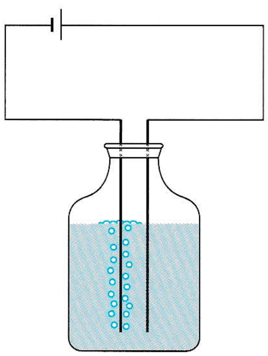


Figure 2

To answer these questions we only need to recall the phenomenon of water electrolysis. The gas released at the cathode is hydrogen. There will be no bubbles if alcohol, glycerin, or kerosene is used, because the products of electrolysis of these substances are quite different.

Bulletin Board

Imagine the Universe!

Dedicated to a discussion about our Universe, the "Imagine the Universe!" web site (http://imagine.gsfc.nasa.gov/docs/homepage.html) contains a wealth of astrophysics information. The site catalogs what we know about the Universe, how it is evolving, and the kinds of objects and phenomena it contains. Just as importantly, it also discusses how scientists know what they know, what mysteries remain, and how we might one day find the answers to these questions.

Features of the site include "Ask a NASA scientist," "Satellites and Data," "Teacher's Corner," "Other Good Resources," and "The Imagine Dictionary." Imagine the Universe! is a service of the High Energy Astrophysics Science Archive Research Center within the Laboratory for High Energy Physics at NASA/Goddard Space Flight Center.

Future physics teacher scholarships

The AAPT Executive Board offers a scholarship for future high school physics teachers. This scholarship is supported by an endowment funded by Barbara Lotze. Undergraduate students in, or planning to enter, physics teacher preparation curricula and high school seniors planning to enter such curricula are eligible. Successful applicants, normally one per year, will receive a stipend of up to $2,000. The scholarship may be granted to an individual for each of four years.

Applications will be accepted at any time and will be considered for recommendation to the Executive Board at each AAPT Winter Meeting. Applications for which all materials, including letters of recommendation, are received by the first day of December will be considered for recommendation at the following January meeting of the AAPT Executive Board.

Request materials from:
Programs Department
American Association of Physics Teachers
One Physics Ellipse
College Park, MD 20740
Phone: (301) 209-3300, ext. 5071
Fax: (301) 209-0845
E-mail: aapt-prog@aapt.org

Young Producers Contest

Earth and Sky Radio Series invites all K–12 students to enter the 1999 Young Producers Contest. Participants submit a 90-second radio program on a science or nature topic of their choosing. Entries will be judged on content (is it accurate?), presentation (is it engaging?), and production (is it clearly heard?).

The five winning teams will have their programs aired on Earth and Sky in April 1999, during National Science and Technology Week. The grand prize winning team will receive a $1,000 U.S. Savings Bond (or equivalent amount in an international check), and the four other winning teams will each receive a $500 bond or international check. Winners will be chosen from a variety of age groups. In addition, all participants will receive a certificate stating that the student is "A Young Producer for Earth and Sky."

Earth and Sky is a daily radio series that broadcasts on over 950 public and commercial radio stations in all 50 states. It is also heard in Canada, the South Pacific, and on many international networks, including Armed Forces Radio, Voice of America, and World Radio Network.

Entries must be postmarked by December 15, 1998. For contest guidelines, samples of past winners, resource materials, and entry forms, e-mail *contest@earthsky.com*, or visit Earth and Sky online at *www.earthsky.com*. You can also send a self-addressed, stamped envelope to Young Producers, P.O. Box 2203, Austin, TX 78768.

River runners

This month's CyberTeaser (B238 in this issue) led contestants on a trickier voyage than the problem first suggested. Fortunately, it was smooth sailing for the following expert navigators, who floated us the first 10 correct responses to our "current" question:

Alex Wissner-Gross (New Hyde Park, New York)
Natalia Toro (Boulder, Colorado)
Leo Borovskiy (Brooklyn, New York)
Jaak Sarv (Tallinn, Estonia)
Bruno Konder (Rio de Janeiro, Brazil)
Jim Grady (Branchburg, New Jersey)
John Beam (Bellaire, Texas)
Melania Drozdzewicz (Thornton, Colorado)
Theo Koupelis (Wausau, Wisconsin)
Quek Dingfeng (Singapore)

Congratulations! Each of our winners will receive a *Quantum* button and copy of the July/August issue. Everyone who submitted a correct answer in the time allotted was entered in a drawing for a copy of *Quantum Quandaries*, our collection of the first 100 *Quantum* brainteasers.

ANSWERS, HINTS & SOLUTIONS

Math

M236

Let E_1 be the point symmetric to E with respect to L (fig. 1). Then $KE = ME_1$. Also, segment $AF = FM$ and segment $AE = KE$ (since the corresponding arcs are equal). If we take point P on the extension of AB past A, then $\angle E_1ML = \angle EKL = \angle EAP$ (the last equality follows from the properties of the angles of an inscribed quadrilateral). Similarly, $\angle PAF = \angle LMF$. Thus, $\angle FME_1 = \angle FAE$, triangles FME_1 and FAE are congruent, and $EF = FE_1$. So, FL, the median of the isosceles triangle EFE_1, is perpendicular to EE_1. (To make the reasoning complete, we should consider configurations other than the one shown in fig. 1).

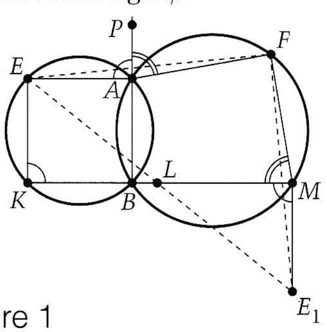


Figure 1

M237

It is easy to see that the required locus includes all the points lying on diagonals AC and BD of the diamond. In fact, if M is a point on AC (fig. 2), then, because the diamond is

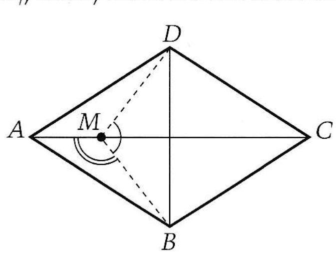


Figure 2

symmetric with respect to AC, we have

$$\angle AMB + \angle CMD = \angle AMD + \angle CMD = 180°.$$

Now let's show that there are no other points in this locus. Suppose that a point M that doesn't lie on diagonals AC and BD satisfies the conditions. Let's draw a circle through points A, B, and M. Denote the points where the circle meets diagonals AC and BD by M_1 and M_2, respectively (fig. 3). Then points M_1, M_2, and M

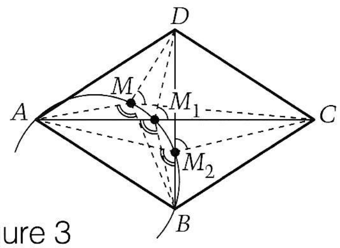


Figure 3

belong to the required locus. And, since $\angle AMB = \angle AM_1B = \angle AM_2B$ (because they are inscribed angles intercepting the same arc), we conclude that $\angle CMD = \angle CM_1D = \angle CM_2D$. Therefore, points C, D, M, M_1, and M_2 lie on one circle, too. That is, we've found two different circles that have three common points, which is impossible.

M238

If $\log_{2n-1}(n^2 + 2) = p/q$, then $(n^2 + 2)^q = (2n - 1)^p$, and thus all the prime factors of the numbers $n^2 + 2$ and $2n - 1$ coincide. So, the fraction

$$\frac{n^2+2}{2n-1}$$

is reducible. Of course, the fraction

$$\frac{4n^2+8}{2n-1} = \frac{4n^2-1+9}{2n-1} = 2n+1+\frac{9}{2n-1}$$

must also be reducible. This means that $9/(2n - 1)$ is a reducible fraction. Moreover, all the prime factors of 9 and $2n - 1$ coincide. There are only two opportunities now: $2n - 1 = 3$ and $2n - 1 = 9$, which simplify to $n = 2$ and $n = 5$. If $n = 2$, we have $\log_3 6 = 1 + \log_3 2$. This number is clearly irrational. If $n = 5$, we get $\log_9 27 = 3/2$.

