QUANTUM

JANUARY/FEBRUARY 1998

US $7.50/CAN $10.50

Springer

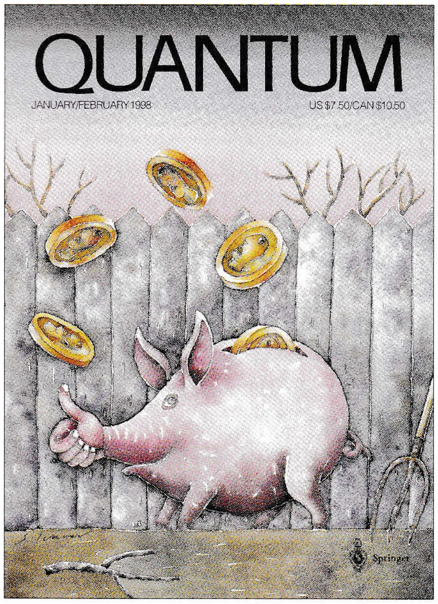
Gift of Lila Acheson Wallace, © 1997 Board of Trustees, National Gallery of Art, Washington

Jarama II (1982) by Frank Stella

JARAMA II IS NAMED AFTER AN AUTOMOBILE racetrack outside of Madrid, Spain. This piece was created as part of the Circuit series by American artist (and racing aficionado) Frank Stella. Each piece in this series is named after a different racetrack. Like the sculpture, a racetrack can assume a complex overlapping structure. Intersections, however, can pose quite a problem to racetrack designers—as well as to the drivers who must navigate them. The designers of planar graphs are also quite concerned with intersections. Discover the complex patterns that emerge from these graphs in this issue of *Quantum*.

QUANTUM

JANUARY/FEBRUARY 1998 VOLUME 8, NUMBER 3



Cover art by Sergey Ivanov

"Invest in yourself" are words of advice that this little pig has chosen not to heed. Instead, all this little sow's ear can hear is the promise of coins in her silk purse. She is willing to bet her bacon on the the toss of a coin, even though it may end up "breaking the bank," so to speak.

To understand why our porcine friend is willing to bet the farm on the outcome of a series of coin tosses, turn to page 20.

Indexed in *Magazine Article Summaries, Academic Abstracts, Academic Search, Vocational Search, MasterFILE,* and *General Science Source.* Available in microform, electronic, or paper format from University Microfilms International.

FEATURES

DEPARTMENTS

Democratizing expert knowledge

Climate change and the public understanding of science

THE WARMING PUBLIC DEbate on climate change affords educators a fascinating opportunity to examine the enigmatic role of science in modern society. While presidents and prime ministers receive briefings on the most up-to-date findings on how greenhouse gases may be altering global meteorological conditions, few ordinary citizens have seen even a single shred from the mountain of documentation produced by the Intergovernmental Panel on Climate Change. Only a handful of inquisitive souls have carefully evaluated the contrasting arguments in august publications such as *Science* and *Nature*.

Recent polls suggest that despite a lack of homework and little formal scientific training, most Americans have already decided whether global warming is a pressing concern or a figment of millennialist imaginations. To assist Americans as they struggle with technical details, President Clinton enlisted 100 ebullient television weather personalities. How has it come to pass that people without even the most basic understanding of the science of climate change sit as judge and jury on the most irascible problem ever to confront humanity?

As science has become omnipresent in everyday life, it has been lifted from the laboratory and out of the exclusive purview of specially qualified personnel. Science has become part of our common public discourse and now serves new societal roles besides functioning as a rigorous technique for interrogating nature. In living rooms, pubs, and bowling alleys, science—or its negation—is now used as an excuse to justify values and agendas. People define themselves in relation to science. They use scientific knowledge to affirm their identities and to impose coherence on the world.

For instance, purported environmentalists unable to distinguish between Fahrenheit and Celsius temperature scales impulsively wrap themselves in the science of climate change—at least since the balance of reported evidence shifted in their favor. In contrast, but equally impetuously, their industrial adversaries concoct elaborate conspiracy theories to discount a growing body of data weighing against their interests. Change the issue to nuclear power or food irradiation, and the sides swap foils with hardly a forethought!

One view of this seemingly confused situation blames reckless politicians for again undermining science by granting an ill-informed public entry into the arena. Chalk up climate change as yet another calamitous episode in which science has been knocked from the straight and narrow by inadvisable outside scrutiny. The conclusion arising from this view is that science, and humanity in general, are poorer as a result.

An alternative perspective recognizes that larger society occasionally appropriates certain types of expert knowledge from the scientific realm. For instance, to better understand the unsettling experiences of urbanization and industrialization during the nineteenth century, the public seized haphazardly upon certain elements emerging in the nascent fields of economics and psychology. By borrowing organizational categories and terminology, people were better able to grasp phenomena such as unemployment and mental illness.

This initial interaction paved the way for a dialectical relationship between experts in these disciplines and the wider public. Though the process is by no means complete, growing numbers of social scientists are coming to realize that they cannot wall themselves off from outside contact for fear that such exposure will contaminate their objectivity. The conventional posture is becoming increasingly untenable, and the best scholars are intimately engaged with the individuals they study. This new type of knowledge production often has a thrust and parry character to it. Experts' findings are regularly tested against lay experience, and this validation process feeds back iteratively into successive rounds of inquiry.

We can view public appropriation of science as the most recent phase of a 400-year process of intellectual maturation. During this time, the natural and physical sciences, for both good and ill, have infused themselves into virtually every sphere of modern life. Recent disputes over concerns such as global warming suggest the public is attempting to draw on this knowledge, albeit awkwardly and with political intent. Such developments mean that science educators must make science more accessible and, through this process, reestablish its relevance in the lives of ordinary people.

Rigor need not be sacrificed in the pursuit of democratizing expert knowledge. Rather, scientists and science educators need to recognize the increasingly central position of their expertise and find meaningful ways to impart their special knowledge to the general public.

—Maurie J. Cohen

Maurie J. Cohen *is Ove Arup research fellow at Oxford Centre for the Environment, Ethics & Society, Mansfield College, Oxford University, United Kingdom.*

What's happening?

Summer study ... competitions ... new books ... ongoing activities ... clubs and associations ... free samples ... contests ... whatever it is, if you think it's of interest to *Quantum* readers, let us know about it! Help us fill Happenings and the Bulletin Board with short news items, firsthand reports, and announcements of upcoming events.

What's on your mind?

Write to us! We want to know what you think of *Quantum*. What do you like the most? What would you like to see more of? And, yes—what *don't* you like about *Quantum*? We want to make it even better, but we need your help.

What's our address?

Quantum
National Science Teachers Assoc.
1840 Wilson Boulevard
Arlington VA 22201-3000

Be a factor in the QUANTUM equation!

QUANTUM

THE MAGAZINE OF MATH AND SCIENCE

A publication of the National Science Teachers Association (NSTA)
& Quantum Bureau of the Russian Academy of Sciences
in conjunction with
the American Association of Physics Teachers (AAPT)
& the National Council of Teachers of Mathematics (NCTM)

The mission of the National Science Teachers Association is to promote excellence and innovation in science teaching and learning for all.

Publisher
Gerald F. Wheeler, Executive Director, NSTA

Associate Publisher
Sergey S. Krotov, Director, Quantum Bureau,
Professor of Physics, Moscow State University

Founding Editors
Yuri A. Ossipyan, President, Quantum Bureau
Sheldon Lee Glashow, Nobel Laureate (physics), Harvard University
William P. Thurston, Fields Medalist (mathematics), University of California, Berkeley

Field Editors for Physics
Larry D. Kirkpatrick, Professor of Physics, Montana State University, MT
Albert L. Stasenko, Professor of Physics, Moscow Institute of Physics and Technology

Field Editors for Mathematics
Mark Saul, Computer Consultant/Coordinator, Bronxville School, NY
Igor F. Sharygin, Professor of Mathematics, Moscow State University

Managing Editor
Kenneth L. Roberts

Associate Managing Editor
Mike Donaldson

Editorial Assistants
Jennifer M. Wang
Cara Young

Art Director
Sergey Ivanov

Editorial Advisor
Timothy Weber

International Consultant
Edward Lozansky

Assistant Manager, Magazines (Springer-Verlag)
Madeline Kraner

Advertising Managers
Paul Kuntzler (Washington office)
Brian Skepton (New York office)

Advisory Board
Bernard V. Khoury, Executive Officer, AAPT
Linda Rosen, Executive Director, NCTM
George Berzsenyi, Professor of Mathematics, Rose-Hulman Institute of Technology, IN
Arthur Eisenkraft, Science Department Chair, Fox Lane High School, NY
Karen Johnston, Professor of Physics, North Carolina State University, NC
Margaret J. Kenney, Professor of Mathematics, Boston College, MA
Alexander Soifer, Professor of Mathematics, University of Colorado–Colorado Springs, CO
Barbara I. Stott, Mathematics Teacher, Riverdale High School, LA
Ted Vittitoe, Retired Physics Teacher, Parrish, FL
Peter Vunovich, Capital Area Science & Math Center, Lansing, MI

Editorial Consultants
Yuly Danilov, Senior Researcher, Kurchatov Institute
Irina Oleynik, Managing Editor, Quantum Bureau

Quantum (ISSN 1048-8820) is published bimonthly by the National Science Teachers Association in cooperation with Springer-Verlag New York, Inc. Volume 8 (6 issues) will be published in 1997–1998. *Quantum* contains authorized English-language translations from *Kvant*, a physics and mathematics magazine published by Quantum Bureau (Moscow, Russia), as well as original material in English. **Editorial offices:** NSTA, 1840 Wilson Boulevard, Arlington VA 22201-3000; telephone: (703) 243-7100; electronic mail: quantum@nsta.org. **Production offices:** Springer-Verlag New York, Inc., 175 Fifth Avenue, New York NY 10010-7858.

Periodicals postage paid at New York, NY, and additional mailing offices. **Postmaster:** send address changes to: *Quantum*, Springer-Verlag New York, Inc., Journal Fulfillment Services Department, PO Box 2485, Secaucus NJ 07096-2485. Copyright © 1998 NSTA. Printed in U.S.A.

Subscription Information:

North America: Student rate: $17.70; Personal rate (nonstudent): $25. This rate is available to individual subscribers for personal use only from Springer-Verlag New York, Inc., when paid by personal check or charge. Subscriptions are entered with prepayment only. Institutional rate: $45. Single Issue Price: $7.50. Rates include postage and handling. (Canadian customers please add 7% GST to subscription price. Springer-Verlag GST registration number is 123394918.) Subscriptions begin with next published issue (backstarts may be requested). Bulk rates for students are available. Mail order and payment to: Springer-Verlag New York, Inc., Journal Fulfillment Services Department, PO Box 2485, Secaucus, NJ 07096-2485, USA. Telephone: 1(800) SPRINGER; fax: (201) 348-4505; e-mail: custserv@springer-ny.com.

Outside North America: Personal rate: Please contact Springer-Verlag Berlin at subscriptions@springer.de. Institutional rate is US$57; airmail delivery is US$18 additional (all rates calculated in DM at the exchange rate current at the time of purchase). SAL (Surface Airmail Listed) is mandatory for Japan, India, Australia, and New Zealand. Customers should ask for the appropriate price list. Orders may be placed through your bookseller or directly through Springer-Verlag, Postfach 31 13 40, D-10643 Berlin, Germany.

Advertising:

Advertising Representatives: (Washington) Paul Kuntzler (703) 243-7100; (New York) Brian Skepton (212) 460-1575; and G. Probst, Springer-Verlag GmbH & Co. KG, D-14191 Berlin, Germany, telephone 49 (0) 30-827 87-0, telex 185 411.

Printed on acid-free paper.

In the planetary net

Seeing the potential in gravitational fields

by V. Mozhayev

THE GRAVITATIONAL FIELD is the weakest of the forces in nature, yet it is often the dominant force in the macroscopic world. Let's examine the gravitational interaction between bodies at rest or moving rather slowly (relative to the speed of light). Such are the conditions when Newton's law of gravitation is valid. This law says that any two material points (that is, bodies whose linear dimensions are much less than the distance between them) with masses m_1 and m_2 are mutually attracted by a force F that is directly proportional to the product of both masses and inversely proportional to the square of the distance between them:

$$F = G\frac{m_1 m_2}{r^2}.$$

The proportionality factor G is known as the gravitational constant and has the value $G = 6.67 \cdot 10^{-11}$ $m^3/kg{\cdot}s^2$.

Newtonian gravitational fields obey the superposition principle—the gravitational force acting on a material point is the vector sum of the gravitational forces due to all other particles, and each of these forces is independent of the others.

Newton's law of gravitation implies that the corresponding gravitational field is a potential field, which means that the field performs no net work when a body moves along any closed trajectory and returns to its starting point. This property of the gravitational field results also in a relationship between the gravitational force F acting on a material point and its potential energy U. In the case of a spherically symmetric gravitational field, this relationship is described by

$$F_r = -\frac{dU}{dR},$$

where F_r is the radial component of the force. Below we consider concrete examples of motions in spherically symmetric fields.

Problem 1. (a) Assuming a zero value for the potential energy at infinity, determine the potential energy of a body of mass m in Earth's gravitational field. Consider Earth to be a homogeneous sphere of mass M_E and radius R_E. Consider the cases when the body is placed inside and outside Earth. (b) To what maximum distance from the Earth's surface could a small body of mass m travel if it was imparted with initial velocity equal to the orbital velocity v_{orb} just above Earth's surface?

Solution. (a) First we consider the case when a body of mass m is an arbitrary distance r from Earth's center and $r > R_E$. In this case the body is affected by the force of gravity $F = -GmM_E/r^2$ and is directed toward Earth's center. Enlisting the formula $F = -dU/dr$, where U is the potential energy of a body in Earth's gravitational field, we have

$$U = -\int F dr + C_1,$$

where C_1 is a constant to be determined from the condition $U(\infty) = 0$. Inserting the formula for the gravitational force results in

$$U(r) = -G\frac{mM_E}{r} + C_1.$$

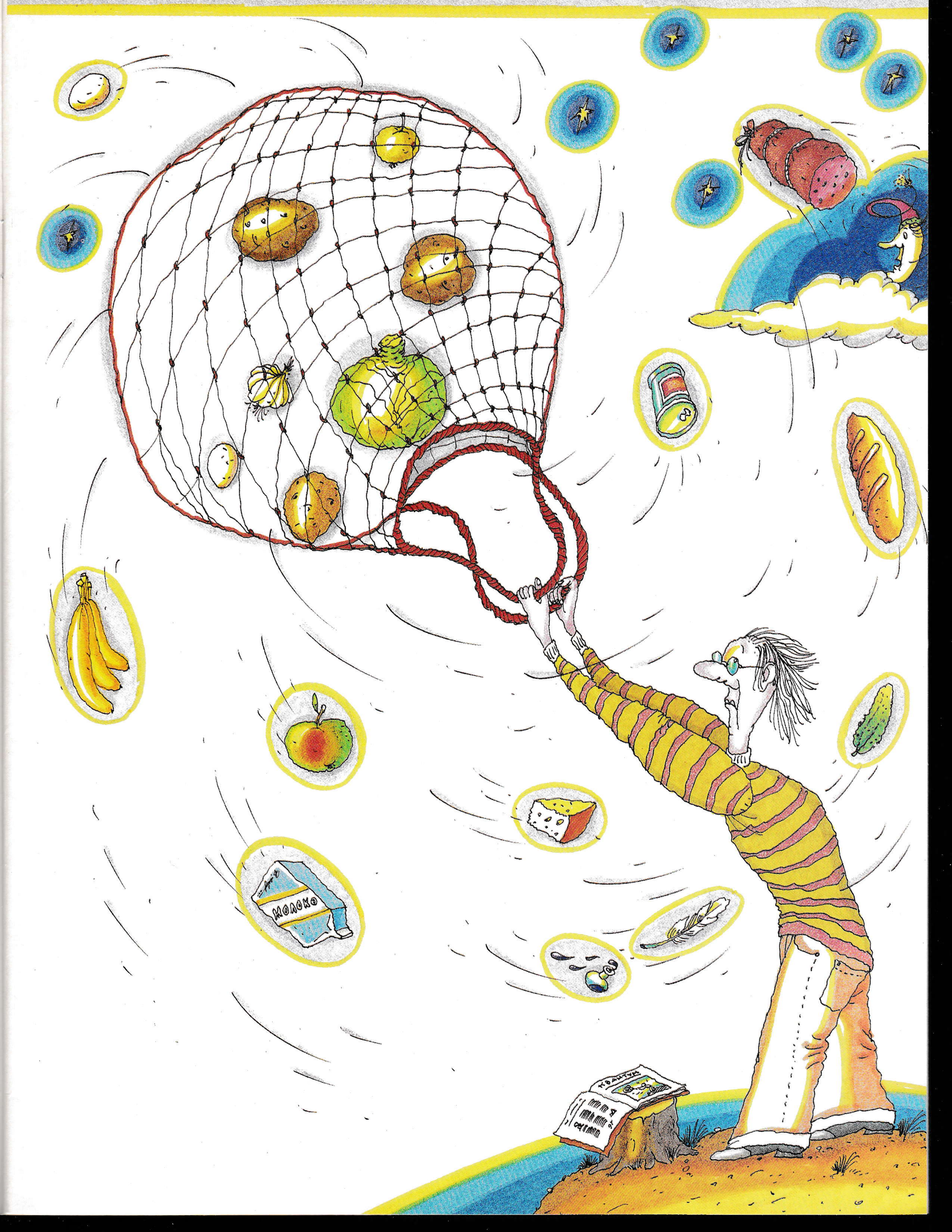
We can thus see that $C_1 = 0$.

Now consider the case when $r < R_E$. The gravitational force inside Earth is given by $F = -GmM_E r/R_E^2$ (prove it). Therefore,

$$U(r) = \int \frac{GmM_E r dr}{R_E^3} + C_2.$$

The constant C_2 is determined from the boundary condition

Art by V. Ivanyuk


МОЛОКО
КВАНТИК

$U(R_E) = -GmM_E/R_E$. After some manipulations we get

$$C_2 = -\frac{3GmM_E}{2R_E}.$$

Therefore,

$$U(r) = G\frac{mM_E}{R_E}\left[\frac{1}{2}\left(\frac{r}{R_E}\right)^2 - \frac{3}{2}\right].$$

The plot of $U(r)$ is shown in figure 1. Evidently, a similar function will describe not only Earth's gravitational field but any gravitational field generated by a ball with a homogeneous density.