M239

Let (x_0, y_0) be the solution of our system. The conditions of the problem imply that the parabolas $y = x^2 + a$ and $x = y^2 + b$ touch each other at the point (x_0, y_0)—that is, their tangents at this point coincide. Let's find the slope of this tangent by taking the derivatives of both functions at this point. For the first one we have

$$y'_{x=x_0} = 2x$$

and for the second one,

$$y'_{x=x_0} = \frac{1}{x'_{y=y_0}} = \frac{1}{2y_0}$$

(here y'_x denotes the derivative of y as a function of x, and x'_y denotes the derivative of x as a function of y). So, $2x_0 = 1/(2y_0)$, and thus $4x_0y_0 = 1$. We see that the point (x_0, y_0) lies on the hyperbola determined by the equation $4x_0y_0 = 1$. Clearly, we should take only those points of this hyperbola that belong to the first quadrant. (To convince yourself that this is so, draw a few examples of parabolas described by the given equations.)

M240

We suppose the student is located at point O and draw a circle of radius 2 km centered at O. It is given that this circle either intersects the border of the forest or is tangent to it, so if we walk out to the edge of the circle, then walk around it, we will encounter the edge of the forest. But this path is too long.

We can improve on this by noting that our path must intersect every tangent to the circle, but not necessarily at the point of contact. We can construct such a path, of length less than 13, as follows. We take a point A situated $4/\sqrt{3}$ miles from O and draw the two tangents to the circle from A (see diagram). Let B be the point of contact of one of these tangents. It is not hard to see that $\angle OAB$ measures $\pi/6$. We proceed from O to

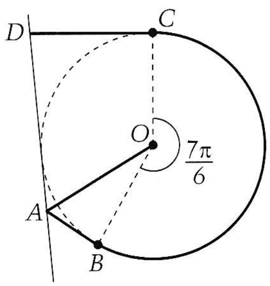


Figure 4

A to B along these line segments, then around the circle to point C so that the arc BC has measure $7\pi/6$ (and length $7\pi/3$ miles). Then we look for the other tangent from A to the circle, and drop a perpendicular from C to this line. If the foot of the perpendicular is D, then we complete our path by walking from C to D. This path intersects each tangent to the circle, and its length is

$$\frac{4}{\sqrt{3}}+\frac{2}{\sqrt{3}}+2\frac{7\pi}{6}+2$$
$$=2\sqrt{3}+\frac{7\pi}{3}+2<2\cdot 1.75+\frac{7}{3}3.15+2$$
$$=12.8543<<13 \text{ miles.}$$

And, since it intersects all the lines tangent to the circle with center at O and a radius of 2 miles, this path would inevitably lead the student out of the forest.

Physics

P236

Imagine that at some moment the vehicle's camera "found" a crater on the lunar surface and sent a "report" on this observation back to Earth. Some period of time will elapse before the vehicle can receive an appropriate command:

$$t=2\frac{l}{c}+\tau$$
$$=2\frac{380\cdot 10^3 \text{ km}}{300\cdot 10^3 \text{ km/s}}+0.1 \text{ s}\sim 2.6 \text{ s}.$$

Here l is the distance between Earth and the Moon, c is the speed of the radio signal, and τ is the time necessary for the engineers to decide on a command.

To estimate the vehicle's speed, we assume that it is less than the speed of a car on a country road (about 20 km/h) by the same factor that characterizes the difference between the time necessary to transmit a command to the lunar vehicle and the reaction time of a driver on Earth. Accordingly, the maximum speed of such a lunar vehicle is

$$v_{\text{L}}\sim v\frac{\tau}{t}\sim 1 \text{ km/h}.$$

P237

If a body moves along a circle of radius r with velocity v under the attractive force of a central body of mass m, the centripetal acceleration v^2/r equals that produced by gravity Gmr^2 (G is the gravitational constant):

$$\frac{v^2}{r}=\frac{Gm}{r^2}.$$

As the period of revolution is $t = 2\pi r/v$, we get a formula for the central body's mass:

$$m=\frac{4\pi^2 r^3}{Gt^2}.$$

By comparing Earth's motion around the Sun (with the period T_0 = 1 year) with the stellar motion about the Galaxy's center, we obtain the total Galaxy's mass M inside the sphere of radius R:

$$M_0=\frac{4\pi^2 R_0^3}{GT_0^2},\ M=\frac{4\pi^2 R^3}{GT^2},$$
$$M=M_0\frac{R^3}{R_0^3}\frac{T_0^2}{T^2}=1.9\cdot 10^{11} M_0.$$

Thus, the invisible mass of Galaxy is

$$\Delta M=M-M_1=4\cdot 10^{10} M_0.$$

P238

The temperature difference between the refrigerator's interior and the surrounding air ΔT and the duration of the idle period τ_2 are related by the formula

$$\tau_2\Delta T=\text{const.} \qquad (1)$$

Another equation is valid for an ideal heat engine:

$$\frac{W}{Q}=\frac{\Delta T}{T},$$

where W is the motor's work and Q is the amount of heat extracted from a body inside the chamber. The work performed by the motor is proportional to the time τ_1 the motor is on

$$W\sim \tau_1,$$

and the amount of extracted heat is proportional to the entire period of the refrigerator's cycle $(\tau_1+\tau_2)$ and to the temperature difference ΔT:

$$Q\sim(\tau_1+\tau_2)\Delta T.$$

Therefore,

$$\frac{\tau_1+\tau_2}{\tau_1}(\Delta T)^2=\text{const.} \qquad (2)$$

Inserting the values of τ_1, τ_2, and ΔT corresponding to the first case, and $\Delta T'$ corresponding to the second case into equations (1) and (2), we find

$$\tau_1'=2 \text{ min},\ \tau_2'=4.1 \text{ min}.$$

The maximum temperature in the room corresponds to the condition $\tau_2''=0$:

$$\Delta T_{\max}=\Delta T\sqrt{\frac{8}{5}}=46.8 \text{ K}.$$

Thus,

$$t_{max} = 34.8°C.$$

P239

We assume that all rods make electrical contact with both rails. The mutual contacts between the rods do not affect the answer to the problem, because the contacting points have the same potential.

The resistance of an individual rod making an angle α with the rails $(0 < \alpha < \pi)$ is given by

$$r_i = \rho \frac{l_i}{S} = \rho \frac{\dfrac{l}{\sin\alpha_i}}{\dfrac{\pi d^2}{4}}.$$

The net resistance between the rails is given by

$$\frac{1}{R} = \sum \frac{1}{r_i} = \sum \frac{\pi d^2 \sin\alpha_i}{4\rho l} = \frac{\pi d^2}{4\rho l} N (\sin\alpha_i)_{mean}.$$

Because the copper wires are long compared to the distance between the rails, we can estimate the mean value of $\sin\alpha_i$ in the following way:

$$(\sin\alpha_i)_{mean} = \frac{1}{\pi}\int_0^\pi \sin\alpha \cdot d\alpha = \frac{2}{\pi}.$$

Thus,

$$\frac{1}{R} = \frac{\pi d^2}{4\rho l} N \frac{2}{\pi} = \frac{d^2 N}{2\rho l},$$

and finally,

$$R = \frac{2\rho l}{d^2 N} \approx 4 \cdot 10^{-4}\ \Omega.$$

Note: We obtained the mean value of net resistance. A particular value of the resistance depends on how the rods have fallen on the rails. Therefore, it will differ from the mean value. But how much will it differ? Try to estimate the scatter of resistance values using a random number generator.

P240

Let time t elapse during the UFO flight from point A (nearest to the observer) to point B, when the UFO is located at angle ϕ. We assume that this period is timed by a watch located at point B. Denoting the altitude of the UFO over Earth by l, we get the path traveled by the UFO from A to B:

$$vt = l \tan\phi \qquad (1)$$

An observer will see the shining object at points A and B somewhat later due to the finiteness of the speed of light c, so his watch will show the time

$$t_1 = t + \frac{l}{c\cos\phi} - \frac{l}{c}. \qquad (2)$$

The velocity of the UFO as measured by the observer is thus

$$v_1 = \frac{dx}{dt_1} = \frac{dx}{dt}\frac{dt}{dt_1} = \frac{v}{\dfrac{dt_1}{dt}}.$$

Differentiating equations (2) and (1) yields

$$\frac{dt_1}{dt} = 1 + \frac{l\sin\phi}{c\cos^2\phi}\frac{d\phi}{dt},$$

and

$$\frac{d\phi}{dt} = \frac{v\cos^2\phi}{l}.$$

Therefore, the velocity we are looking for is

$$v_1 = \frac{v}{1 + \dfrac{v}{c}\sin\phi}.$$

Brainteasers

B236

See figure 5.