Usually the dependence shown in figure 1 is called a "potential well." This term refers to the fact that if the total energy of a body placed in such a field is less than zero, the body will be "trapped" in the well, which means that it cannot escape from Earth to infinity and its motion is bounded. The maximum possible radius of the body is determined by the walls of the well, at which the body's speed becomes zero and the body returns to Earth.

(b) For a fixed value of initial velocity v_0, the body will travel to the largest distance from Earth if its velocity is directed along a radius. This farthest point H can be found using conservation of energy:

$$\frac{mv_0^2}{2} - G\frac{mM_E}{R_E} = -G\frac{mM_E}{R_E + H},$$

from which we get

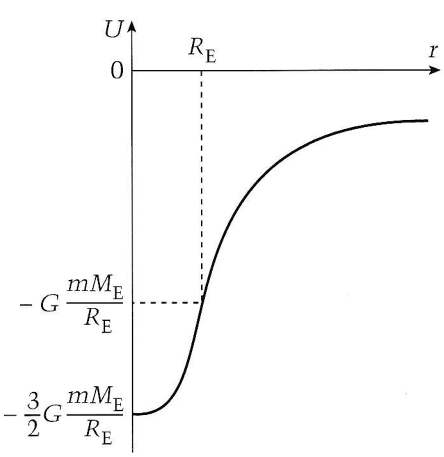


Figure 1

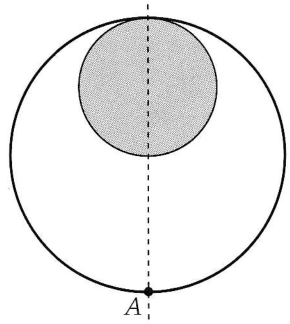


Figure 2

$$H = \frac{R_E}{2GM_E/(R_E v_0^2) - 1}.$$

As the orbital velocity is

$$v_{orb} = \sqrt{GM_E/R_E},$$

the substitution yields

$$H = R_E.$$

Problem 2. The escape velocity for a planet is $v = 12$ km/s. Find the minimal value of escape velocity for a similar planet that has a hole filled with matter that is double the density of the planet's matter $\beta = 2$, (figure 2). The ratio of the hole's radius to the radius of the planet is $\alpha = 1/2$.

Solution. The escape velocity for a planet is the velocity at the planet's surface that corresponds to zero total energy of the body. For a homogeneous planet of mass M and radius R, this condition yields the equation

$$\frac{v^2}{2} - G\frac{M}{R} = 0.$$

In the case of a planet with nonuniform density that contains a cavity filled with a substance of density

$$\rho = \frac{3\beta M}{4\pi R^3},$$

we consider this hole as a superposition of two cavities—one with the planet's normal density

$$\rho_0 = \frac{3M}{4\pi R^3},$$

and another with the density

$$\rho_1 = \frac{3(\beta - 1)M}{4\pi R^3}.$$

The potential energy of a body at the surface of such a planet equals the sum of potential energies in the gravitational fields of a planet of uniform density and of a ball of density ρ_1 and the same radius as the hole's.

The minimum value for the escape velocity will be at that point of the planet's surface where the potential energy has the minimum absolute value (that is, at point A in figure 2). If we call the escape velocity for this point v_1, then the condition for the body's total energy to be zero will be

$$\frac{v_1^2}{2} - G\frac{M}{R} - G\frac{\alpha^3(\beta - 1)M}{(\alpha + 1)R} = 0,$$

Substituting the numerical values for α and β, we get

$$v_1^2 - \frac{13}{6}G\frac{M}{R} = v_1^2 - \frac{13}{12}v^2 = 0.$$

Therefore,

$$v_1 = \sqrt{\frac{13}{12}}v \cong 12.5 \text{ km/s}.$$

Problem 3. Find the length of the semimajor axis of elliptical orbit of a satellite revolving about Earth, provided its total energy (kinetic plus potential) equals E.

Solution. Let Earth be at the left focus F_1 of the elliptical trajectory shown in figure 3. In this case point A (apogee) corresponds to the highest altitude of the satellite, and point P (perigee) corresponds to its lowest altitude. We denote the length of segment PF_1 as r_1 and the length of F_1A as r_2, so the length of the semimajor axis is $2a = r_1 + r_2$.

The total energy of the satellite at point P is:

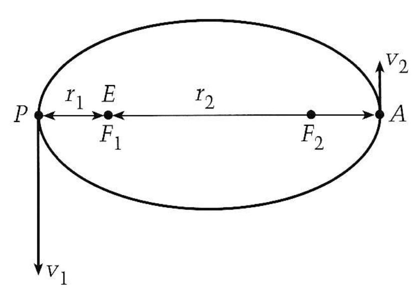


Figure 3

$$\frac{mv_1^2}{2} - \frac{GmM_E}{r_1} = E,$$

where m is the satellite's mass, v_1 its speed, and M_E the mass of Earth. We use Kepler's second law (the law of areas): The radius vector connecting Earth and a satellite sweeps out equal areas over equal periods of time. This law results in the following equation for orbital points A and P:

$$v_1 r_1 = v_2 r_2.$$

Let's call this product L. By expressing v_1 in terms of L and inserting it into the formula of the total energy, we get:

$$r_1^2 + G\frac{mM_E}{E} r_1 - \frac{mL^2}{2E} = 0.$$

This equation has two roots corresponding to points A and P because the coefficient at r_1 and the constant term of this equation are the same for both points. So we get

$$r_1 = -G\frac{mM_E}{2E} - \sqrt{\left(G\frac{mM_E}{2E}\right)^2 + \frac{mL^2}{2E}},$$

$$r_2 = -G\frac{mM_E}{2E} + \sqrt{\left(G\frac{mM_E}{2E}\right)^2 + \frac{mL^2}{2E}}.$$

It immediately follows that the semimajor axis of the satellite's elliptical orbit is

$$2a = r_1 + r_2 = -G\frac{mM_E}{E}.$$

It should be noted that total energy E is a negative value because the overall energy for bounded motion must always be negative. (Note that we assumed at the beginning that the potential energy at infinity is zero).

What is the physical meaning of this formula? At a given (constant) total energy, a satellite may travel along a wide variety of elliptical orbits, but all of them will have the same semimajor axis. If we know the size of this semimajor axis, we can calculate the satellite's total energy. Naturally, the described

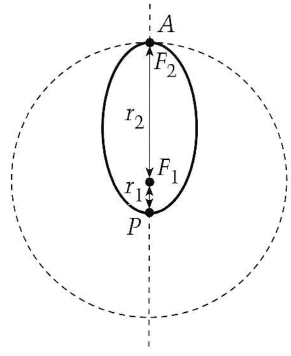


Figure 4

relationship is true not only for Earth's satellites but also for the orbits of planets of the Solar System and their satellites, provided the satellites (natural or artificial) have masses that are much less than that of the central body.

Problem 4. A space vehicle revolves about Earth along an elliptical orbit whose semimajor axis is $2a$. Earth's center is located at the focus F_1 (figure 4). At this time the spacecraft passes the most remote point located a distance r_2 from Earth's center, the spacecraft's booster is ignited for a short time. How should the velocity of the spaceship be changed to make it move along a circular orbit of radius r_2? The acceleration due to gravity at Earth's surface is g, and Earth's radius is R_E.

Solution. Since the new orbit must be circular, the new velocity should be perpendicular to the radius vector connecting Earth's center and the center of mass of the space vehicle, so the change in velocity should be directed along the spacecraft's velocity. Now let's calculate the value and the sign of the change in velocity.

The value of the orbital velocity of a spacecraft in a circular orbit of radius r_2 is found by equating the centripetal force to the gravitational force:

$$\frac{mv_0^2}{r_2} = \frac{GmM_E}{r_2^2}.$$

Therefore,

$$v_0 = \sqrt{G\frac{M_E}{r_2}} = \sqrt{R_E g\frac{R_E}{r_2}}.$$

The velocity of the spacecraft v_A at point A just before the engine starts can be obtained from the relationship between the semimajor axis of the elliptical orbit and the total energy of the spacecraft (see problem 3). In our case this relationship is

$$\frac{mv_A^2}{2} - G\frac{mM_E}{r_2} = -G\frac{mM_E}{2a},$$

from which we get

$$v_A = \sqrt{G\frac{M_E}{r_2}\left(2 - \frac{r_2}{a}\right)} = v_0\sqrt{2 - \frac{r_2}{a}}.$$

Since $r_2 > a$, then $v_A < v_0$. Therefore, to pass from elliptical to circular orbit the spacecraft must increase its velocity by

$$\Delta v = v_0 - v_A = \sqrt{R_E g\frac{R_E}{r_2}}\left(1 - \sqrt{2 - \frac{r_2}{a}}\right).$$

Problem 5. Find the approximate value of the escape velocity that should be imparted to a rocket launched from Earth such that it would leave the Solar System forever. Neglect the influence of the planets of the Solar System. Consider Earth's orbit around the Sun to be circular with a radius $R_{ES} = 1.5 \cdot 10^8$ km and a period of revolution $T = 1$ year. The orbital velocity around Earth is $v_{orb} = 7.9$ km/s.

Solution. Let's divide the rocket's trajectory into two parts. The first part we shall consider in the Earth's system of reference and we'll neglect the heterogeneity of the Sun's gravitational field. Assuming Earth's mass M_E to be infinitely larger than that of the rocket m, we write conservation of energy in the following form:

$$\frac{mv^2}{2} - G\frac{mM_E}{R_E} = \frac{mv_\infty^2}{2},$$

where v is the rocket's velocity at Earth's surface and v_∞ the rocket's velocity at the time it leaves Earth's gravitational field. Let's express the rocket's potential energy in terms of the velocity of circular motion of a satellite around Earth near its surface:

$$G\frac{mM_E}{R_E} = mv_{orb}^2.$$

Then

$$v_\infty^2 = v^2 - 2v_{orb}^2.$$

During the second stage, after the rocket has "left" Earth's gravitational field, we'll consider its motion in the gravitational field of the Sun. In the Sun's system of reference, the velocity of the rocket is a vector sum of velocity $\mathbf{v}_\infty$ and the velocity of the circular motion of Earth around the Sun $\mathbf{V}$. Determine the value of the parabolic (guess why its called "parabolic") velocity v_p that a body in orbit around Earth needs to leave the Solar System forever. According to conservation of energy we have

$$\frac{mv_p^2}{2} - G\frac{mM_S}{R_{ES}} = 0,$$

from which we get

$$v_p = \sqrt{2G\frac{M_S}{R_{ES}}} = \sqrt{2}V.$$

The rocket's minimum velocity v_{min} will occur when its velocity vector is directed along the Earth's velocity—that is, when $v_p = v_\infty + V$. Since $V = 2\pi R_{ES}/T \cong 30$ km/s and $v_{orb} = 7.9$ km/s, $v_{min} = 16.7$ km/s.

Problem 6. Find the minimum additional velocity that can be imparted by a short impulse to a satellite moving in a very high circular orbit around Earth in order to send it to Mars. Assume the orbits of Mars and Earth to be circular. Earth's orbital radius R_{ES} is $1.5 \cdot 10^8$ km, and Mars's orbit radius R_{MS} is 1.52 times larger.

Solution. "A very high circular orbit" means that the radius of the satellite's orbit is much larger than Earth's radius, so we can neglect the velocity of the satellite relative to the planet. However, being Earth's satellite, it moves with Earth around the Sun in a circular orbit with velocity

$$V = \sqrt{\frac{GM_S}{R_{ES}}} \approx \frac{2\pi R_{ES}}{T} \approx 30 \text{ km/s},$$

where M_S is the mass of the Sun and T is the period of Earth's revolution about the Sun. If we accelerate the satellite in the direction of Earth's orbital motion, it will move in elliptical orbits with semimajor axes that are larger than the diameter of Earth's orbit and that grow with the change in velocity. So, the satellite will get to Mars if the maximum remote orbital point intrudes into the circular Martian orbit. Such a trajectory is shown in figure 5 by the dashed line. The semimajor axis of this orbit equals $2a = R_{ES} + R_{MS}$. In this orbit the total energy of a satellite of mass m is

$$E = \frac{m(V+v)^2}{2} - G\frac{mM_S}{R_{ES}}$$
$$= \frac{m(V+v)^2}{2} - mV^2$$
$$= \frac{m\left(v^2 + 2Vv - V^2\right)}{2}.$$

Now we recall the relationship between the semimajor axis of an ellipse and a satellite's total energy:

$$R_{ES} + R_{MS} = \frac{2GM_S}{V^2 - 2Vv - v^2}$$
$$= \frac{2V^2 R_{ES}}{V^2 - 2Vv - v^2}.$$

Simple manipulations yield

$$v^2 + 2Vv - \frac{(R_{MS} - R_{ES})V^2}{R_{MS} + R_{ES}} = 0,$$

which naturally has two roots:

$$v_1 = V\left(\sqrt{\frac{2R_{MS}}{R_{MS} + R_{ES}}} - 1\right) \cong 2.95 \text{ km/s}$$

and

$$v_2 = -V\left(1 + \sqrt{\frac{2R_{MS}}{R_{MS} + R_{ES}}}\right) \cong -62.95 \text{ km/s}.$$

As we accelerate the spacecraft in the direction of Earth's velocity, the first root is the solution of our problem. Notice, however, that "en passant" we solved one more problem—the second root is also correct and describes the acceleration of the spacecraft in the opposite direction. Note that v_2 is exactly equal to v_1 plus double the velocity (60 km/s) of Earth's orbital flight around the Sun.

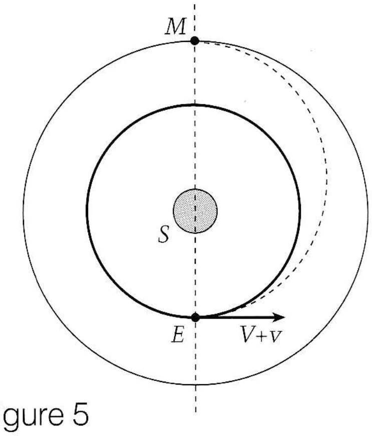


Figure 5

Exercises. (1) Imagine a narrow well drilled from Earth's surface to its center. A body falls into this well from infinity, where it had zero velocity. What velocity will this body have at the center of Earth? Consider Earth to be a homogeneous sphere of radius R_E. At Earth's surface the acceleration due to gravity is g. Hint: Infinity here means a large distance from Earth where the attraction between a body and Earth is negligible but where both bodies revolve around the Sun as a single entity.

(2) The escape velocity for a planet is $v_0 = 10$ km/s. Find the minimal value of the escape velocity in a similar planet that has a hole (figure 2) filled with matter of half the planet's density. The ratio of the hole's radius to the radius of the planet is 0.5.

(3) A spacecraft moves about Earth in an elliptical orbit with semimajor axis a. Earth's center is located at the focus F_1 of the ellipse (figure 4). When the spacecraft is at point P (perigee), and when the distance from Earth's center is r_1, the rocket engine is started. How should the spacecraft's velocity be changed at this point to place it in a circular orbit of radius r_1? Take as given the values of Earth's radius R_E and the acceleration due to gravity at Earth's surface.

(4) A shot is fired from a satellite moving in a circular orbit with velocity v_0. The direction of the shot makes an angle $\phi = 120°$ with the satellite's velocity. What should the bullet's velocity relative to the satellite be for the bullet to fly to infinity?

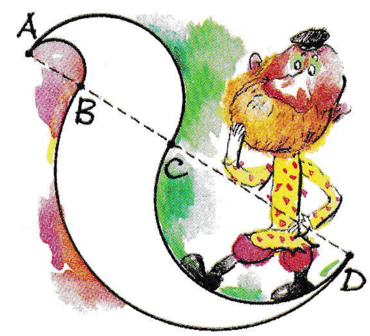
Just for the fun of it!

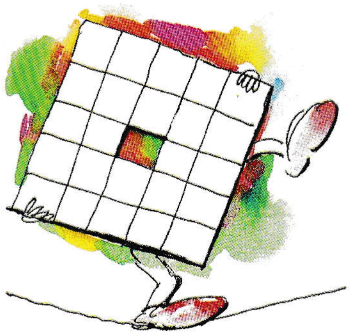
B221

Chipping in. John, Jim, and Gerry went to a baseball game. On the way John bought five bags of potato chips, Jim bought two bags, and Gerry didn't buy any. During the game they all ate the chips, each one eating as much as the others. After the game, Gerry figured out how much the bags of chips cost and handed over \$1.40. How much money should John get?

B222

The weight of gilt. The figure at the right represents the gilded area of a pattern. (The border of this figure consists of four semicircles with diameters *AB, AC, BD,* and *CD*.) The weight of the gold paint needed to cover the figure depends on the area of the figure. The artist knows only two numbers: the lengths of *AD* and *CD*. Can he calculate the area of this figure?





B223

Navigation squared. A checker starts on the lower left square of the board shown in the figure. How many different paths are there for it to move from there to the upper right square? The checker may move in only two directions: upward and to the right.

B224

Connect-the-dots cubed. Can you draw a six-segment broken line through all the vertices of a cube?

Art by Pavel Chernusky

B225

Touching points. Two wheels roll toward each other with identical angular velocity. At the moment of collision they contact each other at the same points that touched the ground before they began rolling. Could the radii of the wheels differ?

ANSWERS, HINTS & SOLUTIONS ON PAGE 47

Planar graphs

Can you make the connections?

by A. Y. Olshansky

ELECTRIC CIRCUITS, TRAFfic schemes, geographical maps, structural formulas of molecules, and family trees are examples of graphs, in which certain points are connected by curves that signify some relationship between the points. In the following article we deal with those properties of planar graphs that do not change under continuous transformations—in other words, with topological properties of graphs.

The principal facts of planar topology, established in the eighteenth and nineteenth centuries (Euler's and Jordan's theorems, for example), are fully visual. More surprising are the more recent observations, such as the nice theorem by A. A. Klyachko and the simple but elegant lemma of J. R. Stallings (see sections 6 and 7 below), published in 1993 and 1987 respectively in connection with some combinatoric group theory problems.

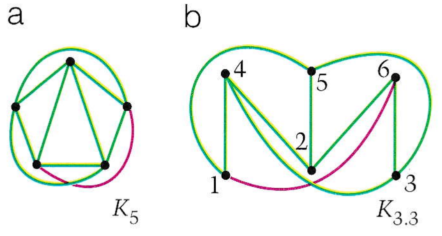


Figure 1

Graph theory can be applied, for example, in the design of computer circuit boards. The connections among the chips must lie flat along the board. We can solve these problems in a graph-theoretic setting. For instance, suppose you are given five points on a plane. Is it possible to connect each of them to all the rest by a system of nonintersecting curves? In figure 1a all but one pair of points are connected by green lines, the latter being connected by the red line. Unfortunately, this red line crosses one green curve. But would it be possible to avoid intersections if we adopted another kind of planar connection?

A similar problem to consider is the old one about three wells. There are three wells (4, 5, 6) and three farmsteads (1, 2, 3) whose owners are at odds. To keep the peace, we must make paths from each farm to each well such that the paths don't cross anywhere (fig. 1b).

To answer these and other questions, let's first clarify the properties of a line on a plane.

1. Planar curves

A circle, border of a square, or the path *1–4–2–6–3–5–1* in figure 1b comply with our intuitive idea of a simple closed curve on a plane—a

line without self-intersections, which we can draw if we return to the point where we began without lifting the pencil from the paper. If a curve's endpoint does not coincide with its beginning, we'll call the curve an arc.

This intuitive concept of a closed curve is quite sufficient to understand our further exposition. But for readers who are not satisfied with this visual explanation, we offer a formal definition. It generalizes the following simple observation: We can obtain the border of a square by a continuous transformation of its circumscribed circle (for example, if we project every point onto the nearest side of the square).

In the general case, we consider an arbitrary continuous map f of the circle O with unit radius onto some set C on a plane such that the following properties hold:

(a) f is a one-to-one map—that is, each point X of the circle has one image $Y = f(X)$ in C, and conversely, each point $Y \in C$ has exactly one inverse image $X \in O$ $(F(X) = Y)$;

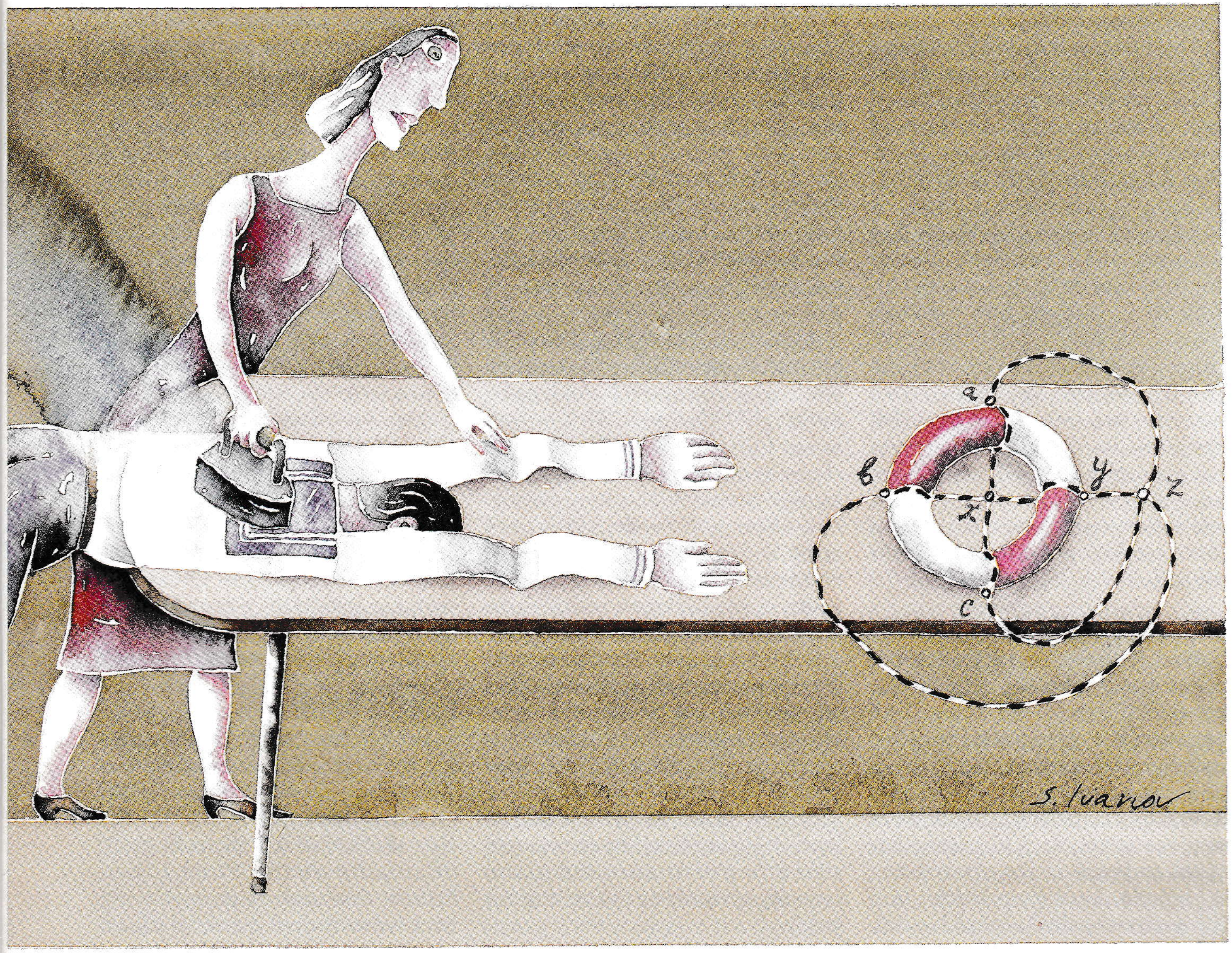
(b) Map f is (uniformly) continuous. This means that for any positive number ε, however small it is, we can find a number $\delta > 0$ such that for any two points X_1 and $X_2 \in O$ situated at a distance less than or equal to δ from each other, the distance between their images $Y_1 = f(X_1)$ and $Y_2 = f(X_2)$ is less than ε.

If there exists a map $f: O \to C$ with properties (1) and (2), we say that set C is a simple closed curve. When a variable point X moves around O, the corresponding point $Y = f(X)$ goes around curve C.

The definition of a planar arc d differs from that given above in only one detail: We should take standard segment [0, 1] instead of the circle O. When the point $X \in [0, 1]$ varies from 0 to 1, its image $Y = f(X)$ "runs across" arc d. The images of the numbers 0 and 1 are the endpoints of arc d.

2. Jordan's theorem

A standard unit circle divides a plane into two components: the interior component, consisting of all points (x, y) satisfying the inequality $x^2 + y^2 < 1$, and the exterior component, defined by the condition $x^2 + y^2 > 1$. It is impossible to connect a point $P(x_1, y_1)$ in the interior component with a point $Q(x_2, y_2)$ in the exterior component with an arc that does not cross circle O. This is not difficult to see, if one uses an argu-



Art by Sergey Ivanov

ment by continuity. Suppose a point (x, y) varies along an arc connecting P to Q. Then the distance between (x, y) and the origin varies continuously (it is measured by the function $\sqrt{x^2+y^2}$). Since it takes on a value less than 1 at P and a value greater than 1 at Q, it must take on a value exactly equal to 1 at some intermediary point R. This point R must then be on the circle.

Jordan's theorem states that any simple closed curve C on a plane shares this property of circles. That is, the set of all points on the plane that do not lie on C splits into two domains: the interior (or bounded) part E_1 and the exterior (unbounded) part E_2. Any two points from one domain can be connected by an arc that belongs entirely to this domain, yet it is impossible to connect a point in E_1 to a point in E_2 by an arc that doesn't intersect C.

This theorem is complemented by Schenfliss's theorem: Any point $Z \in E_1$ can be connected with each point $Y \in C$ by an arc, all of whose points (except for Y) lie in E_1. A corresponding statement can be made for E_2.

Rigorous proofs of Jordan's and Schenfliss's theorems suddenly turn out to be very difficult and are often omitted in introductory topology courses. We shall not give them here either. We are partly justified by the fact that the statements of these two theorems are clear on an intuitive level. Besides this, for our purposes it is enough to consider only broken lines with a finite number of links and not arbitrary curves (but even in this case Jordan's theorem is not trivial!).

3. Graphs in a plane and on a sphere

First of all, let's agree to consider only finite graphs. We define a planar graph Γ as a set $V = \{v_1, v_2 \ldots\}$ of points on the plane, called vertices, together with a set $E = \{e_1, e_2 \ldots\}$ of links connecting some of the points in V. A link connecting two different vertices is an arc, and a link that starts and ends at the same vertex is a simple closed curve (such a link is sometimes called a *loop*). No link

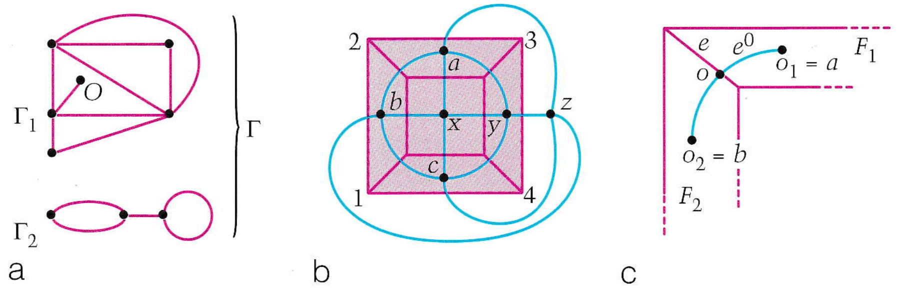


Figure 2

may have a point in common with any other link (except its endpoints).

A graph Γ is called *connected* if any two vertices can be connected by a continuous path consisting of several links of the graph. For example, the graph in figure 2a is not connected: It breaks down into the two connected *components* Γ_1 and Γ_2. In what follows, we will consider only planar graphs with at least one link.

In accordance with Jordan's theorem, every simple closed curve (including broken lines) composed of the links of a planar graph Γ breaks the plane into two domains, and thus the whole graph Γ decomposes into several domains, or faces. The red graph in figure 2b consists of five interior faces (in pink) and one exterior (unbounded) face—the planar domain with the inner border *1–2–3–4–1*.

Jordan's and Schenfliss's theorems provide an opportunity to associate with each planar graph Γ another planar graph Γ^0 *dual* to it. This is done in the following way. We begin by choosing a point within each face of graph Γ (points a, b, c, x, y, z in fig. 2b): These will be the vertices of graph Γ^0. Let e be a common edge of two faces F_1 and F_2 of graph Γ with points o_1 and o_2 chosen within them respectively. We then choose a point o on e different from its endpoints and draw arcs o_1–o and o_2–o lying within appropriate faces. (Figure 2c is a detail of graph 2, showing this construction for the two faces F_1 and F_2 with vertices $o_1 = a$ and $o_2 = b$ inside them.) We construct an edge of Γ^0, which we label e^0, by drawing an arc o_1–o–o_2 connecting vertices o_1 and o_2 and crossing over edge e of the original graph.

The graph Γ^0, which is dual to the red graph Γ, is drawn in fig. 2b with blue lines. By definition, the number of vertices in graph Γ^0 equals the number of faces in Γ, the number of edges in Γ^0 equals that in Γ (the reader is invited to check this), and the number of faces in Γ_0 equals the number of vertices in Γ.

It is easy to imagine how to put a planar graph on a sphere. Conversely, every graph Γ on a sphere has a planar realization: We call a point O within one of its faces the north pole and project the spherical graph Γ from O on the plane, tangent to the sphere at the south pole. In particular, if we "inflate" the faces of a cube to a sphere (one of the ways to do so is to project the cube's edges onto the surface of a circumscribed sphere from its center) and then apply to the obtained spherical graph the procedure of stereographic projection described above, we get the red graph shown in figure 2b. One of the faces of the cube (in fact, one that is situated under the north pole) corresponds to the unbounded face of planar graph 2b. Similarly, the vertices, edges, and faces of each convex polyhedron turn into the vertices, edges, and faces of some planar graph. (The reader can check, for example, that in the planar development described here, the graph of an octahedron is dual to that of a cube).

4. Euler's formula

Let N_V, N_E, and N_F be the number of vertices, edges, and faces of a polyhedron P respectively. For instance, if P is a cube, then $(N_V, N_E, N_F) = (8, 12, 6)$; if it is an octahedron, $(N_V, N_E, N_F) = (6, 12, 8)$; for a tetrahedron, this triplet equals (4, 6, 4); for a dodecahedron (fig. 3a) it equals

(20, 30, 12); for an n-lateral pyramid it is $(n + 1, 2n, n + 1)$; and for an n-lateral prism it is $(2n, 3n, n + 2)$. One can check that in each of these cases

$$N_V - N_E + N_F = 2. \quad (1)$$

Is this just a curious coincidence or a manifestation of a general rule?

It turns out that Euler's formula (1) holds for all planar graphs Γ, and thus for all convex polyhedrons, since it is possible to put them down on the plane such that one of their faces turns into an unbounded domain. It is easy to prove Euler's formula by induction on N_E, the number of edges of a planar graph Γ. Suppose that $N_E = 1$. Then there are two possibilities: The only edge of Γ can be either a loop or a simple arc. In the first case there is only one vertex and, according to Jordan's theorem, two faces. In the second case $(N_V, N_E, N_F) = (2, 1, 1)$. We can see that relation (1) holds in both cases.

If $N_E > 1$, there are again two cases:

(a) Graph Γ contains a vertex o of degree 1—that is, a vertex that belongs only to one edge e, and e is not a loop (like vertex O in graph Γ in fig. 2a). Then graph Γ′, which appears if we remove edge e and vertex O from graph Γ, is connected, and in it

$$N'_V = N_V, N'_E = N_E - 1, N'_F = N_F - 1. \quad (2)$$

Since $N'_E = N_E - 1$, we can assume that Euler's formula is already proved for Γ′: $N'_V - N'_E + N'_F = 2$. If we substitute N'_V, N'_E, and N'_F in this formula in accordance with (2), we obtain Euler's formula (1) for Γ.

(b) Every time we finish traveling along some edge $e = e_1$, we can continue our movement along another edge e_2. The number of edges in graph Γ being finite, we will inevitably go along the edges we've already passed if the number of continuations described above is large enough. Thus, in this case we can find a simple closed path consisting of several consecutive edges e_1, e_2, …, e_k of graph Γ (it is possible that $k = 1$). Jordan's and Schenfliss's theorems imply (and moreover, it is intuitively clear), that if we remove edge e from graph Γ (so that the vertices at its endpoints remain unmoved), the number of faces will decrease by 1—that is, in graph Γ′ obtained in this way

$$N'_V = N_V, N'_E = N_E - 1, N'_F = N_F - 1.$$

Now one can finish the inductive reasoning just as in the first case.

5. Planar graphs

We now apply Euler's formula to certain spherical or planar graphs. These are graphs such that the number of edges we must pass to go around any of its faces (including the outer one in the case of a planar graph) is greater than or equal to some fixed $n > 2$. If we go around every face and sum up the numbers of edges we've passed, we'll count at least nN_F edges. Since in this way we'll count each edge twice, we obtain the inequality

$$2N_E \geq nN_F.$$

Now let's multiply relation (1) by n and add it to this inequality. We get

$$N_E \leq \frac{n(N_V - 2)}{n - 2}. \quad (3)$$

Inequality (3) proves useful whenever we try to determine whether any "abstract graph" can be drawn on a plane or sphere. By "abstract" graph we mean a collection of finitely many vertices connected by links that have no common points except their endpoints, whether or not it can be drawn on the plane.

It is easy to show that we cannot put graph K_5 (fig. 1a) on a plane so that no self-intersections appear. In fact, there are no loops or multiple edges (two or more distinct edges with common ends) in it, and thus each of its faces would have at least three boundary edges if this graph was put on a plane. But, putting $n = 3$ in formula (3), we see that it does not hold for graph K_5 since the graph has 5 vertices and 10 edges.

And if we assume that graph $K_{3,3}$ (fig. 1b) has a planar realization, we can apply formula (3) to it with $n = 4$ since there are no closed paths consisting of three edges in it either. (For instance, having passed along three consecutive edges starting from a farm, we'll inevitably come to a well.) So, we once again arrive at a contradiction with (3) since $N_V = 6$ and $N_E = 9$ here.

It's evident that any graph containing a smaller graph (subgraph) that cannot be put on a plane is nonplanar itself. Also, all graphs homeomorphic to K_5 and $K_{3,3}$ (that is, all graphs that appear if we decompose every edge e of K_5 or $K_{3,3}$ into several new edges by putting additional vertices on it) are nonplanar.

Pontrjagin and Kuratowski's famous theorem says that there are no other obstructions to planarity:

Graph Γ is planar if and only if it contains no subgraphs homeomorphic to graph K_5 or to graph $K_{3,3}$.

We've proved that the condition of this theorem is necessary. It is much more difficult to prove that it is sufficient (see section 3).

Another interesting application of formula (3) appears if we take $n = 6$. It is well known that bees build their honeycomb so that each cell (interior face) is a hexagon. Would they be able to put the honeycomb on a sphere and keep to this rule? Formula (3) would give us the inequality

$$N_E < \frac{3}{2} N_V. \quad (4)$$

On the other hand, there are at least three edges coming out of each vertex. Thus $3N_V \leq 2N_E$ (we put the coefficient 2 on the right since we count every edge twice when we count the edges coming out of all the vertices.) This contradicts equation (4), so it is in fact not possible to have a honeycomblike construction covering an entire sphere.

Moreover, this reasoning shows that there are no convex polyhedrons such that each of their faces has at least six sides. This is why soccer balls are made from both hexagonal and pentagonal patches.

Problem: Derive the inequalities connecting the number of vertices and the number of edges of a ball made of k pentagonal and l hexagonal patches and show that $k \geq 12$.