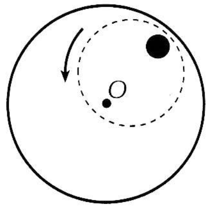


Figure 5

B237

Of the 14 children with brown eyes, what is the smallest number that have dark hair? Well, if the remaining 6 children (without brown eyes) have dark hair, there must be 9 children with both brown eyes and dark hair. Of these 9, how many must weigh more than 80 lbs.? Again, there are 3 students who do not weigh more than 80 lbs., and if these are among the 9, there must be 6 more with three of the four characteristics. Finally, there are only 2 students who are not more than 4 feet tall, so there must be 4 students (of the 6 with three characteristics) who have all four characteristics.

B238

Let V be the speed of the boat and v be the speed of the current. Then the distance between the boat and the raft grew at the rate $(V + v) - v = V$, when the boat was going to B (here $V + v$ is the velocity of the boat, taking the speed of the flow into consideration). When the boat was going from B to A, the distance between it and the raft decreased at the same rate: $(V - v) + v = V$. So, when they met, the time during which the distance between them increased was equal to the time during which it decreased: 1 hour.

B239

One can divide each bar, regardless of the percentage of gold in it, in the proportion 1:2:3.

B240

Snow is composed of many ice crystals, so the Sun is reflected from a vast number of small mirrors. Some of them send light directly to our eyes, and when we move, one set of mirrors is replaced by another. We perceive this as sparkling.

What's wrong?

1. The weight of the watermelons decreased twofold, to 5 tons. Many people think this a miracle.

2. It is a mistake to add 3 dollars to 13. If we do so, we count the 3 dollars squandered by the servant twice. In fact, $13 = 10 + 3$, where 10 dollars is the money received by the general and 3 dollars is the money wasted by the servant.

3. If the line BB_2 passes through the point C, our reasoning is false (fig. 6). Angles CBB_2 and CB_2B are

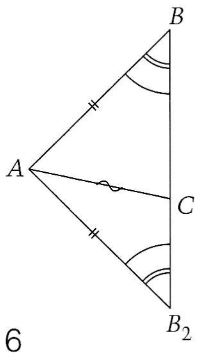


Figure 6

equal, but they are equal to 0. In fact, the reader is invited to examine those cases where two noncongruent triangles have the same two sides and nonincluded angle. She or he will find that it is in exactly these cases that the line BB_1 passes through point C. Thus, we cannot use the feature of an isosceles triangle.

4. There is a case in which the centers of both circles mentioned in the conditions coincide with the center of the parallelogram. Then the parallelogram becomes a rectangle. Therefore, the problem has a second answer: 90°.

5. The problem does not say that the equation has no other roots except p and q. The problem has one more answer: $p = q = -1/2$.

6. When we transform the equation in this way, we narrow down the domain of the functions that appear in it, and the following series of solutions is lost: $x = \pi/2 + \pi n$.

7. We can easily check that the numbers 1/2 and 1/4 satisfy the equation. These two solutions correspond to the points (1/2, 1/4) and

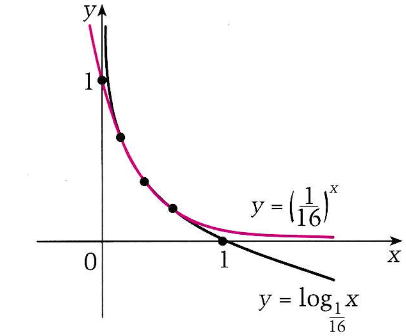


Figure 7

(1/4, 1/2) on the graphs of the functions $y = \log_{1/16} x$ and $y = (1/16)^x$, which are symmetric to each other with respect to the bisector of the first and the third quadrants. Besides this, these graphs intersect at this bisector. Thus, the equation has at least three solutions. As a matter of fact, both graphs cling tightly to the coordinate axes (see fig. 7), so it is quite possible that they intersect more than once.

It is not difficult to prove with the help of differential calculus that the equation has exactly three solutions. In general, an equation $\log_a x = a^x$ has no more than three solutions. (The proof is based on the well-known theorem that says that the derivative of a function vanishes at least once between any two zeros of the function.)

8. All the sections of a cone that pass through its vertex are isosceles triangles, whose equal sides are all the same (they are lateral elements of the cone). If α is the angle at the vertex of an axial section and ϕ is the angle between the lateral sides of an arbitrary section, then $0 < \phi \le \alpha$. But the area of such a section is proportional to $\sin \phi$. Therefore, if $\alpha \le 90°$, then the axial section is the one with the largest area. But, if $\alpha > 90°$, then the section with the largest area is the section for which $\phi = 90°$. The conditions of the problem mean that $\alpha > 90°$ and $\sin \alpha = 1/2$, from which we conclude that $\alpha = 150°$.

9. The correct conclusion is that the second sphere lies completely within the first one so that 5 is just the ratio of the surface of the second sphere to that of the first. Thus the ratio of their radii is $\sqrt{5}$. On the other hand, if the ratio given in the conditions were less than 4 and were equal, for example, to 3, then the problem would have no solution.

10. Let's consider a quadrilateral $ABCD$ in which $AB = BC = 10$ and $AD = DC = 6$ (fig. 8). The angles at the vertices A and C are equal. The area of the quadrilateral is a maximum when these angles are right. Thus the greatest possible area of the base is 60. But this is less than $112/\sqrt{3}$, the area we found when we solved the problem (it is easy to check this).

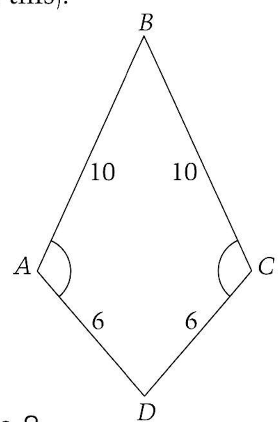


Figure 8

So, does the problem have no solution? The statement of the problem does not imply that the projection of the vertex of the pyramid falls exactly in the center of the circle inscribed in $ABCD$. This implies only that it falls at a point equidistant from the lines AB, BC, CD, and DA and that this distance is $7/\sqrt{3}$. It is possible that this point lies outside $ABCD$. Denote such a point by O_1 (fig. 9). Then the area of $ABCD$ could be represented as the sum of the areas of triangles ABO_1 and BCO_1 (they are equal) minus the areas of triangles CDO_1 and

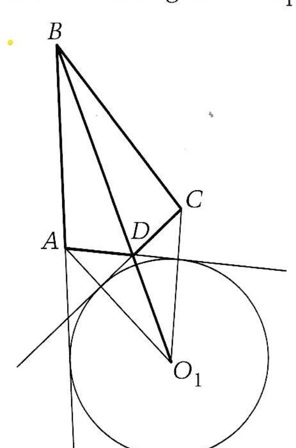


Figure 9

ADO_1 (they are also equal). That is, it equals $(10-6)\cdot 7\sqrt{3} = 28/\sqrt{3}$. Thus, the volume of the pyramid is $196/\sqrt{3}$.

Gradus

Problem 1. If we let $x = 1$, the value of $150x^2 - 77x - 73$ is 0. Thus $x - 1$ is a factor. Using division of polynomials, or otherwise, we quickly find out that the other factor is $150x + 73$.

Problem 2. The given polynomial vanishes when $x = -1$. Thus one factor is $x + 1$, and the other factor turns out to be $x^2 + 14x + 1$.

Problem 7. The polynomial vanishes when $x = -1$. This allows us to find the factorization

$$(x + 1)[x^2 + (a - 1)x + 1].$$

Problem 8. Since the polynomial vanishes when $x = a$, one factor is $x - a$. Thus we have the factorization $(x - a)(x^2 + 2a)$.