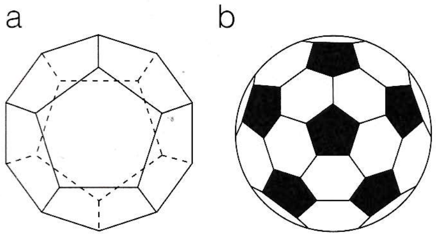


Figure 3

(Hint: The number of edges is $(1/2)(5k + 6l)$, and the number of vertices is not greater than $(1/3)(5k + 6l)$.) It would be possible to make the ball out of 12 pentagonal patches—just like the dodecahedron, composed of 12 pentagons—but this sort of ball wouldn't be round enough. In reality, soccer balls are made of 12 pentagonal and 20 hexagonal patches sewn together along 90 edges (fig. 3b). Find the number of its vertices using Euler's formula.

6. One-way traffic

Stallings's lemma says that in a city that has only one-way streets there must exist a block such that one can drive around it without going the wrong way.

A rigorous mathematical setting of this lemma is as follows: Let Γ be an oriented planar graph—that is, a planar graph in which a direction, symbolized in figure 4 by an arrow, is assigned to each edge (we shall not get deeper into the mathematical details here). Let's call this graph "correct" if it does not contain a vertex such that all the edges adjacent to it "come out of" the vertex (such vertices are called *sources*) or "come into" it (such vertices are called *sinks*). If the traffic scheme described above is reasonable, then the graph corresponding to it will be correct; otherwise drivers will be forced to violate the rules.

We will prove Stalling's lemma using methods different from those in section 2. That is, we want to show that a correct graph Γ must contain an interior face that one can walk around, following the orientation of its sides. An example is the pink face in figure 4.

We start with the following observation: Having walked along some edge e_1, we can always continue along another edge e_2 since, by assumption, the endpoint of edge e_1 is not a sink. The number of edges in graph Γ being finite, there will inevitably be repetitions in the sequence $e_1, e_2, \ldots$, which means that there exists a closed oriented path p in graph Γ. We can assume that this path is chosen without self-intersections, otherwise we could substitute p for some part of it. The red path in figure 4 is an example.

According to the Jordan curve theorem, p divides the plane into two domains. The interior domain O contains several faces $F_1, F_2, \ldots, F_k$ of graph Γ (in fig. 4, $k = 5$). Assume that path p is chosen so that the number of faces in O will be the fewest possible. To prove Stallings's lemma, it's enough to show that $k = 1$ (that is, that p goes around exactly one face).

Assuming that $k > 1$, we arrive at the conclusion that some edges of graph Γ must lie inside the domain O. Since graph Γ is connected, one of these edges f will have a common vertex o with border p of domain O (fig. 4). We will consider only the case when f "goes out" of o. The opposite case (when f "comes into" o) can be deduced from this one if we invert the orientation of all edges in Γ.

Because end o_1 of edge $f = f_1$ is not a sink, there must exist an edge f_2 coming out of it. Continuing this construction, we obtain a path q, composed of edges $f_1, f_2, \ldots$ (the blue path in fig. 4), inside domain O. There will be no self-intersections in q, as we can see from the minimality condition imposed upon path p. Therefore, the oriented path q will reach the border p of domain O at some other vertex o'. Path q breaks the closed path p into two parts, p_1 and p_2. One of these parts will come out of o' and go into o. This part, together with path q, forms a simple closed path encircling a domain in which the number of faces is less than k.

This contradiction with the choice of path p and the number k shows that the above assumption is wrong (because $k = 1$).

7. Collisions are inevitable

In Klyachko's theorem we'll deal not only with traffic rules but also with dynamic models of motion on a sphere. The reader can try to find a planar interpretation of this theorem.

In what follows, Γ is a connected, nonoriented graph on a sphere around whose faces points are moving clockwise (possibly with nonconstant velocities). To distinguish them from stationary points of the graph, let's call the moving points "cars." Thus, a car moves along the border of every facet. When we pose the problem more accurately, we'll prove that car accidents are inevitable in this situation.

It is easy to check this statement when there is only one vertex and one edge e (a loop) in the graph. For instance, if e is an equator that divides sphere S into northern and southern hemispheres (fig. 5a), then one car must go around the northern hemisphere, moving clockwise along the equator (for an imaginary observer standing at the north pole) and the other car will go clockwise around the southern hemisphere (but from the point of view of an observer standing at the south pole).

The inevitability of accidents for

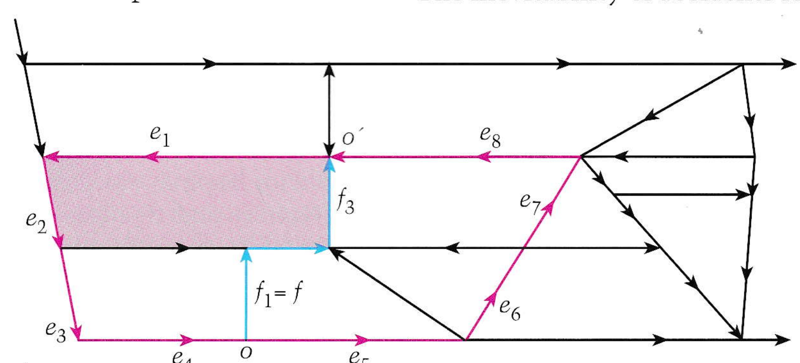


Figure 4

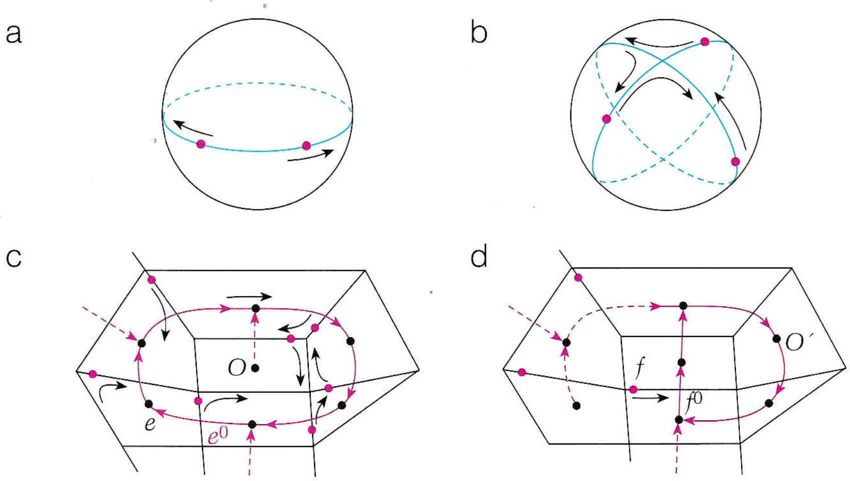


Figure 5

a graph with two vertices, which cuts the sphere into a number of "slivers" along several meridians, (fig. 5b) is not so evident, and the reader might try to organize the movement without accidents (alas, in vain). It is even less evident that the statement is true for graphs with a greater number of vertices.

Let's make some further assumptions in the setting of this problem. First of all, we assume that graph Γ has no vertices of degree 1 (otherwise one would be able to avoid collisions by "hiding" all the time at the only edge adjacent to this vertex). Second, the time it takes for any car to traverse an edge is bounded from above by a constant $T > 0$, which is the same for all cars. Otherwise, for example, some of the cars could move with a motion that is "infinitely damped," and might not ever get very far from their initial positions. Third, when a car comes to a vertex, it should immediately proceed along the next edge. (One can relax this condition a bit and allow the cars to stop on the vertices, but only for a limited amount of time.) Eventually, every car must always go clockwise around its face; reverse movement is not allowed. (For a car traveling "clockwise," the face assigned it always lies to the right of the direction of the car's movement. We'll refrain from a more detailed explanation of this notion.)

If cars go around the faces of a spherical graph Γ so that all the assumptions made above hold, the traffic is said to be *regular*.

Klyachko's theorem. No regular traffic, regardless of the initial positions of cars, can go on forever. In other words, after some time two cars will inevitably collide.

We'll prove this statement by contradiction in several steps (note that in *(a)* it is set a bit more generally than here).

(a) It's convenient to think that at every moment of time $t > 0$, there is no more than one car at a vertex of graph Γ. One can always satisfy this condition by a very small local change of the schedule of movement so that no new collisions occur.

(b) At every moment $t > T$, there is no edge with two cars standing on it at points different from its endpoints. In fact, since each car goes clockwise around its face, cars must go in opposite directions along the common edge of these faces. But this means that they will soon meet or that they met at some previous moment $t' > t - T > 0$ and now are driving away from each other.

(c) Let's consider the dual graph Γ^0. Suppose that at the moment $t > T$, the i^{th} car, going around face F_i of graph Γ (the black graph in fig. 5c), is in edge e of this face (but not at a vertex). Then we shall pick out edge e^0 of graph Γ^0, crossing edge e. Let's orient edge e^0 so it crosses edge e from right to left as one looks toward the direction of the i^{th} car's movement. Because of the results of section 2, there should be no trouble concerning the choice of orientation for edge e^0.

So, for each nonsingular moment of time $t > T$, when no car is standing at a vertex, we have constructed an oriented graph Δ_t (in fig. 5c the edges of this graph are red), consisting of all vertices of graph Γ^0 and all edges of this graph that we've picked out, with their orientation defined above. (So, the number of edges in Δ_t = the number of cars = the number of faces in Γ = the number of vertices in Γ = the number of vertices in Δ_t.)

(d) For each vertex of Δ_t there is one (and only one) edge e^0 of the same graph coming out of this vertex since, in accordance with the definition, this vertex lies inside a face F_i of graph Γ and at any nonsingular moment t, the i^{th} car, which goes clockwise around F_i, is on one of its sides.

As in section 6 above, we can find a simple closed oriented path p consisting of the edges $e = e_1, e_2, \ldots$ of graph Δ_t (the solid red path in fig. 5c). We'll designate as O the spherical domain with border p such that p goes clockwise around it.

(e) What happens when the i^{th} car goes through a vertex from edge e to the next one? It's clear that edge e^0, corresponding to e, should be removed from graph Δ_t so that the new graph $\Delta_{t'}$ (if it is defined in a unique way) contains instead of e^0 another oriented edge f^0 (fig. 5d). But edge f^0 will not leave domain O since the car goes clockwise around F_i. We can extend the oriented path, beginning with edge f^0, until we obtain a closed path p' in graph $\Delta_{t'}$, encircling a domain O' that contains fewer vertices of graph Γ (or faces of Γ^0) than O (see the solid red path in fig. 5d).

(f) The number of vertices of the graph Γ, enclosed by the sequence of paths $p, p', p'', \ldots$ constructed in this way, cannot decrease infinitely. Thus there must exist a moment $t_0 > T$ such that the graph Δ_{t_0} is not defined in a unique way. This can happen only if two cars are on the same edge of graph Γ at the same time. In view of step (b), this implies that there must have been, or must soon be, a collision.

Waves beneath the waves

Exploring the sound science behind oceanic investigations

by L. Brekhovskikh and V. Kurtepov

LIKE A LONG-DISTANCE telephone network for whales, oceanic acoustic waveguides allow sound to propagate for thousands of kilometers. Besides providing a convenient way for denizens of the deep to stay in touch, these underwater sound pathways allow researchers to collect vast amounts of data for large areas of ocean.

We have an acute interest in knowing as much as possible about the ocean's conditions, the motion of its waters, and the structure of its floor. This knowledge is used for surface and underwater navigation, finding regions of maximum biological productivity, accounting for the ocean's effect on climate and weather, and many other purposes.

Using sound waves to probe the ocean is an increasingly important oceanographic method as the limitations of traditional data collection become apparent. Many research ships from different countries continuously navigate the oceans. They record the conditions of water at different depths and the characteristics of the atmosphere above the ocean and of the ground underneath it. While these data are very important, they provide too little information to describe the current state of all the world's oceans with enough detail. To do this the data flow would need to be increased a thousand times, which can be done only with the help of satellite-based devices.

Data acquisition systems based on artificial satellites and space stations can rapidly collect data about many characteristics of the ocean over vast areas. Measurements of intrinsic oceanic radiation in the infrared and centimeter wavelength ranges make it possible to record ocean surface temperatures. By analyzing the diffusion of centimeter electromagnetic waves on the ocean's surface, we can determine the characteristics of surface waves, surface currents, and near-surface winds.

By analyzing the time necessary for an electromagnetic pulse to travel to the ocean's surface and back to the satellite, we can measure variations in the ocean's elevation over large areas. This method made it possible to discover a "dent" of about 23 m in the ocean's surface, located over the Puerto Rico Depression of the ocean floor.

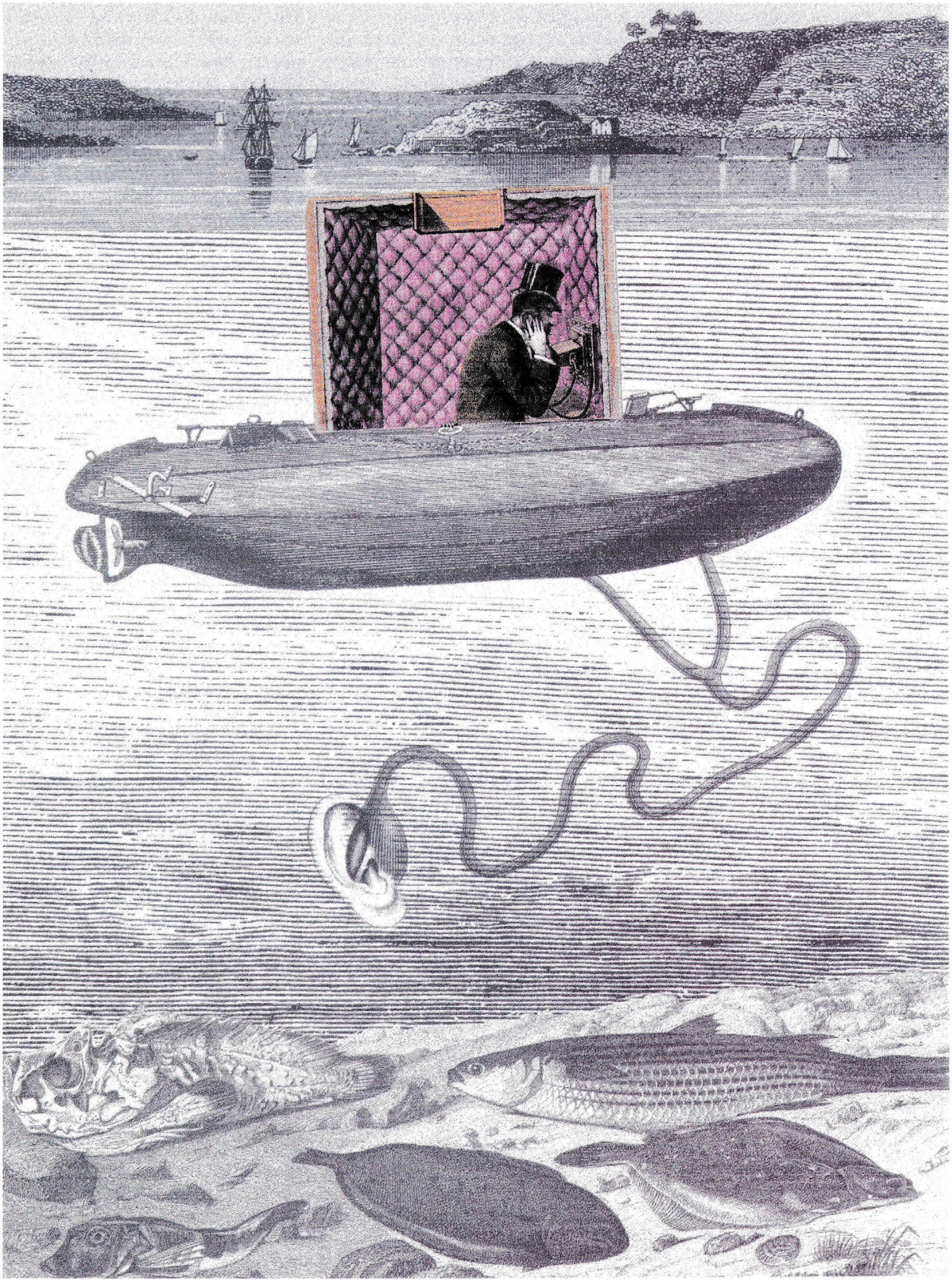
All these data, however, deal with the ocean's surface. And we can guess why: Electromagnetic waves just can't penetrate seawater to any appreciable depth. Light fades away within dozens, and a powerful laser beam only penetrates a few hundred meters.

But acoustics provides a tool to scan the oceanic depths. Only sound waves can propagate in water for very large distances. Indeed, in oceanic experiments the sound of comparatively small underwater explosions was detected by scientists at a distance of 22,000 km!

Let's recall a discovery made half a century ago. In 1946 Russian scientists in the Sea of Japan observed an interesting phenomenon. When antisubmarine bombs were exploded at a depth of about 100 m, the sound waves propagated without marked fading for hundreds of kilometers. A detailed analysis of the experimental data showed that this phenomenon results from a dependence of the speed of sound in the ocean upon depth.

More specifically, the speed of sound in seawater varies with temperature, salinity, and hydrostatic pressure. During the tests in the Sea of Japan the salinity was almost independent of depth, so it did not affect the experimental results. With an increase in depth from zero to about 300 m, the speed of sound fell due to a drop in temperature. At

Collage by Vera Khlebnikova



larger depths the variation in temperature was negligibly small, about 0.3–0.5°C. However, a further increase in depth (the maximum depth in the Sea of Japan is 3,700 m) led to a marked increase in hydrostatic pressure that caused a gradual increase in sound velocity. All these factors produce a particular dependence of sound speed on depth.

An example of such a dependence (a vetical profile of the sound speed) is shown in figure 1. The minimum sound velocity is reached at a depth of 300 m. The speed of sound increases both above and below this level. What results from such a profile of sound speed?