Problem 9. The polynomial vanishes when $x = y$, so we get the factorization

$$(x - y)(x^3 - 5x^2y - 5xy^2 - y^3).$$

Problem 11. Since $(a - b)^3 = -(b - a)^3$, the expression vanishes when $a = b$, and also when $a = c$ and $b = c$. So once again we can write

$$(a - b)^3 + (b - c)^3 + (c - a)^3 = (a - b)(b - c)(a - c)M,$$

where M is a polynomial in a, b, and c. It is somewhat surprising, but still true, that the original expression is quadratic (and not cubic) in a, and so is the expression

$$(a - b)(b - c)(a - c).$$

And of course (by symmetry) the same holds for b and c. It follows once more that M is a constant, and some judicious plugging in of numbers (try $a = 3$, $b = 2$, $c = 1$) will show that $M = -3$.

Problem 12. The expression vanishes if $a = -b$, if $a = -c$ and if $b = -c$. Thus it has factors $(a + b)(b + c)(a + c)$. An argument similar to those used in the previous solutions lets us conclude that

$$(a + b + c)^3 - (a^3 + b^3 + c^3) = 3(a + b)(a + c)(b + c).$$

Problem 13. Following the hint, we note that the equation is quadratic in x. Furthermore, it is true of $x = a$, $x = b$, and $x = c$. If any two of these values are equal, the right side of the given equation has no sense. Thus we are looking at a quadratic equation satisfied by three different numbers, which must therefore be an identity.

Problem 14. As in problem 13, we can think of this as an equation in x, and once again it is quadratic. It is satisfied when $x = a$, b, or c, and exactly the same reasoning as in problem 13 leads to the desired conclusion.

Problem 15. We will show that both assertions in the problem (for exponent m and for exponent n) are equivalent to the statement that there are two "opposite" numbers among a, b, and c (that is, $a = -b$, or $b = -c$, or $c = -a$).

Certainly, if the set $\{a, b, c\}$ contains two opposite numbers, then for any odd exponent k,

$$\frac{1}{a^k + b^k + c^k} = \frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k}.$$

Let us prove the converse.

Following the given hints, we suppose the cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a^k, b^k, and c^k. Then, since

$$\frac{1}{a^k + b^k + c^k} = \frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k},$$

we know that $r = pq$, so that the equation has the form

$$x^3 - px^2 + qx - pq = 0.$$

The left side vanishes when $x = p$, and so factors into $(x - p)(x^2 - q)$. Thus one root of the equation is $x = p$. This means that $a^k + b^k + c^k$ has one of the values a^k, b^k, or c^k. If $a^k + b^k + c^k = a^k$, then $b^k + c^k = 0$, so $b^k = -c^k$, and (since k is odd), $b = -c$. A similar conclusion follows if $a^k + b^k + c^k = b^k$ or c^k.

If this solution is difficult to read, try formulating the problem with $m = 1$ first, then look at the general situation.

Problem 16. Method I, using hint 1. When $x = -(y + z)$, a simple computation shows that the polynomial vanishes. So $x + y + z$ is a factor, and the other factor can be obtained by long division.

Method II, using hint 2. A computation will show that

$$x^3 + y^3 + z^3 - 3xyz$$

is indeed equal to

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Then, computing with determinants, we find

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} - \begin{vmatrix} x+y+z & y & z \\ x+y+z & x & y \\ x+y+z & z & x \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix}$$

$$= (x+y+z)\left(x^2+y^2+z^2-xy-yz-xz\right).$$

Problem 17. As in problem 16, we write

$$a^3 + b^3 + c^3 - 3abc = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Multiplying the first column by 100, the second by 10, and adding these to the third column, we find that

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 100a+10b+c & b & c \\ 100c+10a+b & a & b \\ 100b+10c+a & c & a \end{vmatrix},$$

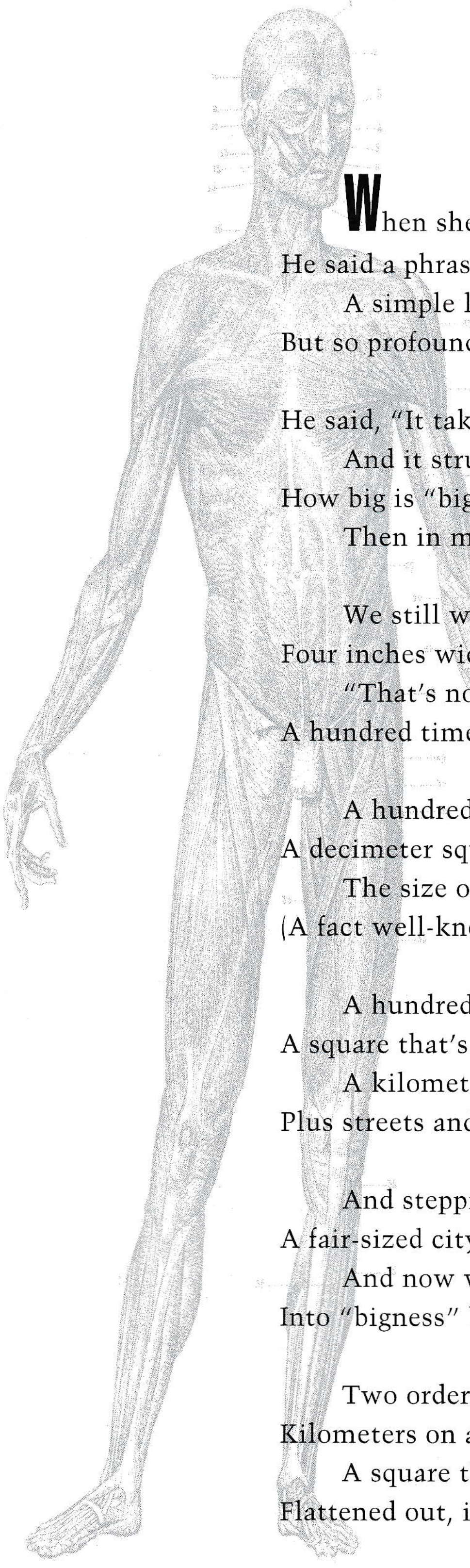
so

$$a^3 + b^3 + c^3 - 3abc = \begin{vmatrix} \overline{abc} & b & c \\ \overline{cab} & a & b \\ \overline{bca} & c & a \end{vmatrix}.$$

And since the numbers $\overline{abc}$, $\overline{bca}$, and $\overline{cab}$ are all divisible by n, so is $a^3 + b^3 + c^3 - 3abc$.

How big am I, really?

by David Arns



When sheepishly apologizing to a friend I'd wronged,
He said a phrase that got me started thinking:
A simple little phrase that I had heard my whole life long,
But so profound, I stood agape and blinking.

He said, "It takes a big man to admit that he is wrong,"
And it struck me: What, exactly, does that mean?
How big is "big?" And to what grouping does that word belong?
Then in my mind I saw the following scene:

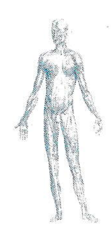
We still were shaking hands, and I beheld the hand I shook,
Four inches wide it was in breadth of beam.
"That's not real big," I thought, "But let's just see how it would look
A hundred times as large..." Thus went my dream.