Figure 2 shows sound beams leaving an acoustic radiator at a depth of 100 m and propagating to a receiver located at a depth of 300 m and placed 184 km from the source. Due to sonic refraction the beams bend and return periodically to the horizontal axis corresponding to the minimum sound speed. In doing so, the entire family of beams (some of them are shown in fig. 2) neither contacts the ocean floor where the sound could be absorbed nor goes to the ocean surface where the sound could be dissipated by various irregularities. As a result, the sound travels to the receiver with very little loss in intensity via pathways known as oceanic acoustic waveguides. This phenomenon makes it possible to record sound signals thousands of kilometers away from their source.[1] Acoustic waveguides thus explain the propagation of sound over long distances observed in 1946 in the Sea of Japan. It turns out that waveguides are present in any sea or ocean, provided it is deep enough.

The emitted sound signals travel via different waveguides and arrive at the receiver at different times. Any changes in the sound speed profile, even small ones, lead to fluctuations in the arrival time of sound coming via an individual waveguide. These fluctuations can be used to characterize the particular waveguides. Since the properties of water fully determine the arrival times of the signals, the signals are an extremely sensitive indicator of their watery medium.

It turns out that the speed of sound depends not only on depth but also on the horizontal position. In fact, the speed of sound in a horizontal slice of the ocean can only be considered to be constant only within a range of a few tens of kilometers. The most important role in the horizontal variation of the speed of sound is played by the oceanic vortices discovered by Russian scientists. This discovery was the main result of the "Polygon" experiment carried out in 1970. When studying the strong current associated with the trade winds in the Atlantic Ocean (which was considered particularly stable), the researchers found that this current contains giant watery vortices hundreds of kilometers in diameter that are similar to atmospheric cyclones. Further studies showed that such vortices occur in virtually the entire ocean. The centers of these giant vortices travel at speeds of up to 300 m/h, but the rotational motion is 10 times faster. The passage of an intense vortex sometimes results in changes that can completely break an acoustic transmission or, on the other hand, produce a sonic "mirage"—that is, it can make audible usually inaudible sources of sound.

Some years ago an idea was discussed to begin long-term (about one year) observations of the large-scale variability of the ocean in an area of about 10^6 km^2 by taking advantage of the high sensitivity of sound signals to variations in the water's properties. To do this we would need to place a system of anchored acoustic generators and receivers at various depths along the equatorial perimeter of the area and then measure fluctuations in arrival times of signals traveling via all possible waveguides. As we mentioned, these signals carry information about the irregularities met on the way. In this way we can scan the depths to obtain an image of an entire volume of water. No fleet of scientific ships can perform such a huge number of simultaneous measurements. By recording the time differences in received signals we should be able to recreate the sound speed field—that is, determine the value of sound speed at different locations in the water area. This method is known as acoustic tomography (from the Greek word τομοσ—a slice or hunk). Analysis of these data provides information about vortices.

The principle method of reconstructing a sound speed field is as follows. The region under study is divided into N volume cells whose size

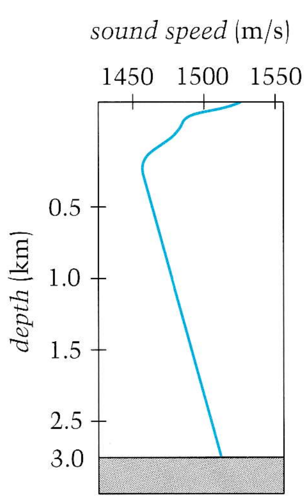


Figure 1

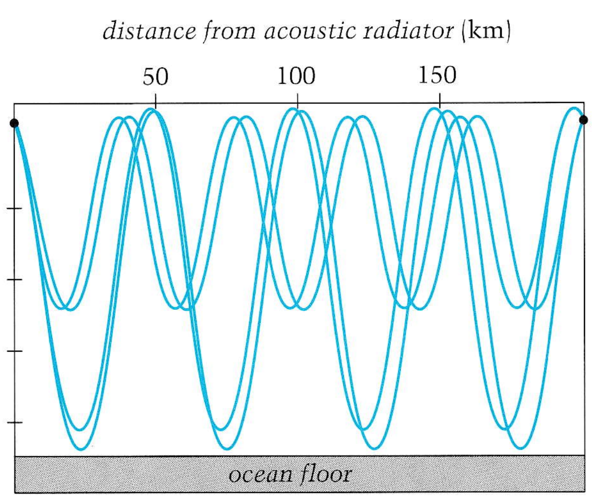


Figure 2

[1]The propagation of sound waves in the oceanic acoustic waveguides is considered in detail in the paper of A. Varlamov and A. Malyarovsky, "The Oceanic Phone Booth," in the May/June 1993 issue of *Quantum*, pp. 37-39. See also the contest problem "Sea Sounds" by Arthur Eisenkraft and Larry D. Kirkpatrick (pp. 34–36, March/April 1996). —Ed.

(tens of kilometers horizontally and hundreds of meters vertically) is small in comparison with that of the vortices. The speed of sound inside the nth cell at moment t, depth z, and horizontal coordinates (x, y) can be presented as

$$c_n(x, y, z, t) = c_0(z) + \Delta c_n(x, y, z, t),$$
$$n = 1, 2, \ldots, N.$$

Here, $c_0(z)$ is the known value of sound speed in the absence of vortices, and Δc_n is the change in the sound speed in the nth cell, caused by the vortices. We will be determining the values of the Δc_n.

The profile $c_0(z)$ corresponds to a set of M waveguides (beams), which in the absence of vortices connect all the sources with all the receivers. These "reference" beams differ from the "real" ones that propagate in the real medium with vortices. The trajectories of the reference beams and the periods t_{0m} ($m = 1, 2, \ldots, M$) of sound propagation along them are calculated by computers. When the values $|\Delta c_n|$ are far less than c_0 (which is a common situation) the real beams do not differ markedly from the reference ones. Denoting the duration of signal propagation along mth reference beam by t_m, we obtain an approximate equation for fluctuations of the arrival times $\Delta t_m = t_m - t_{0m}$ in the following form:

$$\Delta t_m = \sum_{n=1}^{N} E_{mn} \Delta c_n .$$

Considering all the possible beams, we get a set of M equations relative to Δc_n ($n = 1, 2, \ldots, N$). The coefficients E_{mn} can be calculated if the trajectory of the mth reference beam and the geometry of the nth cell are known. For the cells not "visited" by the mth beam, $E_{mn} = 0$. By measuring fluctuations of sound signal arrival times Δt_m and solving the system of linear equations, we obtain the values of Δc_n for every cell. This is the basic calculational procedure, though in practice there are additional technical and mathematical problems to cope with.

If we succeed in reconstructing the sound speed distribution $c_n(x, y, z, t)$, we can obtain the approximate values of the temperature, salinity, and density of water through relationships known in seawater physics. Information on ocean temperature for a large area of water helps determine the heat content of the ocean, which is a prerequisite for weather forecasting. Although this is by no means trivial, acoustic tomography can do much more. By measuring the difference in the times necessary for signals to travel to and from two acoustically connected points, we can find the average projection of the ocean current onto the direction of the sonic track. Researchers hope to apply this approach to measuring the large-scale circulation of the ocean, which can't be achieved by traditional methods. There are global projects to embrace huge areas of the world's oceans with a tomographic network. This would mean a breakthrough in the study of the interactions between the ocean and the atmosphere, which is the key problem in the theory of climate formation on Earth.

Another prospective application of acoustic tomography relates to the study of the ocean floor. This is not a simple task—these researchers are separated from their subject by a column of water several kilometers deep. The available methods, seismic profiling of the ocean floor and deepwater drilling, are cumbersome and expensive. However, there are alternative approaches.

The arsenal of scientific hydroacoustic devices is becoming more and more sophisticated with autonomous ground stations (AGS). These stations, equipped with sound receivers (hydrophones), specialized computers, and magnetic tape recorders, can sink to the ocean floor and work there. When a research ship moves away from an autonomous ground station, the station records the noise of the ship's engines. In accordance with the laws of sound propagation in

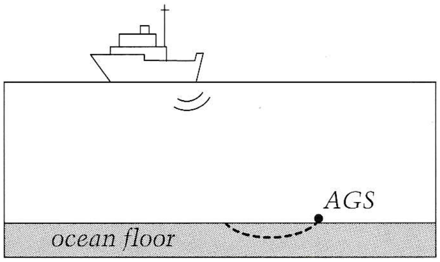


Figure 3

the ocean and within sedimentary rocks, the signals received by the autonomous ground stations travel via water and bottom pathways, the latter resulting from sound refraction in the ocean floor (fig. 3). These signals are recorded against the background noise produced by the vessel and by the sounds dissipated at the ocean surface and ocean floor irregularities. When the ship moves, the "trajectories" of water and bottom beams gradually change. The relative time lags of the engine noise arriving via both beams also varies.

Dividing the oceanic sedimentary rocks into arbitrary layers with temporarily unknown values for sound speed, we can incorporate these values into a system of equations similar to what we obtained earlier in the problem of oceanic tomography. Instead of Δt_m, this system will operate with the autonomous ground station-measured values of the arrival time lags of engine noise at different distances between the ship and the autonomous ground station. Solving this system, we reconstruct the sound speed profile in the sedimentary rocks. After data collection is complete, the station can be retrieved, and the experiment can be repeated at another location.

Acoustic methods are also widely used to study small-scale irregularities and processes in the ocean. For example, the distortions of a sound signal dissipated by the ocean floor carry information about the unevenness of the floor-water boundary and about the irregularities in the sedimentary layers. In the future, this method may lead to acoustic visualization of the ocean floor.

The gambler, the aesthete, and St. Pete

Can you flip your way to a fortune?

by Leon Taylor

IT'S A SLOW WINTER DAY. THE DOWNtown theaters are rerunning *Nightmare on Pennsylvania Avenue,* and football season is over. But here's a way to slay the time. I have a spot of change from buying this morning's *News-Free Press*. Suppose that I keep tossing a dime until it comes up heads. End of game. If I get heads on the first toss, I will pay you \$1; if I get heads on the second toss, \$2; if on the third toss, \$4; et cetera, et cetera, et cetera, as the King of Siam would say. Generally, if I get heads on the nth toss, I will pay you $\$2^{n-1}$. (Yawn.) Now, how much will you stake to partake of this diverting game of chance?

Well, if you think like a computer, you will stake your wallet, your MasterCard, your condo keys, and your Christmas air ticket to Tahiti. The expected value of this bore of a bet strolls off toward infinity.

Unexpected expected value

Hard to believe? Let's go figure. The expected value of the bet is your best mathematical guess of its worth. To compute it, you will think about everything that could happen, how likely it is to happen, and what it would be worth to you if it did happen. An example: Suppose that I had agreed to flip the coin just once. Then you would have a 50% chance of getting heads, with the grand prize of \$1, and you would have a 50% chance of getting tails, worth zilch. Heads or tails: That seems to cover all the bases. So the expected value of the bet is $(1/2)\$1 + (1/2)\$0 = \$.50$.

But I didn't agree to flip the coin just once. I agreed to keep flipping it, regardless of swollen thumbs, until it came up heads.

If that blessed event occurs on the second toss, then you will get \$2. But to get heads on the second toss, we must have gotten tails on the first toss. What is the probability of tails, then heads? Well, one toss doesn't affect the other, so we must reflect separately on each. On the first toss, the probability of tails is 50%. On the second toss, the probability of heads is 50%. We thus have half a chance of getting to the second toss and then half of that chance of getting a head. The probability of getting our first head on the second toss, then, is

$$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}.$$

The expected payoff on that throw is (1/4)\$2 = \$.50.

Onward. What's the probability that we'll get the first heads on the third toss? We must have gotten tails on the first two tosses, and then heads. So the probability of the first heads on the third toss is

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}.$$

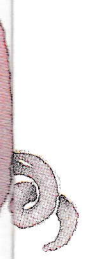
You see the pattern: The probability that the first heads will appear on the nth toss is $(1/2)^n$.

Finally, to compute the expected value of the bet, add up the expected payoffs for all the tosses that could occur. After all, you might get lucky; a head might not appear until the tenth toss—or the hundredth. We (and your tax accountant) must take all of these possibilities into account. Let's save some paper (all the paper in the world, in fact) and use a series:

$$\sum_{n=1}^{\infty}\left(\$2^{n-1}\right)\frac{1}{2^n}=\$.50+\$.50+\$.50+\ldots. \qquad (1)$$

This is an infinite number of 50-cent pieces. More than the Milky Way can ever hold.

Why do we obtain this bizarre result? On any toss, the prospect of getting heads and the payoff to it offset one another. You have a chance of 1/2 of getting heads on the first toss, but the payoff would be only \$1. You have a chance of only 1/1024 of getting the first heads on the tenth toss, but the payoff would be \$512. For any toss, the expected payoff is just \$.50. But you could get the first heads on any of an infinite number of tosses. The bet embraces this infinity of possibilities, so what we might loosely call its "expected value"—we can't really compute it—diverges to infinity.



Art by Sergey Ivanov

It is sorely tempting to look at that series and say, "Of course its value isn't finite. It has an infinite number of terms!" But suppose that you could win only \$1 no matter when you obtain the first heads. Then the expected value of the bet would be

$$\sum_{n=1}^{\infty}(\$1)\frac{1}{2^n}=\$.50+\$.25+\$.25+\ldots. \qquad (2)$$

This too is an infinite series, but the terms get smaller and smaller—shrinking, in a sense, to an atom, a neutron, a quark.... Remarkably, this series sums to just \$1. See for yourself: Punch the problem into your calculator and let it run for a few hours. Or days. Or try this:

Problem 1: Take in hand the formula for the sum of a geometric progression that is infinite. Use the formula to prove that (2) converges to \$1.

Now look back again at (1). That series, which drifts into infinity, is a real showstopper. A bet with an infinite payoff!

But you would not really fork over your wealth for that bet with an infinite payoff. Why not?

A more realistic game

Perhaps you have trouble believing that I am infinitely rich. (So does my wife.) Then, even for a long trail of tails, all I can do is pay you what I have, and that will limit the payoff that you expect. The French mathematician Poisson shows this with a nifty example. Suppose, for convenience, that the bet pays off 2^n for heads on the nth toss. Also suppose that I have only so much money—let's call this amount M. Then think about this game: I will agree to toss the coin N times or until I get heads, whichever comes first. And I will pay you only what I can. Of course, we can make N as large as we want.

To understand how my vast fortune affects the value of getting heads after many coin tosses, we need a way to relate my fortune to the number of tosses. For reasons that will become clear (I hope), Poisson would write down my fortune as $M = 2^v(1 + h)$. Here, v is an integer that we can later compare to the number of coin tosses, and h is a number such that $0 < h < 1$. First we'll find a v that will make M approximate my fortune, then we'll pick h to make the match more exact. And now, haul out your calculator from under the dirty clothes....

Problem 2: Suppose that my fortune, M, is \$67.5 million (fat chance!). What are v and h?

Whatever v we pick, look at what happens on the first v tosses. Your expectation of the bet for those tosses is

$$\left(\frac{1}{2}\right)2+\left(\frac{1}{4}\right)4+\ldots+\left(\frac{1}{2^v}\right)2^v=1+1+\ldots+1=v.$$

Remember, $M = 2^v(1 + h)$, and that's larger than 2^v. If I have to toss the coin at least v times before you get heads, then I will have to pay you all that I have, which is $2^v(1 + h)$. So the expected value of the bet (not just of the first v tosses) is

$$v+2^v(1+h)\left(\frac{1}{2^{v+1}}+\ldots+\frac{1}{2^N}\right) \qquad (3)$$

or

$$v+(1+h)\left(1-\frac{1}{2^{N-v}}\right). \qquad (4)$$

As N approaches infinity, your expectation of the bet settles down between $v + 1$ and $v + 2$. For instance, if my fortune is \$67.5 million, then you will stake no more than \$27 or \$28.

Problem 3: How do we know that (3) is right?

Problem 4: Use the formula for the sum of a finite geometric progression to show that (4) flows from (3).

Problem 5: Show that if my fortune is \$67.5 million, then you will stake no more than \$27 or \$28 to wrest it from me.

Problem 6: Show that as N approaches infinity, your expectation of the bet in (4) settles between $v + 1$ and $v + 2$.

Those calculated bets—\$27 or \$28—do look plausible. But one recalls an unsigned remark about Poisson: "When he had to choose, as between two opposing ideas, the one that he would dignify with an application of his analysis, he generally made the wrong choice." Even when, say, Caesar's Palace has a fortune smaller than that of the New York Federal Reserve, its chance of going broke need not be the reason that you will stake just a few dollars for its bet. The French mathematician Joseph Bertrand argued that, for the purposes of the game, the house could always limit its largest possible payoff by redefining the units in which it paid off. Suppose that its fortune is \$600,000, or 60,000,000 pennies. Then, for heads on the first toss, it might pay just a penny, or a speck of copper, or a grain of sand, or a molecule of hydrogen. "The fear of insolvency may be reduced without limit," Bertrand said. Still, whether or not you break the bank, the basic question remains: Why would most of us pay only a few dollars for a bet with an expected value of many thousands of dollars?

You bet your life

Another approach to the problem is to look at similar bets in the real world and ask why people won't stake their all to take them. Consider an agent that never seems to fear insolvency—the government. It issues tickets in a lottery. Ticket 1 pays \$1 if heads comes on the first toss. Ticket 2 pays \$2 if heads doesn't come until the second toss. Ticket 3 pays \$4 if heads doesn't come until the third toss. Ticket 4 pays \$8 if heads doesn't come until the fourth toss. And so forth. For 50 cents a pop, you might buy tickets 1 and 2. But would you buy ticket 50,000,000? Not according to Antoine-Augustin Cournot, the founder of mathematical economics. He pointed to the French lottery, in which one drew five numbers out of ninety and bet on various combinations. The lottery had withdrawn, for lack of sales, the option of a bet on a particular combination of five numbers. "One imagines," Cournot wrote archly, "that there must be a limit to the smallness of chances."

Maybe some probabilities are too small to perceive. In the eighteenth century, the probability that a 56-year-old man would die overnight was 1 in 10,000. Most men in the prime of their lives give nary a gloomy thought to the prospect of dying before breakfast, so one could reasonably take the probability of 1 in 10,000 as beneath notice, said the French naturalist Buffon. When you calculate the expected value of my bet, you will set this probability, and smaller ones, equal to zero. (You may wonder why Buffon did not select the probability that a man in his twenties would die overnight. Perhaps it was because Buffon fought a duel at that age, as a student in Angers. With the years, he would grow more prudish, to the point of describing gambling as *le mal épidémique.*)

As a man of the world who regarded infinity as *une idée de privation,* Buffon seemed destined to devise the proposition that some events were too unlikely for us to worry about. Upon hearing of it, Edward Gibbon guffawed. "If a public lottery were drawn for the choice of an immediate victim," wrote the English historian, "and if our name were inscribed on one of the ten thousand tickets, should we be perfectly easy?"

Would a million tickets soothe the modern mind? Federal regulators limit concentrations of hazardous substances to imposing a risk of death of no more than one in a million during 70 years of daily ingestion. So do we perceive smaller probabilities than our ancestors did? Perhaps. Or perhaps we are more afraid of dying. The selection of a threshold probability begins to look arbitrary. Condorcet proposed, as a threshold, the risk of sailing on the packet from Dover to Calais. That was before the French Revolution. (Considering Condorcet's fate, he would surely have picked, as the smaller risk, that of sailing from Calais. He perished in the French prisons.) Would we now regard the prospect of drowning in the English Channel too small to entertain while beating toward France?

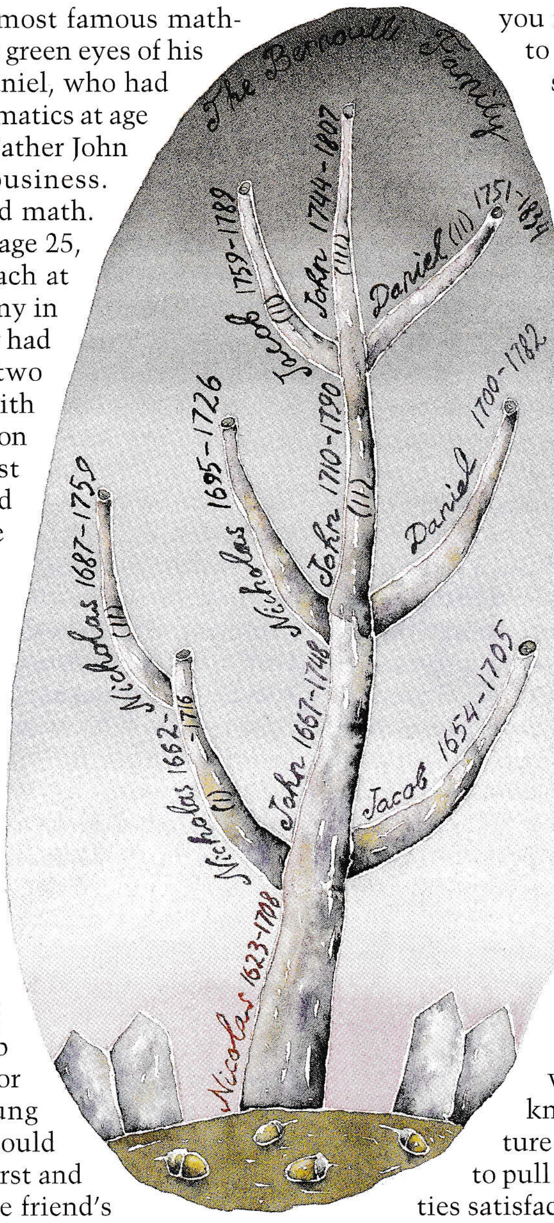
Family feud

Finally, and most simply, try an experiment in the laboratory of your mind. Suppose that you did face a house that could pay off an infinite pile of dough. Would you risk all your income to take its bet? Me neither.

But why do people refuse to take a bet with an infinite payoff—that is, with an infinite mathematical expectation? The most famous solution to that problem came from Daniel Bernoulli, who pondered the question as posed by his cousin Nicholas. The Bernoullis were a distinguished, and bedeviled, family of mathematicians. As Protestants in the Spanish Netherlands in the days of the Spanish Fury, they had fled Antwerp during the massacres by Catholics in 1583. They eventually resettled in the Swiss city of Basel, where they commenced to subtly persecute one another.

For the Bernoullis were riven by jealousy. Daniel's rebellious uncle, Jacob, became an astronomer, taking on the motto *Invito Patre, Sidera Verso:* Against my father's will, I turn to the stars. Jacob was also a superb mathematician. It was he who had first proposed how to calculate the probability of an event such as obtaining the first head on the tenth toss. He secretly taught mathematics to his younger brother John, who crossed his father's wish that he go into business. John proved to be an adept student—too adept for Jacob, who suspected that his brother was plotting to usurp him as the mathematician at the University of Basel. Maybe Jacob was half-right: When he died in 1705, John won his old job.

Arrogant, regal, aloof, the most famous mathematician of his day, John cast green eyes of his own upon his talented son Daniel, who had begun taking lessons in mathematics at age 11 from his teenage brother. Father John tried to force Daniel into business. Daniel preferred medicine and math. Luckily, he had prospects. At age 25, he was lured from Basel to teach at the new St. Petersburg Academy in Russia. However, the academy had no students, other than the two that every professor brought with him. For a mathematician bent on research, in an era when most mathematicians had to spend most of their time teaching, the academy was Elysium. But St. Petersburg—Peter's new capital shaped out of the Neva swamps at the cost of thousands of lives, including that of the architect, who came under the knout of the tsar—was young, rowdy, and raw. The court politics were hard to take, and as the Academy's backers expired, its future clouded. With illness as an alibi, Bernoulli quit the city after eight years, in 1733, virtually shaking the dust off his sandals. He yielded his job with grace (a rare commodity for a Bernoulli) to a hearty young friend from Switzerland who could dash off a paper between the first and second calls to his dinner. The friend's name was Leonhard Euler.



The aesthetic Daniel had disdained the gambling life in holy Russia. Yet from it he seems to have gained an insight into the puzzle that a gambler might pay little for a bet of seemingly infinite value: The gambler does not think of money but of its utility, of its power to procure felicity.

Suppose that each dollar more adds less and less to your satisfaction. You get more satisfaction out of the first dollar that you spend than out of the hundredth; more out of the hundredth dollar than out of the millionth. Then even if the bet pays off an infinite number of dollars, the satisfaction of that much money to you can be modest. You might pay just a few dollars for it.

Bernoulli argued that one's satisfaction in a gain would relate proportionally to the size of the gain but inversely to one's fortune. You think more highly of a $1,000 windfall than of a $10 windfall, but the $1,000 means more to you if you are a pauper than if you are a millionaire. Let x be the amount of money that you now have and let dx be an addition to your money. (Nota bene: dx is one symbol, not the product of d and x.) Let y be your satisfaction in money and let dy be the addition to your satisfaction. Then, Bernoulli argues,

$$dy = \frac{k \cdot dy}{x} \qquad (5)$$

for some positive number k.

To put (5) in English unadorned: The change in your satisfaction varies directly with the change in your money, and it varies inversely with the amount of money that you have. For example, if you are as rich as Croesus (in the days before Cyrus the Great stripped him), and if your grandmother bestows upon you a birthday check for $5, then your satisfaction will increase just a bit; dy will be a smidgen. A lot, though, depends on k. This is a positive constant that expresses one's savvy as a consumer, one's ability to cope with rapid rises in wealth. The well-educated may have high k values: You will enjoy a windfall of a million dollars more if you know just what to do with it. To capture these ideas, Bernoulli uses calculus to pull out of (5) a logarithmic function that ties satisfaction to fortune:

$$y(x) = k \log x + c,$$

where c is a constant. In fact, let's specify the constant as the product of -1, k, and the log of your level of fortune before taking the bet, a fortune we'll call α:

$$y(x) = k \log x - k \log \alpha.$$

or

$$y(x) = k\log\left(\frac{x}{\alpha}\right). \quad (6)$$

Here, y conveniently expresses your relative gain (or loss) in felicity as your fortunes wax (or wane). In the manner of the callow economists, we'll call this function the *utility* of the amount x of the money that you gain. If your fortune remains at α, then your y-odometer will remain stuck at zero, for $y = k \log 1 = 0$.

Back to the bet. For simplicity, let's set $k = 1$. You begin with fortune α. If you get heads on the nth toss, then you will add to α the amount 2^{n-1}. Your glorious new fortune x will be $\alpha + 2^{n-1}$. The utility of that fortune to you, $y(x)$, will be $\log[(\alpha + 2^{n-1})/\alpha]$. The probability that you will get the fortune is $1/2^n$. Accounting for every toss on which the first heads can occur, we can define the expected utility U to you of undertaking the bet:

$$U = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\log\left(\frac{\alpha + 2^{n-1}}{\alpha}\right). \quad (7)$$

What would you pay to take this bet? Surely you would pay no more than you thought the bet would be worth to you. We've already seen that this amount is less than the mathematical expectation of the basic St. Petersburg bet, which is infinite. The question is whether we can use that notion of utility to come up with a more precise estimate of what the bet would be worth to you in terms of satisfaction. It may help to approach the question by the back door. Suppose that you had the right to take the bet for free. Then how much would I have to pay you to induce you to yield your right? It would have to be some amount—let's call it D—that, you think, would leave you as satisfied as the bet would. Suppose that you add D to the fortune α that you already have. Then—remember (6)!—the utility of your new fortune will be $\log[(\alpha + D)/\alpha]$. You won't give up your bet unless this utility is at least as great as what you expected to receive from the bet. We must pick D so that $\log[(\alpha + D)/\alpha]$ equals the value of (7). To do this, let's put (7) in a more palatable form. Since $\log(x_1/x_2) = \log x_1 - \log x_2$, we can write

$$U = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\log\left(\alpha + 2^{n-1}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\log\alpha.$$

Since $\log \alpha$ is a constant, we can set it outside the summation:

$$U = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\log\left(\alpha + 2^{n-1}\right) - \log\alpha\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right).$$

But (manna from heaven!) that last series just sums to 1, so

$$U = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\log\left(\alpha + 2^{n-1}\right) - \log\alpha. \quad (8)$$

Finally, since we want to treat D as an argument in a log function, let's put as much of (8) into one log function as we can, by putting that sum of functions under one log roof. Since $\log x_1 + \log x_2 = \log x_1x_2$, and $r \log x = \log x^r$, we can convert (8) into

$$U = \log\left[\prod_{n=1}^{\infty}\left(\alpha + 2^{n-1}\right)^{\frac{1}{2^n}}\right] - \log\alpha. \quad (9)$$

Finally, we have our expression for D:

$$D = \prod_{n=1}^{\infty}\left(\alpha + 2^{n-1}\right)^{\frac{1}{2^n}} - \alpha.$$

Problem 7: Check that, as promised, $\log[(\alpha + D)/\alpha]$ equals the value of (9).

By Bernoulli's reckoning, if you begin with a fortune of \$100 (that is, α = \$100), then you will pay no more than about \$4 for the bet (that is, D will be about \$4. When I tried out Bernoulli's equation on my micro home companion, QBasic, I got \$4.39). In other words, you will give up the bet for offers that well exceed \$4.

Problem 8: Verify Bernoulli's estimate that if you begin with a fortune of \$100, then you will pay no more than about \$4 for the bet.

Problem 9: If you begin with a fortune of \$1,000, about how much will you pay?

In effect, Bernoulli had usurped the old view of expectation, which legal scholars of the seventeenth century had developed to chip away at the church's tenet that gambling and usury were unjust. They had sought to compute a just return for bankers and insurers who shouldered risk. Justice, as they saw it, meant that all would charge the same price for a risk—its mathematical expectation. In this exegesis of expectation, one peers into the future like a stone-faced judge. By contrast, in Bernoulli's theory, one deliberates on the future like an anxious merchant. The value of the risk varies from one person to another; not all will pay the same price. That argument nettled Nicholas Bernoulli, a professor of Roman and canon law. He objected that his cousin had failed "to evaluate the prospects of every participant in accord with equity and justice." Daniel replied, quite simply, that his theory "harmonized perfectly with experience." He was calibrating the math to fit the world.

What's it really worth?

Fast forward by more than a century. Among economists, the Young Turks arrive, brandishing their scimitars of calculus. They cut up, with relish, the teaching of the Old Turks that the value of a good depended on the amount of labor that went into its making. No, they say, the value of a good depends upon the consumer's satisfaction in it, especially in the last unit consumed. The Young Turks call that unit satisfaction *marginal utility,* and they make it their touchstone. Curiously, they credit the concept not to Daniel Bernoulli but to an English philosopher, Jeremy Bentham.

By the twentieth century, economists were finally coming to grips with the sage from St. Petersburg. Among them, Karl Menger raised a red flag. What if you raised the bet from 2^{n-1} on the nth toss? You could do this by specifying a log function that had no bounds. As the amount of money, x, loped off into infinity, so did the utility from money, $U(x)$. For instance, it was quite possible to find a sum of money x_n such that $U(x_n) = 2^{n-1}$. But in that case, the utility of the bet was

$$\sum_{n=1}^{\infty}\left(2^{n-1}\right)\left(\frac{1}{2^n}\right) = \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \ldots .$$

Welcome back to infinity.

The obvious solution was to set a ceiling on the amount of satisfaction that one could receive from any amount of money—that is, to bound the utility function. In (6), this amounts to specifying a maximum value for $y(x)$—say, a million utils for any income that is at least $10 million. For example, a gambler with this utility function would reap no pleasure from receiving more than $10 million. So he would pay no more than $10 million for any bet. In fact, Bernoulli knew of that solution. His paper reprinted a remark by yet another Swiss mathematician, Gabriel Cramer, that the value function really should be bounded. (Cramer had a knack for unknotting problems. Cramer's rule, for using matrices to solve simultaneous equations, still helps us model everything from hurricanes to world economies.) Unfortunately, Cramer concocted a function in which the total value of money equaled the total amount of money until one had amassed, say, $10 million, after which additional amounts of money were worth nothing. Marginal value suddenly swooped from one to zero. Surely the descent should be more gentle than that.

A utility function that graphs as a hyperbola will do the trick more neatly than Cramer's function. Let w be wealth and let Z be the state of bliss. Then scribble down the function

$$U(w) = \frac{Zw}{Z + w}. \tag{10}$$

This, for a utility function, is a cream puff. See for yourself!

Problem 10: In (10), show that $U(w)$ gives us zero utility for zero wealth; marginal utility of one for zero wealth; and a smooth, asymptotic hang-glide to Z in utility as wealth increases.

What does all this have to do with today? Well, why do societies invest so little in accumulating knowledge? Consider the dire straits of pure mathematics—the basalt of science and technology. Pure math enables economies to grow, yet we spend little to support it. The same was true in Daniel's day. Although rulers recognized that mathematics would improve the navigation of their ships and the precision of their cannons, few would pay much for it. The short list begins with Frederick of Prussia and ends with Catherine of Russia. (Theirs was money well spent, on mathematicians obsessed. While paging through the Aeneid, Euler chanced upon this line: "The anchor drops, the rushing keel is stay'd." He dropped his book, seized his pen, and modeled the swaying of the ship.) What whetted the jealousy of the Bernoullis was their perpetual scramble for a handful of jobs and prizes: John Bernoulli tossed one son out of the house for winning a prize from the Academy of Sciences in Paris that John had vied for. "After all," remarks a sparkling historian of math, E. T. Bell, "if rational human beings get excited about a game of cards, why should they not blow up over mathematics, which is infinitely more exciting?"

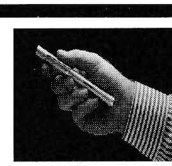
For the nation, investment in basic research is a gamble, a groping for high-payoff, low-probability ideas, a pursuit of heads on the hundredth toss. If expected payoffs were all that mattered, then we would be fools not to spend more on research. But bend your ear to Bernoulli: What matters is not the amount of money that we might make—it's the satisfaction that we would derive. Suppose that we don't have much use for another half billion bucks. Then we might well prefer to keep the income that we have to risking part of it on a gamble.

For that reason, appeals to finance more research because it will someday pay off in cold cash won't stir the nation. What may open hearts and wallets, however, is an appeal to ante up for research because it is a gamble, a historic thrill. Think of the race to the Moon!

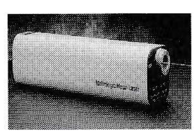
ANSWERS, HINTS & SOLUTIONS ON PAGE 50

METROLOGIC® EDUCATIONAL LASERS

Helium-Neon Lasers
ranging from .05 to 5.0mW
NOW AVAILABLE IN
GREEN (.05mW only)



Laser Pointer 3mW
Only $49.95
Optics Kit $20.00



Various Kits Also Available
* speed of light
* build a laser kit
* laser optics lab
* physical optics lab
* power meters

1-800-667-8400
In New Jersey 609-228-8100
90 COLES ROAD
BLACKWOOD, NJ 08012
EDUCATIONAL DISCOUNTS AVAILABLE

Circle No. 1 on Reader Service Card

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M221

Seven-digit must. Find the length of the longest geometric progression, all of whose terms are seven-digit positive integers.

M222

Triangular heights. In triangle ABC, $\angle BAC$ equals α. The circle inscribed in the triangle touches its sides at points K, L, and M, and M lies on BC. Show that the ratio of the length of altitude MM_1 of triangle KLM to the length of altitude AA_1 of triangle ABC equals $\sin(\alpha/2)$.

M223

Radical proof. Prove that

$$\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}.$$

M224

Solvable system. Solve the following system of equations:

$$\begin{cases} \dfrac{x+y}{1+xy} = \dfrac{2-x}{1-2x} \\ \dfrac{x-y}{1-xy} = \dfrac{3+y}{1+3y}. \end{cases}$$

(For a hint, read the first three lines of our printed solution.)