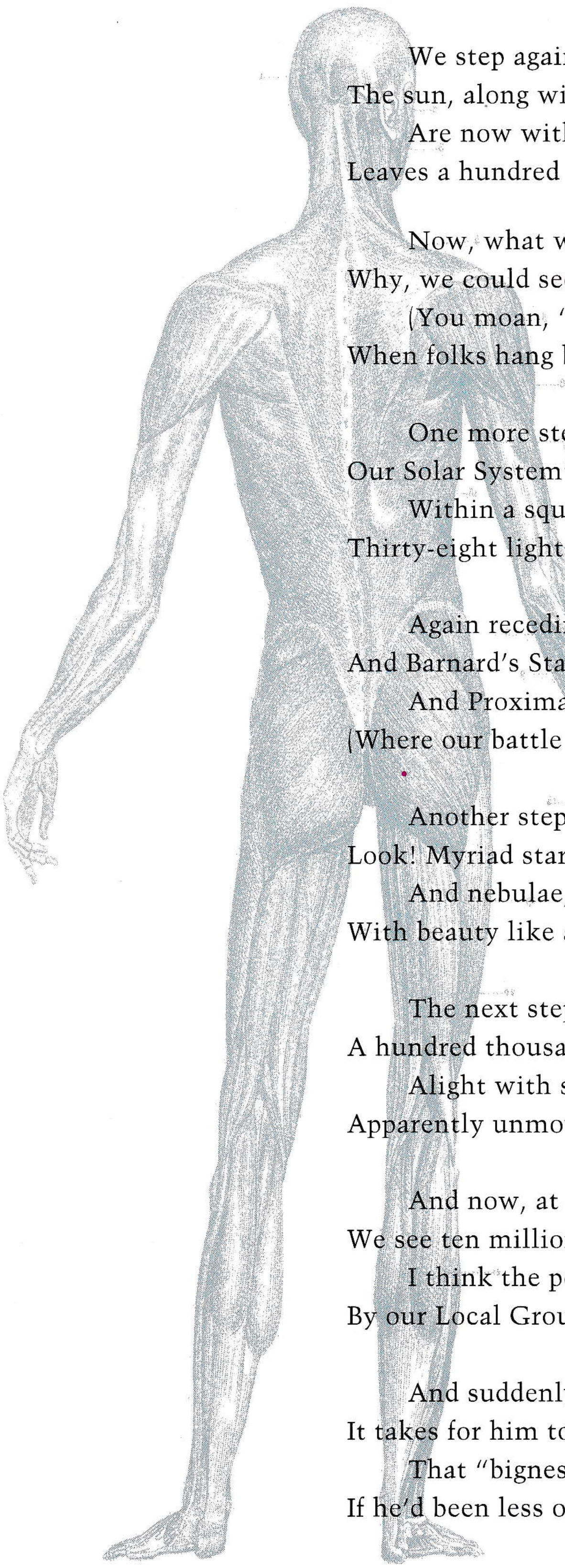
A hundred times as long and a hundred times as wide,
A decimeter square would roughly be
The size of half a tennis court: ten meters on a side
(A fact well-known by Andre Agassi).

A hundred times as big again, and what would be in view?
A square that's several city blocks in size,
A kilometer square containing buildings old and new,
Plus streets and homes, delightful to the eyes.

And stepping back another step, one hundred times as large,
A fair-sized city fits within its border.
And now we're getting large enough, our rash and headlong charge
Into "bigness" becomes quite an arduous order.

Two orders more of magnitude, arriving at ten thousand
Kilometers on a side, and we've unfurled
A square that, when it's stretched out to the limits it allows and
Flattened out, it almost covers up our world.





With one more step, receding back, enlarged a hundred times,
("Stay with me, please," you hear me importune),
The more we see, you'll notice, as our viewpoint ever climbs:
We've just contained the orbit of the Moon.

We step again: the square is five light-minutes on each side;
The sun, along with Mercury and Venus
Are now within our field of view; our stimulating ride
Leaves a hundred billion meters in between us.

Now, what would happen if we took another backward step?
Why, we could see th' entire Solar System!
(You moan, "Must we continue on this ride?" My answer's "Yep!"
When folks hang back, I offer to assist 'em.")

One more step back, and now we see the emptiness around:
Our Solar System's just a tiny dot
Within a square where almost nothing else is to be found
Thirty-eight light-days of next to naught!

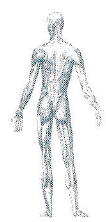
Again receding back along our logarithmic line,
And Barnard's Star is now in evidence,
And Proxima Centauri and, of course, Wolf 359
(Where our battle with the Borg was quite intense).

Another step: Our square is now a thousand light-years broad:
Look! Myriad stars, and clusters of the same!
And nebulae, appearing painted by the hand of God,
With beauty like a multicolored flame.

The next step, our penultimate: We see the Milky Way,
A hundred thousand light-years, stem to stern.
Alight with stellar objects in a dazzling array
Apparently unmoving and eterne.

And now, at last, we take our final weary step arrears:
We see ten million light-years at a glance;
I think the point's been driven home—at least it so appears—
By our Local Group within this great expanse.

And suddenly, that phrase comes back: about how "big" a man
It takes for him to say that he was wrong—
That "bigness" seems quite silly now; he's surely smaller than
If he'd been less offensive all along.



David Arns *is a graphics software documentation engineer for Hewlett-Packard in Fort Collins, Colorado, and also operates a small business designing and creating web sites. In his spare time he dabbles in poetry on scientific themes.*

INDEX

Volume 8 (1997–98)

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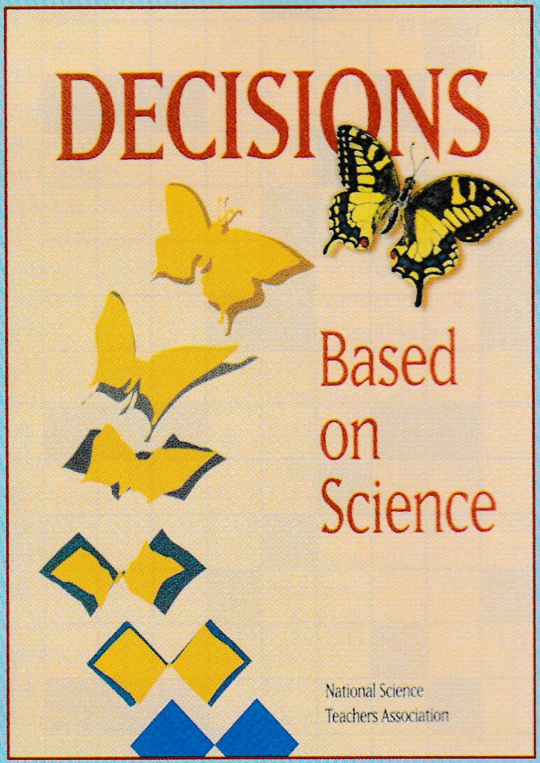