M225

Infinite progress—perhaps. The set of natural numbers is divided into two parts. There are no three-member arithmetic progressions in one of them. Must the other part have an infinite arithmetic progression? (A. Skopenkov)

Physics

P221

Rod hits a wall. A rod of length l = 10 cm slides with velocity

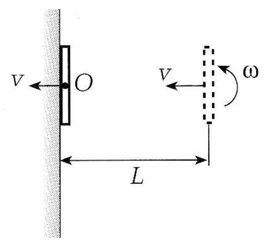


Figure 1

v = 10 cm/s and rotates on a smooth horizontal surface. What angular velocity is necessary for the rod to hit the wall flatwise (see fig. 1) if it was oriented parallel to the wall at a distance L = 50 cm. (I. Poterayiko)

P222

Short life of a soap bubble. A girl blows a soap bubble using a long tube. After inflating the bubble, she opens the end of the tube, which causes the bubble to collapse in some time τ. What is the lifetime of a bubble with twice the diameter? Assume that the air moves slowly inside the tube and the properties of the soap film are identical in both bubbles. (D. Kuptsov)

P223

Such a simple circuit. The circuit diagram in figure 2 contains ideal batteries and identical ammeters. What will a voltmeter show in this circuit? What can the resistance of the ammeters and the voltmeter be? Remember, sometimes actual

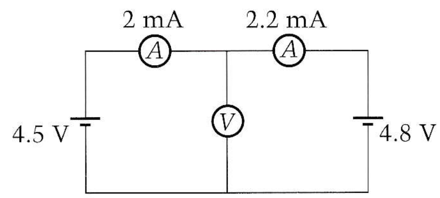


Figure 2

devices are far from ideal! (A. Zilberman)

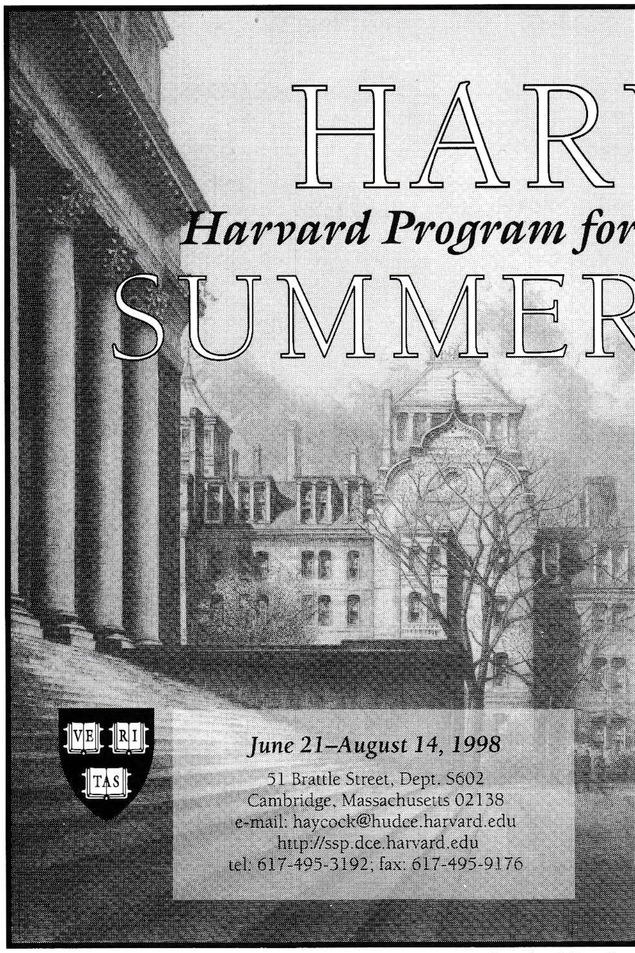
P224

Electric discharge in a gas. At room temperature, only a small portion of neon (Ne) atoms are broken down into electrons and ions during an electric discharge in a rarefied neon gas. Neon's atomic mass is $4 \cdot 10^4$ times larger than that of an electron. The mean free path of the electrons (that is, the mean average distance an electron moves between collisions) is l = 0.1 mm. The gas is in an electric field E = 10 V/cm. Determine the mean kinetic energy of the electrons and their corresponding temperature. The Boltzmann constant is $k = 1.38 \cdot 10^{-23}$ J/K, and the electric charge of an electron is $e = 1.6 \cdot 10^{-19}$ C. (D. Kuptsov)

P225

Shining thread. A source of light is a thin thread of length l = 10 cm located on the principal axis of a converging lens with focal length f = 5 cm and diameter D = 1 m. The near end of the thread is located a distance a = 10 cm from the lens. Find the minimum size of the illuminated spot on a screen placed on the other side of the lens perpendicular to the lens's principal axis. (A. Zilberman)

ANSWERS, HINTS & SOLUTIONS ON PAGE 45



HARVARD

Harvard Program for Secondary School Students

SUMMER SCHOOL

VE RI TAS

June 21–August 14, 1998

51 Brattle Street, Dept. S602
Cambridge, Massachusetts 02138
e-mail: haycock@hudce.harvard.edu
http://ssp.dce.harvard.edu
tel: 617-495-3192; fax: 617-495-9176

The Harvard Summer School offers 1,000 qualified high school students who have completed their junior or senior year the opportunity to earn college credit. They sample college life and live and learn with interesting students from around the world.

Secondary School Program students may tour New England colleges, learn about college selection and admissions, and attend a college fair. They use Harvard's libraries and laboratories, visit its museums, attend dances, concerts, and movies; and take part in the orchestra, chorus, pops band, a talent show, a trivia bowl, mealtime language tables, and many intramural athletic games.

Located in Cambridge, the ultimate college town, Harvard is just four subway stops from downtown Boston. Atlantic Ocean beaches, historic New England villages, and Walden Pond are closeby.

Circle No. 2 on Reader Service Card

"Seeing is believing ..... A *great* way to turn theoretical discussions into practical laboratory exercises."

Dr. Andrew J. Leavitt, Department of Chemistry, West Georgia College

New Pricing

Many students have the opportunity to view atoms in their classrooms, shouldn't yours?

- *High performance, student-ready research/teaching instrument for atomic resolution microscopy (both high school and college environments).*
- *Affordable - new pricing and financing options now available!*
- *Pre-defined experiments with student workbook for easy adaptation into your lesson plans. Includes theory and laboratory exercises with pre-qualified sample set.*
- *Windows-based True Image™ Software provides student experience with state-of-the-art data collection, analysis, and digital image processing.*

Call 1-800-425-8085 ext. 2200 for more information regarding Burleigh's educational systems. Ask about the New METRIS™!

burleigh

Instructional Scanning Tunneling Microscope™

Burleigh Instruments, Inc.
Burleigh Park, P.O. Box E
Fishers, NY 14453-0755
Tel. (716) 924-9355
Fax. (716) 924-9072
e-mail spm@burleigh.com

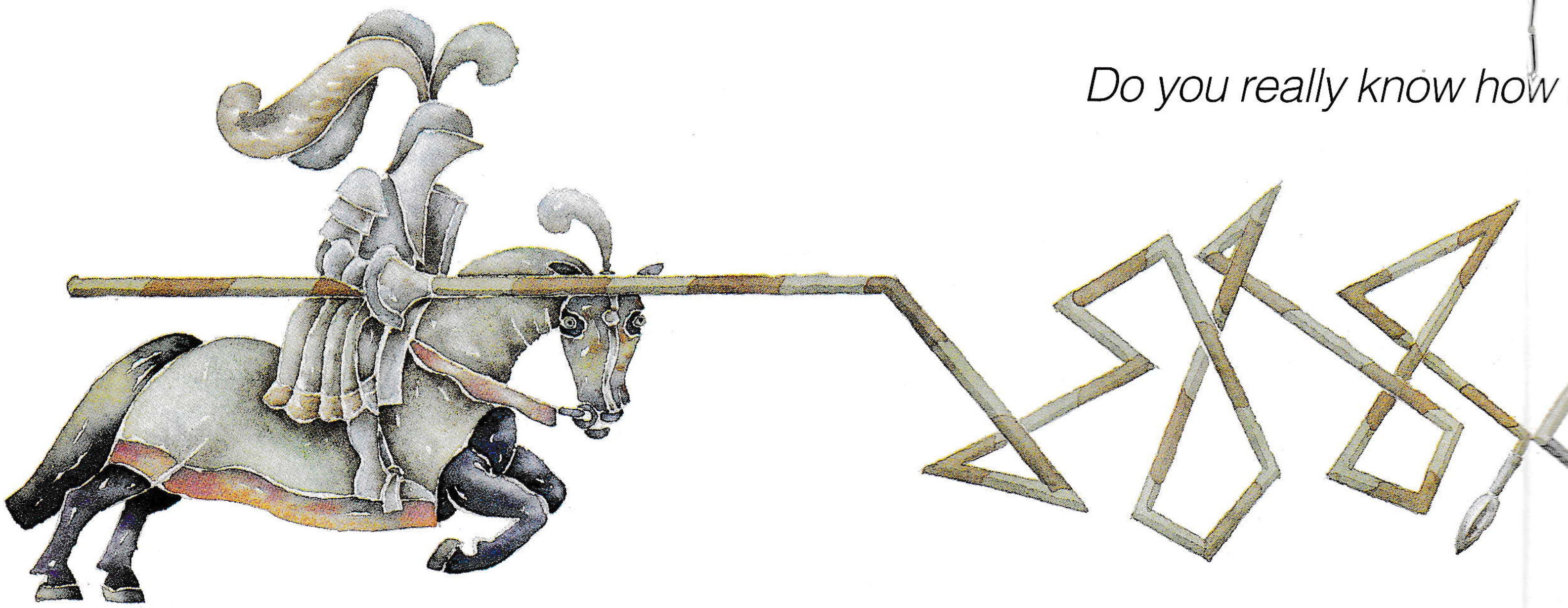
The Power of Precision in Scanning Tunneling Microscopy.

Circle No. 3 on Reader Service Card

"Because of their size, the motion of the tiny bits of matter cannot be seen."—*Mikhail Lomonosov*

Molecular

Do you really know how s



SOME STROKES OF THE GENeralized portrait of molecules: They are so small that if everybody on Earth donated one billion molecules, the total mass of the collected matter would be no more than a few billionths of a gram...

They are so numerous that if a glass of "labeled" molecules was poured into an ocean, then after a long period of time a glass of water from any ocean would contain no less than 200 labeled molecules...

They are so crowded that a molecule in a gas under usual conditions has 10 billion collisions per second with other molecules...

They are so quick that a vertically moving air molecule could reach an altitude of about 10 km before being stopped by the force of gravity, if it didn't collide with other molecules...

However, in spite of their small size and brisk motion, molecules betray their presence in a number of ways. We hope your solutions to the following problems will not permit the molecules to hide from eyes armed with modern knowledge.

Questions and problems

1. Why is the volume of a mixture of water and alcohol less than the sum of the volumes of water and alcohol taken individually?

2. Why does a highly inflated and tightly closed rubber ball shrink over the course of several days?

3. A tube with walls made of a microporous material is placed inside a reservoir from which the air is pumped out. If the tube contains a mixture of gases, the gas that accumulates in the reservoir has a higher concentration of the lighter molecules. Why?

4. There are three isotopes of hydrogen with mass numbers of 1, 2, and 3. Ions of which isotope move more rapidly to the cathode during electrolysis?

5. One face of one end (B) of a glass slide is covered with a layer of copper. The slide is suspended on a thread as shown in figure 1. It does not move in air, but when chlorine is substituted for the air, the slide rotates with side B moving forward. Explain this phenomenon. Hint: The chlorine molecules are adsorbed by copper, but they are reflected by glass.

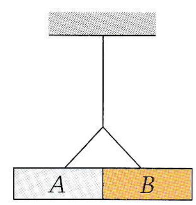
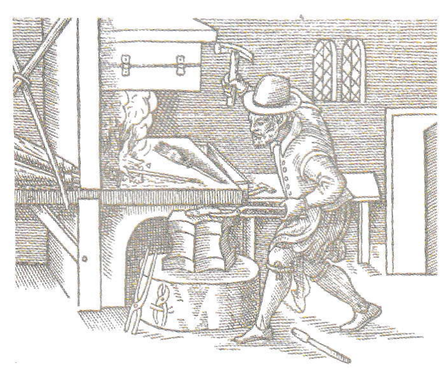


Figure 1

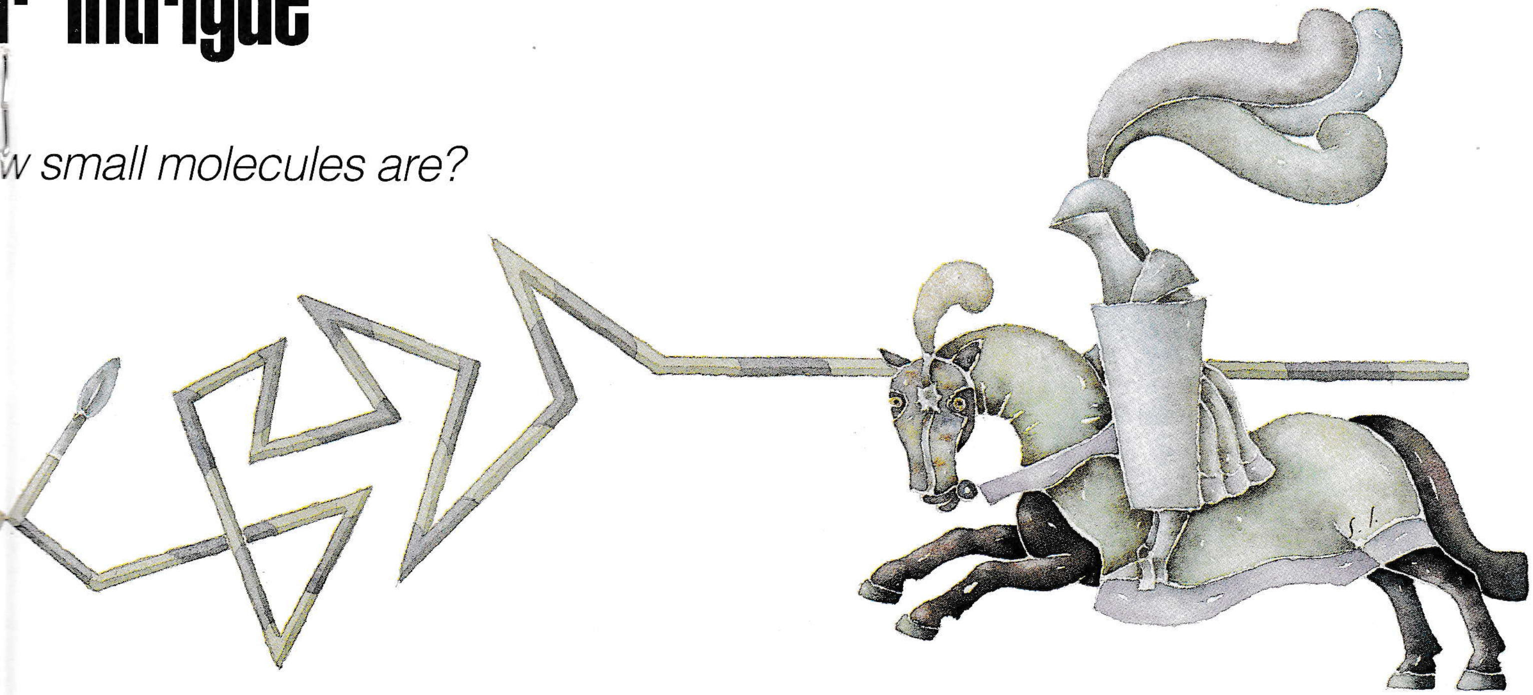
6. To weld one piece of iron to another, a blacksmith heats both



r intrigue

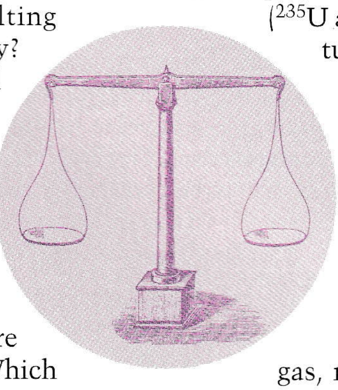
w small molecules are?

"The origin of things is not visible to the human eye."—*Lucretius*

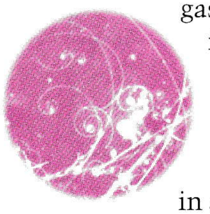


pieces white-hot in the flame of a forging furnace, places one piece over another on the anvil, and hammers them together with strong strokes. The resulting joint is strong. Why?

7. Two identical vessels are set on each side of an equal-arm balance. One of the vessels is filled with dry air, the other with wet air of the same pressure and temperature. Which vessel is heavier?



8. In what atmospheric layers does air behave more like an ideal gas—near the Earth's surface (the troposphere) or "up above the world so high" (that is, in the stratosphere)?

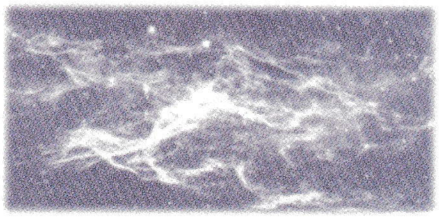


9. The magnetic field in a Wilson cloud chamber is uniform. Why does the track of a particle have a varying (continuously decreasing) curvature?

It is interesting that...

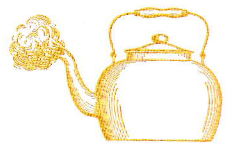
...in the Tennessee River valley a giant fabric was constructed that separated two isotopes of uranium (^{235}U and ^{238}U). The gaseous mixture of the two isotopes circulated throughout the chambers with porous walls. The lighter molecules diffused through the walls more rapidly than the heavier molecules, which made it possible to separate the isotopes.

...in clouds of interstellar gas, researchers found not only comparatively simple molecules, such as water and ammonium, but also complex organic compounds. The compounds were detected by emission and absorption spectral lines in the radio frequency range.



Microexperiment

Heat water in a kettle just to the boiling point and turn the burner off. Why does a strong vapor jet immediately shoot upward from the kettle even though no vapor could be seen before?



—Compiled by A. Leonovich

***Quantum* articles about molecular kinetics:**

1. Kikoin, A., "Temperature, heat, and thermometers," May 1990, pp. 16–21.
2. Schreiber, H. D., "Below absolute zero," January/February 1997, pp. 23–27.
3. Stasenko, A., "An ideal gas gets real," September/October 1993, pp. 42–43.
4. Stasenko, A., "Love and hate in the molecular world," November/December 1994, pp. 10–13.
5. "What a commotion," May 1990, pp. 32–33.

ANSWERS, HINTS & SOLUTIONS ON PAGE 48

Hyperbolic tension

A handier way of measuring a superficial coefficient

by I. I. Vorobyov

THE USUAL WAY OF MEASuring the coefficient of surface tension of a liquid is based on how far the liquid is drawn into a capillary tube. However, the capillary tubes and a microscope to measure their internal diameter are not always readily available. Fortunately, two glass plates can be used in place of the capillary tubes.