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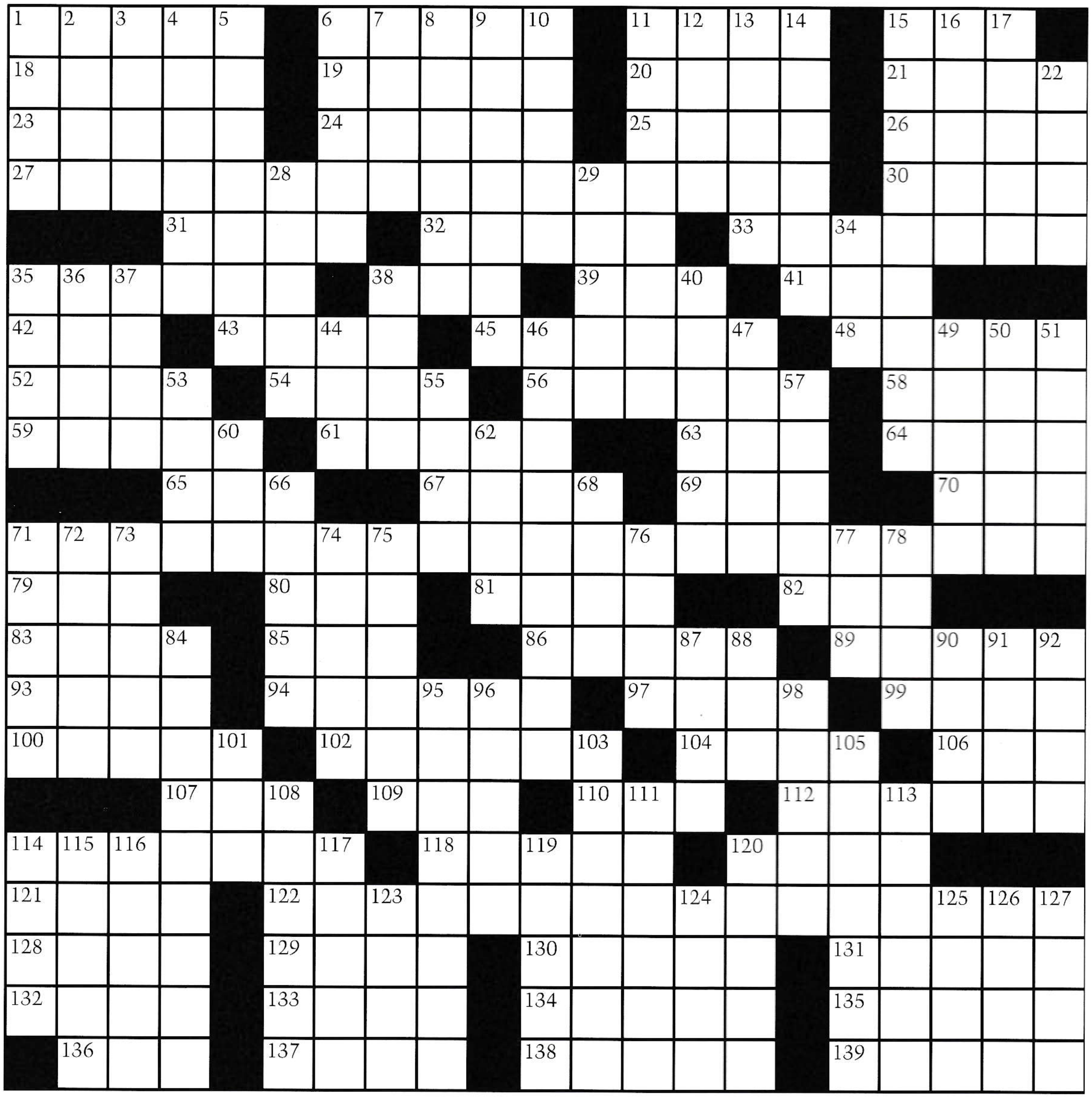
Criss × cross science

by David R. Martin



Across

1 Sorrow
6 Young male horses
11 Width times length
15 Asian country (slang)
18 Sumerian moon god
19 Thin as ___
20 A geometric shape: abbr.
21 Length unit
23 Flower oil
24 Writer Bret
25 French director Jacques ___ (1908–1982)
26 Quality: suff.
27 Phonograph inventor
30 Mallophaga
31 First garden
32 Astronomer ___ Pannekoek (1873–1960)
33 Hertzsprung-___ diagram
35 Directionless quantity
38 Muhammad ___
39 Zip
41 Brother or sister
42 100 square meters
43 French artist ___ Tanguy (1900–1955)
45 Homes
48 Type of boom
52 10^{12}: pref.
54 1946 physiol. Nobelist Antonio ___ Moniz
56 Of human waste
58 Corned beef ___
59 Organic compounds

61 ___ circle (equator, e.g.)
63 That woman
64 Roster
65 Solution: abbr.
67 Australian poet ___ Hope
69 Collection of anecdotes
70 ___-isomer
71 Cyclotron inventor
79 ___ group (of topology)
80 Poet's before
81 Geologist Reginald A. ___ (1871–1957)
82 Resinous insect secretion
83 Egg cell
85 Poet's even
86 Family member
89 First ___ (QB's concerns)
93 Round: pref.
94 Finished second
97 X followers
99 Lithium hydroxide: abbr.
100 Eldest son of Cain
102 Begins
104 Coral ridge
106 Nitrilotriacetic acid
107 Possesses
109 Sphere
110 "___ Got a Secret"
112 Archaeologist Richard
114 ___ relativity
118 Kind of thread
120 ___ group (chem. group)
121 ___ sphincter (certain muscle)
122 Electricity pioneer
128 Makes lace
129 Indonesian islands
130 Practitioner: suff.
131 New England state
132 Opp. of endo
133 Race: comb. form
134 Metric mass unit
135 ___ malaria
136 Trig. function
137 Type of carpet
138 ___-Recklinghausen disease
139 Dice roll

Down

1 Small insect
2 A Dravidian cave temple
3 Interested in
4 Type of paint
5 Electromagnetic induction discoverer
6 Newspaper editor Abraham ___ (1869–1951)
7 Type of exam
8 Of a young insect
9 Uranus satellite
10 Frozen rain
11 ___ series of elements
12 Inlets
13 Ozone temperature
14 Gland sac
15 Quantum physics pioneer
16 Astronomer ___ Cannon (1863–1941)
17 1350 to Caesar
22 Foot part
28 Be a waiter
29 Semiconductor atom
34 Sibling: abbr.
35 Satisfy
36 Spring: comb. form
37 Of aircraft
38 Esker
40 "___ Weapon"
44 Zygote
46 1939 chem. Nobelist Adolf ___
47 Tuscan commune
49 Sodium cyanate
50 Characteristic of: suff.
51 Pursue
53 Wings
55 Otariidae member
57 Swimming stroke
60 Stannous sulfide
62 "When I was ___"
66 Precipitous
68 451 to Brutus
71 Run away
72 Torn apart
73 Nerve: comb. form
74 Hershiser and namesakes
75 1975 physiol. Nobelist ___ Dulbecco
76 ___ and terminer
77 One hundredth of a gray
78 Environmental sci.
84 Light speed researcher
87 Concern
88 Sense organ
90 Blink
91 Musical sound
92 Chaise
95 Archetypical psychologist
96 Er_2O_3
98 City in Alabama
101 1968 physiol. Nobelist ___ Gobind Khorana
103 Element 14
105 1965 physics Nobelist Richard ___
108 Swords
111 Geophysicist Felix ___ Meinisz (1887–1966)
113 Type of metal
114 Logic circuit
115 Make into law
116 1963 chem. Nobelist Giulio ___
117 Port near Edinburgh
119 Strike hard
120 Synthetic fiber
123 Okinawan seaport
124 Temple (archaic)
125 Wood: comb. form
126 Arrow poison
127 Beatty and Rorem

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE MAY/JUNE PUZZLE

C	F	D	A		A	L	D	E		U	K	A	S	E
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C	L	A	D	E		A	R	T	E		E	R	R	S

Barn again

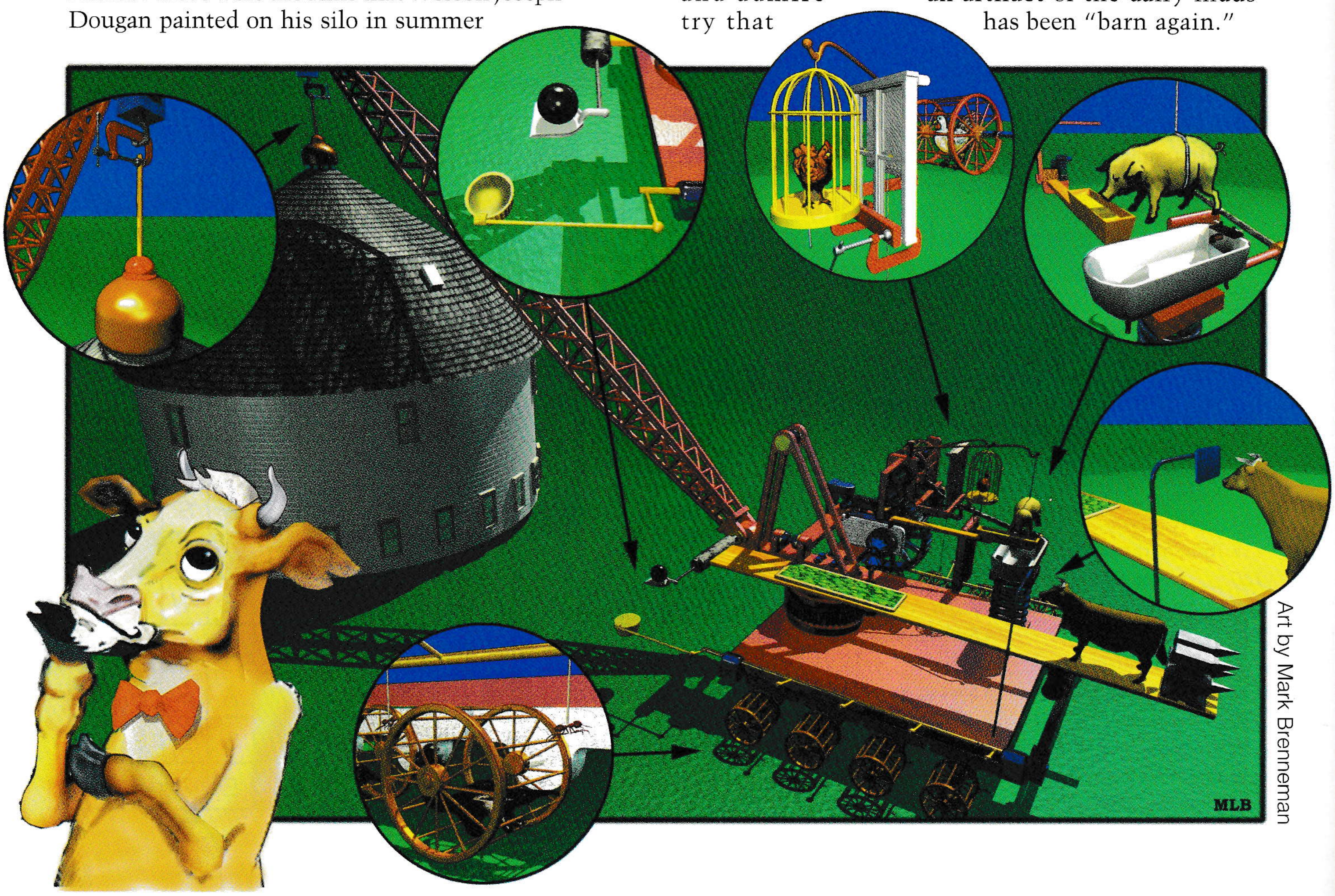
by Dr. Mu

WELCOME BACK TO COWCULATIONS, THE column devoted to problems best solved with a computer algorithm. This year marks the 150th anniversary of Wisconsin's statehood. The sesquicentennial celebration is a time to reflect on our past and those who first immigrated to this fertile land in the Midwest to carve out a living. Their aims were modest:

1. Good Crops,
2. Proper Storage,
3. Profitable Livestock,
4. A Stable Market, and
5. Life as Well as a Living.

At least those were the aims that Wesson Joseph Dougan painted on his silo in summer 1911, as he finished building a magnificent round barn on his dairy farm in Beloit, Wisconsin. Two generations of Dougans earned a good living delivering their milk to the babies of Beloit and in the process led a full life.

Today, the Dougan Round Barn, like so many barns scattered around Wisconsin, is worn out, weather-beaten, and idle. It has lost its paint but not its charm or its supporters. In fact, the "Friends of the Round Barn" have crafted a plan to save the round barn and restore it to its former glory. They will have it moved from its present spot to a plot next to the Wisconsin State Information Center on Interstate 90. There visitors coming into the state will be able to stop and admire an artifact of the dairy industry that has been "barn again."



Art by Mark Brenneman

To move a 68-foot-diameter round barn with a cement silo in the center is not trivial. Mover Bob will do the lifting but wants you to cowculate the smoothest move. A smooth move from the Dougan Farm to the Information Center has the following properties:

(1) It avoids hills and valleys as much as possible,
(2) It avoids any sudden changes in elevation, and
(3) It is the shortest in length.

The order of these properties is important. Thus, if there is more than one move that satisfies property (1), then (2) is used. If there is still more than one move that is best in properties (1) and (2), then (3) is used.

The Dougan Farm (DF) is a 20×20 square array of elevations. The Round Barn is at {1, 1}, and the Information Center is at {20, 20}. A Move to the Information Center is a list of coordinates $\{i, j\}$ starting at {1, 1} ending at {20, 20} and connected by east-west or north-south moves. Here is the formal definition of properties of a Move according to the description above.

(1) A hill is the highest point of the Move relative to the elevation at {1, 1}. A valley is the lowest.

```
hill = Max[DF(Move(i)) - DF(Move(1)),
  2<=i<=20]
valley = Min[DF(Move(i)) - DF(Move(1)),
  2<=i<=20]
```

A smooth Move should have the lowest possible **(Max[hill,Abs[valley]])**.

Instructions for moving the round barn in the illustration at left are given by the artist, Mark Brenneman.
(1) Heavy-duty no-slip plunger is attached securely to vent on barn roof.
(2) Window shade is raised to start the "Rooster Drive with Chicken Ignition" (pat. pending). Speed is controlled by the height of the shade.
(3) As the rooster runs, the cable is reeled in and raises the barn.
(4) Intermediate gear moves rack that positions slop trough in front of pig.
(5) Pig eats slop, gains weight, and eventually...
(6) Breaks the rope holding it up and falls into spring loaded bathtub.
(7) When the bathtub goes down, it rocks a lever, which releases the clutch.
(8) Now the Rooster Drive no longer raises the barn; it turns a bevel gear, causing the whole upper assembly to pivot.
(9) As the upper assembly pivots, the cow on the cantilever swings away from behind the blinder.
(10) Now the cow moves ahead to graze on the grass near the fulcrum, reducing the moment and causing the cantilevered board and crane arm to tip slightly.
(11) This causes the bowling ball in the dustpan to drop into the basket, which reestablishes balance and...
(12) Lowers plastic ants into view of the anteaters.
(13) As the anteaters chase the ants, the machine and barn roll on to their next destination and the window shade may be lowered...

(2) If more than one Move has the same value for 1, then check the slope.

```
slope = Max[Abs[DF(Move(i))-DF(Move(i+1))],
  1<= i<= 19]
```

A smooth Move has the smallest possible slope.

(3) If more than one Move scores the same on (1) and (2), then check the Move length. Make Length[Move] as small as possible.

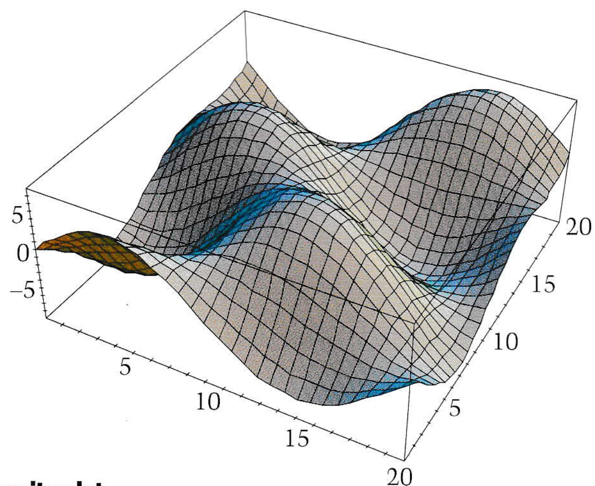
If there is more than one smooth Move in the sense of (1), (2), and (3), then pick one. Show the smooth Move, and its smoothness.

```
{hill, valley, slope, length}
```

Dougan Farm

Use the following topological map for the Dougan Farm.

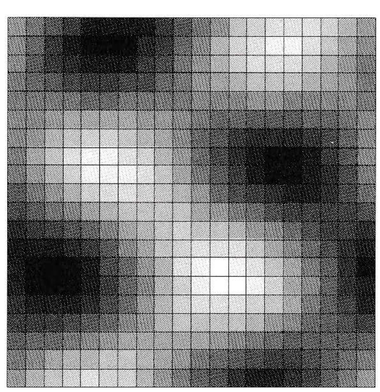
```
(f[x_, y_] := 7. Sin[x/3.] Cos[y/2.] - 5.
  Cos[x/3. ] Sin[y/5.]
DF = Transpose[Table[Floor[f[x, y]], {x, 1,
  20}, {y, 1, 20}]];
Plot3D[f[x, y], {x, 1, 20}, {y, 1, 20},
  PlotRange -> All, PlotPoints -> 30]
```



Density plot

In the density plot that follows, the light squares are the higher elevations and the dark squares are the lower elevations.

```
elevations = ListDensityPlot[DF, Frame ->
  False]
```



A Move

Let's examine a typical Move of the round barn.

```
Move = {{1, 1}, {2, 1}, {3, 1}, {3, 2},
{4, 2}, {5, 2}, {6, 2}, {7, 2}, {7, 3},
{7, 4}, {7, 5}, {7, 6}, {8, 6}, {8, 7},
{8, 8}, {8, 9}, {9, 9}, {10, 9}, {10, 10},
{10, 11}, {10, 12}, {10, 13}, {10, 14},
{10, 15}, {10, 16}, {11, 16}, {12, 16},
{13, 16}, {14, 16}, {15, 16}, {16, 16},
{17, 16}, {18, 16}, {19, 16}, {19, 17},
{19, 18}, {20, 18}, {20, 19}, {20, 20}};
```

We first define the **hill, valley, jump,** and **length** of a Move in *Mathematica*. The **hill** value is the largest difference in elevation between all locations in the Move and the starting elevation. The **valley** measures the largest negative drop in elevation.

```
hill = Max[DF[[#[[1]], #[[2]]]] - DF[[1, 1]]
  &/@ Move]
```

6

```
valley = Min[DF[[#[[1]], #[[2]]]] -
  DF[[1, 1]] &/@ Move]
```

-10

The jump number measures the largest increase or decrease between two consecutive locations in the Move.

```
jump = Max[Abs[#[[1]] - #[[2]]]&/@
  Partition[DF[[#[[1]], #[[2]]]]&/@
  Move, {2}, {1}]]
```

4

Finally, Length measures the total number of steps in the Move.

```
Length[Move]]
```

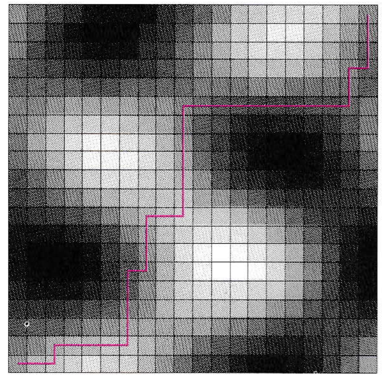
39

We put it all together by defining a smooth function for any Move.

```
smooth[Move_] := Module[{hill, valley,
  slope},
  hill = Max[DF[[#[[1]], #[[2]]]] -
  DF[[1, 1]] &/@Move];
  valley = Min[DF[[#[[1]], #[[2]]]] -
    DF[[1, 1]] &/@Move];
  jump = Max[Abs[#[[1]] - #[[2]]]&/@
    Partition[DF[[#[[1]], #[[2]]]]&/@ Move,
      {2}, {1}]];
Print["{hill,valley,jump,length} = ",
  {hill, valley, jump, Length[Move]}]]
```

Visualize the Move by placing it on the Density Plot.

```
Show[elevations, Graphics[{Thickness[ .01],
  RGBColor[1, 0, 0], Line[Move - .5]}]],
smooth[Move]
```



In case you thought it would be easy to consider all possible Moves, there are over 35 billion of them with the shortest possible Length. This is cowculated in *Mathematica* with the Binomial function.

```
Binomial[38, 19]
```

35345263800

Cow 11

Write a program that finds the smoothest Move for the Dougan Round Barn.

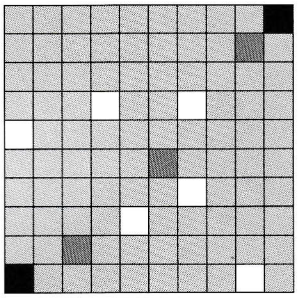
Come this Winter, with the first snow.
Move this barn, and take it slow.
Find a way to the freeway station,
Whose change is small in elevation.
When you've found it, turn it in,
And help restore this barn again.

—Dr. Mu

Solution to COW 9

In COW 9 you were asked to find how many milk routes are possible from the farm to town that deliver the milk to each customer and never go through a snowdrift. Here is how Cream County was laid out with the farm at {1, 1} and the town at {10, 10}.

```
customers = {{2, 3}, {5, 6}, {9, 9}};
snowdrifts = {{3, 5}, {4, 7}, {1, 9}, {7, 4},
  {7, 7}, {6, 1}};
n = 10;
road[1, 1] = 0;
road[n, n] = 0;
road[x_, y_] := 1 /; MemberQ[customers,
  {x, y}]
road[x_, y_] := 3 /; MemberQ[snowdrifts,
  {x, y}]
road[x_, y_] = 2;
CreamCounty = Array[road, {n, n}];
ListDensityPlot[CreamCounty, Frame -> False]
```



Road key: black = farm and town, white = snowdrifts, dark gray = customers, light gray = all the rest.

The key points to observe about the solution are:

(1) If a customer is located at $\{i, j\}$, then there are no routes that go through any $\{x, y\}$ where $(x < i$ and $y > j)$ or $(x > i$ and $y < j)$. Sketch these regions on paper to convince yourself that this is true.

(2) These are no routes that go through the snowdrifts.

(3) There is one route to $\{1, 1\}$ and other locations straight east or straight north home that have routes through them.

(4) If routes$[x, y]$ = the number of routes through $\{x, y\}$, then routes$[x, y]$ = routes$[x - 1, y]$ + routes$[x, y - 1]$. This happens since the only way to go through $\{x, y\}$ is to come from $\{x - 1, y\}$ or $\{x, y - 1\}$.

Using these observations, the solution is constructed in *Mathematica* as follows:

(a) Define the noRoutesQ predicate that tests whether (1) is true or false at a point {x, y}, ($\wedge$ denotes AND, $\vee$ denotes OR).

```
noRoutesQ[x_, y_, i_] := (x < customers
  [[i, 1]] ∧ y > customers[[i, 2]] ∨
  (x > customers[[i, 1]] ∧ y < customers
  [[i, 2]])
```

(b) Set the routes$[x, y] = 0$ at all locations $\{x, y\}$ where noRoutesQ is true for any customer.

(c) Put zeros at the snowdrifts.

(d) Define the recursive relationship between routes through $\{x, y\}$ and those through $\{x - 1, y\}$ and $\{x, y - 1\}$.

(e) Set routes$[x, y] = 1$, east or north of home.

(f) Display the routes through all locations in Cream County. Read off the answer of 1122 routes into town.

```
  Clear[routes]
routes[x_, y_] := 0 /; Or @@
  Table[noRoutesQ[x, y, i], {i, 3}]
routes[x_, y_] := 0 /; MemberQ[snowdrifts,
  {x, y}]
routes[x_, y_] := 1 /; x == 1 [Or] y == 1
routes[x_, y_] := routes[x, y] = routes[x
  - 1, y] + routes[x, y - 1]
Reverse[Array[routes, {n, n}]] //
  MatrixForm
```

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 561 & 1122 \\
0 & 0 & 0 & 0 & 0 & 33 & 66 & 198 & 561 & 561 \\
0 & 0 & 0 & 0 & 0 & 33 & 33 & 132 & 363 & 0 \\
0 & 0 & 0 & 0 & 0 & 33 & 0 & 99 & 231 & 0 \\
0 & 0 & 0 & 0 & 0 & 33 & 66 & 99 & 132 & 0 \\
0 & 0 & 3 & 12 & 21 & 33 & 33 & 33 & 33 & 0 \\
0 & 0 & 3 & 9 & 9 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 6 & 0 & 3 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

A correct solution was submitted by Benjamin Karas, a freshman at Case Western Reserve University.

And finally...

In commemoration of Wisconsin's Sesquicentennial, a copy of "Stories from the Round Barn" by Jackie Dougan Jackson (http://www.uis.edu/~jjackson/barnbook.htm), granddaughter of W. J. Dougan, will be sent to the person who submits the smoothest move.

Send your solution to drmu@cs.uwp.edu. Past solutions are available at http://usaco.uwp.edu/cowculations. If competitive computer programming is your smooth move, stop by the USA Computing Olympiad web site at http://usaco.uwp.edu. Take a look at the 1998 USA team of four students who were just selected to represent the United States at the 10th International Olympiad in Informatics to be held in Setúbal, Portugal, September 5–12, 1998.

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