Begin by submerging the plates parallel to each other in a vessel of water. Slowly bring them close to each other, maintaining the parallel orientation. When the plates are very close together, water will rise between them due to the force of surface tension (fig. 1). From the height y and width d of the wall of water between the plates, we can easily find the coefficient of surface tension σ. Indeed, the force due to the surface tension is $F = 2\sigma L$, where L is the length of a plate (the factor 2 appears because the wall of water is pulled up by both plates). This force counterbalances the weight of the wall of water with mass $m = \rho L d y$, where ρ is density of water. Thus,

$$2\sigma L = \rho L d y g.$$

Therefore,

$$\sigma = \frac{1}{2}\rho g d y. \qquad (1)$$

Our setup makes it possible to perform an interesting experiment. Let's push the plates together at one end and leave a small separation at the other (fig. 2). Water will rise between the plates and form a wonderfully regular surface (of course, you should use clean, dry glass plates). It is not difficult to understand that the vertical cross-section of this surface is a hyperbola. Indeed, insert the separation (how it depends on x) into equation (1) instead of d. As we can see from the similar triangles in figure 2,

$$d = D\frac{x}{L}.$$

Here D is the separation at the end of the plates, L is again the length of the plates, and x is the distance from the reference point (located on the line where the plates are in contact) to the position where we calculate the height of the water level and clearance between the plates. Therefore,

$$\sigma = \frac{1}{2}\rho g y D\frac{x}{L},$$

or

$$y = \frac{2\sigma L}{\rho g D}\cdot\frac{1}{x}. \qquad (2)$$

Equation (2) describes a hyperbola.

To carry out this experiment, use glass plates that are 10 by 20 cm. Use a match to fix the separation and a cuvette from a photo lab as a vessel. The height of the water level can be conveniently measured if you affix graph paper to the outer surface of one of the plates. When you have a curve drawn by the water, make sure that it is a hyperbola. Recall that the area of any rectangle under the hyperbola is the same (fig. 3).

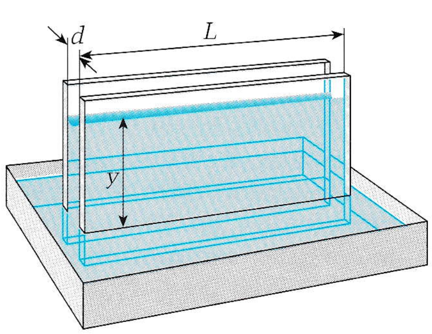


Figure 1

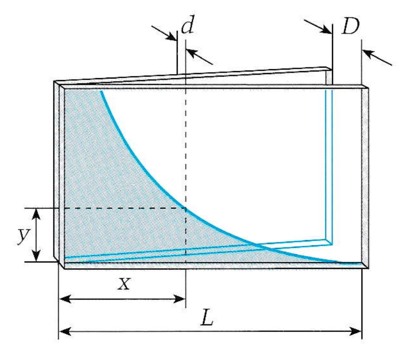


Figure 2

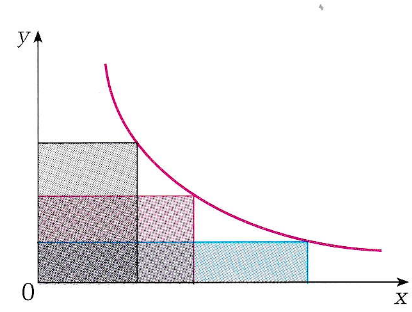


Figure 3

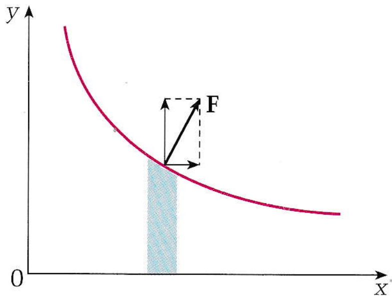


Figure 4

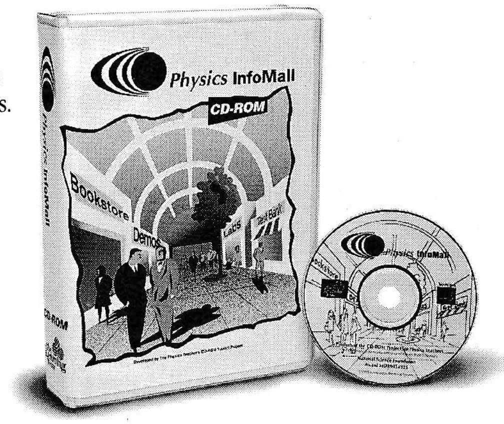
Other instructive experiments can be carried out with our setup. For example, you could use a thermometer to measure the water's temperature and find the dependence of surface tension on temperature. You could also study the effect of dissolving substances in the water on this physical phenomenon.

At last, think about this question: The direction of the surface tension **F** is perpendicular to the line where the water surface contacts the glass (fig. 4). The vertical component of this force is counterbalanced by the weight of the water column. What force counterbalances its horizontal component?

Welcome to Content!

Welcome to Physics InfoMall. Welcome to over 35,000 pages, 11 complete books, 6 study guides and the contents of 19 textbooks. The Physics InfoMall *is the largest collection of physics resources ever assembled for students and teachers.* You can offer students more than 1000 demos and labs to help them conduct experiments ... 3000 articles for researching concepts ... and 100 equations linked to problems and graphics.

Use it to prepare and print background materials ... and supplement Problem Sets with more than 3000 problems and solutions. ***Call for a free extensive guide to planning your Physics program using the Physics InfoMall.***

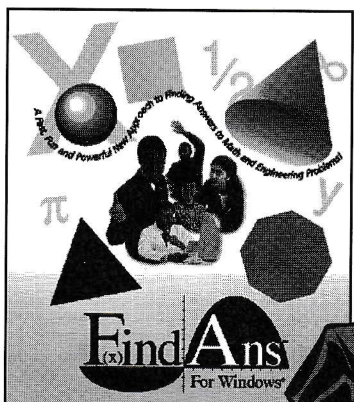


The Learning Team

For more information or to order call: 1-800-793-TEAM
Fax 914-273-2227 http://learningteam.org

Circle No. 4 on Reader Service Card

SOLVE GRAPH! PUBLISH



FindAns™ is your complete mathematics software solution! More than a graphing calculator, FindAns combines high-level computational power with scientific wordprocessing. Tap into **over 50 mathematical tools**, from calculus, trigonometric and geometric functions to vectors and complex systems of equations.

special $89 offer
if ordered by 3/31/98

FindAns offers a unique approach to problem solving where your observations play a critical role in every solution. Students and professionals alike can use FindAns to solve and express mathematical problems with graphs, colorful drawings, text and symbols.

FindAns is designed to help you solve problems and communicate information of a mathematical nature, from homework assignments to scientific studies.

FindAns requires Microsoft Windows 95 or Windows 3.1, and 8 MB of RAM.

Order by March 31, '98, to save $40 and receive a free, FindAns sweatshirt!

FindAns For Windows

www.bsesoftware.com
Ph/Fax (773) 264-0708

BSE
Bradbury Software Enterprises, Inc.
BSE, Inc.
9205 S. Michigan Ave.
Chicago, IL 60619

Circle No. 5 on Reader Service Card

Does your library have QUANTUM?

If not, talk to your librarian! *Quantum* is a resource that belongs in every high school and college library.

"It should be in every high school library [and] in most public libraries ... we owe it to our students to make *Quantum* widely available."—**Richard Askey, Professor of Mathematics at the University of Wisconsin, Madison**

Share the QUANTUM *experience!*
To order, call 1 800 SPRINGER (1800 777-4643)

PHYSICS CONTEST

Local fields forever

"What doth gravity out of his bed at midnight?"
—Shakespeare, Henry IV

by Arthur Eisenkraft and Larry D. Kirkpatrick

AN ASTRONAUT'S DAY MAY be more intense than yours, but the physics is basically the same. You get in an elevator and feel a bit heavier as the elevator accelerates upward. The astronaut's shuttle accelerates upward and the astronaut feels three times as heavy. You jump off a step and experience weightlessness for less than a second. The astronaut lives on the space shuttle or *Mir* and experiences weightlessness for days or months.

How do we comprehend these sensations of feeling heavier or lighter? Can we distinguish between accelerations and gravitational fields? The elevator problem is a classic in elementary physics: If an elevator accelerates a 60 kg student upward at 3 m/s^2, what is the student's weight? If you are standing on a bathroom scale in a stationary elevator (as only a dedicated, socially secure physics student would), the scale supports you by applying a force equal in magnitude to Earth's gravitational force. The force exerted by the scale is what we call your "weight." When the elevator begins to ascend, the scale pushes on you with an additional force—enough to accelerate you.

$$\sum F = ma$$
$$F_s - F_g = ma$$
$$F_s = ma + F_g$$
$$F_s = m\left(a + a_g\right)$$

$$F_s = (60 \text{ kg})\left(3 \text{ m/s}^2 + 9.8 \text{ m/s}^2\right)$$
$$= 588 \text{ N},$$

where F_s is the force of the scale and F_g is the gravitational force.

The scale reads 588 N, or 180 N more than your normal weight. But it's not just the scale's reading. You really do feel heavier while the elevator accelerates upward. You experience the same sensations that you would have on a planet with a stronger gravitational pull.

If we move to two dimensions, we have a more interesting effect. As your car accelerates, you are pressed against the seat. It is almost as if another planet appeared behind your car. You now experience the gravity of Earth and the apparent gravity connected to the acceleration. If the car accelerates at 9.8 m/s^2, you will experience the vector sum of the effects of the gravity and the acceleration. Ignoring the prior knowledge that cars are horizontal, the two accelerations are indistinguishable.

An interesting experiment to perform involves a helium balloon tied to the seat in your car. As the car accelerates forward, the helium balloon will lean forward. There are two distinct ways of explaining why. The first involves the inertia of the air. An acceleration forward compresses the air in the rear of the car. The increased pressure in the rear of the car forces the balloon forward to the area of lower pressure. Alternatively, we can imagine the acceleration of the car as being equivalent to a gravitational field pointing backward. We can find the effective gravity by finding the vector sum of the two gravitational forces. The helium balloon will point opposite this vector sum in the same way that it points opposite the Earth's downward pull. We can call the vector sum of the gravitational fields the "local field."

Both approaches, pressure differences and local fields, can be used to explain the motion of the helium balloon. What's so nice about the field interpretation is that the angle of the balloon can provide an instantaneous calculation of the car's acceleration. If the angle is 45°, the car must be accelerating at 9.8 m/s^2.

We live on an accelerating planet. Earth is rotating on its axis, and all objects on Earth must have a centripetal force pulling them toward the axis of rotation. Objects at the equator require a large centripetal force in comparison to objects in the United States. Objects at the North Pole require no centripetal force at all. What force is responsible for this centripetal force? It must be gravity since gravity is the only force present. If a component of gravity is required for the centripetal accelera-

Art by Tomas Bunk


THE GLOBE
Elevator
Elevator
TOMAS BUNK

tion, then the remaining gravitational force must be considered the object's weight. Assuming a spherical Earth, the weight of an object at the equator (what a bathroom scale would read) would be less than the weight of that same object at the North Pole.

The bathroom scale in the elevator had a different reading because of its acceleration. We can use this weight as a measure of the local field. Similarly, we can consider the local field of Earth at each latitude. The local field is equal to the vector difference of that gravitational field and the centripetal acceleration. These local field effects are quite real. The astronauts sense them, we sense them, and Earth senses them. The local field defines the direction an object falls and the perpendicular to the surfaces of liquids. Over time, the local field has actually changed the shape of Earth!

Our contest problem begins by asking you to find some local fields on an idealized spherical Earth.

(1) Calculate the local field at the equator, the North Pole, and at 40° latitude.

(2) Determine the angular deviation of the local field at 40° latitude from the radial line toward the center of Earth.

(3) The local field at the equator is along a radial line; the local field at the North Pole is also along a radial line. For all other latitudes, the local field deviates from the radial direction. For which latitude is the deviation of the local field from the radial line the greatest? Calculate this deviation.

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington, VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space.

A physics soufflé

We asked our readers in the July/August issue of *Quantum* to analyze the atmosphere of an unknown planet using an uncalibrated graph of pressure versus time generated as a probe descended vertically. We begin by examining a small rectangular slice of the atmosphere with a horizontal cross-sectional area A and thickness Δy. The force due to the pressure differences on the slice in the vertical direction must support the weight of the air within the slice. Therefore,

$$P_l A - P_u A = \rho A \Delta y g,$$

where P_u and P_l are the pressures at the upper and lower surfaces, ρ is the mean density of the air, and g is the value of the local gravitational field. Thus,

$$\Delta P = \rho g \Delta y.$$

If we divide both sides by a small increment of time Δt, we can find how the pressure changes with time:

$$\frac{\Delta P}{\Delta t} = \rho g \frac{\Delta y}{\Delta t} = \rho g v,$$

where v is the speed of the probe.

Because we do not know the density of the gas, we use the ideal gas law

$$PV = nRT$$

to find that

$$\rho = \frac{nM}{V} = \frac{PM}{RT},$$

where n is the number of moles of gas, M is the molecular weight, R is the gas constant, and T is the absolute temperature of the gas. This leads us to

$$v = \frac{\Delta P}{\Delta t} \cdot \frac{RT}{PMg}.$$

We use this relationship to find the speed of the probe just before it hits the surface. We find the surface value of $\Delta P/\Delta t$ by calculating the slope of the pressure-time curve just before it flattens out. We estimate a value of 0.060 ± 0.006 units/s. Using values of $T_s = 400$ K, $P_s = 60$ units from the graph, $M = 44 \cdot 10^{-3}$ kg/mol, $g_s = 9.9$ N/kg and $R = 8.3$ J/mol·K, we obtain $v_s = 7.6 \pm 0.8$ m/s, where the uncertainty in the speed is due to our uncertainty in the slope. We note that the unit of pressure, and therefore the calibration, does not matter because the units in the ratio $\Delta P/P$ cancel. Therefore, the pressure calibration is not needed for this measurement.

At a constant speed it takes the probe

$$t = \frac{h}{v_s} = 2000 \pm 200 \text{ s}$$

to fall from a height $h = 15$ km. At $t = 2000$ s, $P = 15.0$ units, $g = 9.8$ N/kg at altitude, and $\Delta P/\Delta t = 0.012$ units/s, we get

$$T = \frac{PMgv_s}{R\dfrac{\Delta P}{\Delta t}} = 490 \text{ K}.$$

Using the other times and the corresponding pressures and slopes, we estimate that there is an uncertainty of 90 K or more in the temperature. Therefore, we are not able to determine the temperature very accurately.

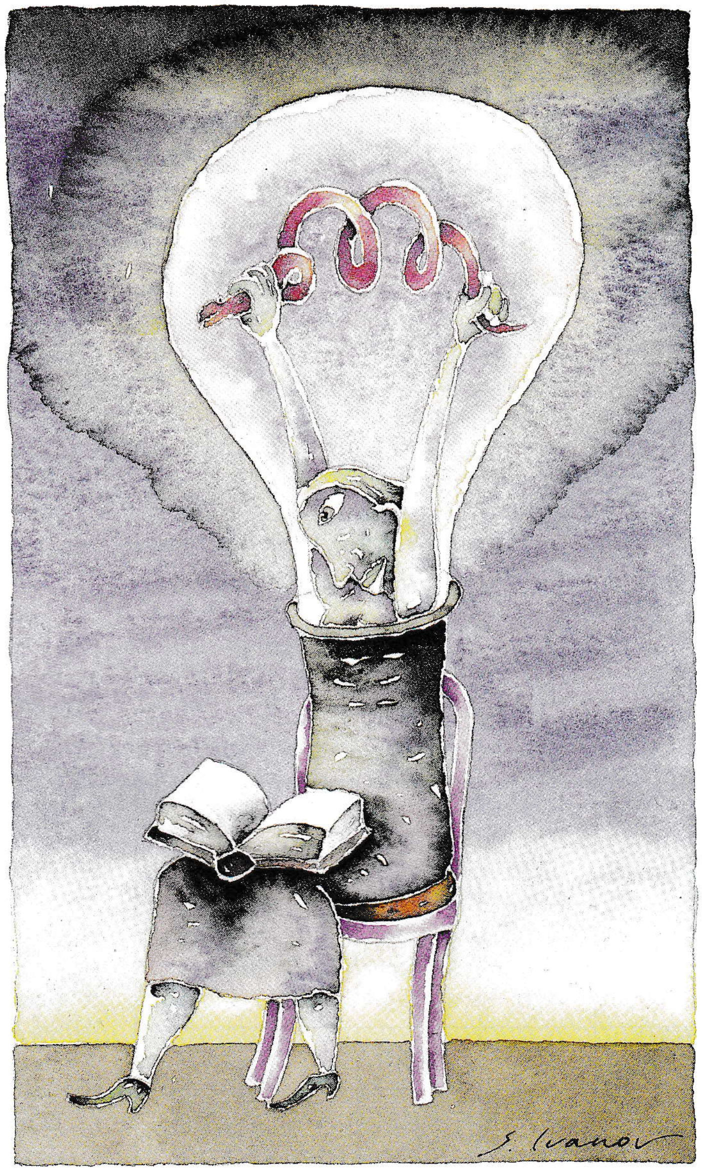
A significant factor in the analysis is the assumption that the probe has a constant speed for the entire 15 km. If the probe had reached terminal speed before it reached an altitude of 15 km, the probe should slow down due to the increased air resistance as the atmospheric density increases. If the probe takes 20% longer to complete the descent from 15 km, the calculated temperature decreases to about 430 K. If the probe had not reached terminal speed at an altitude of 15 km, the descent time could be quite a bit shorter, resulting in a much higher temperature. We conclude that we really do not know the temperature very well and that if the mission is to be repeated, the mission scientists should be capable of measuring the probe's speed profile.

Finally, we note that we obtained a higher temperature at altitude than at the surface. This differs from Earth's atmosphere, where the temperature at an altitude of 15 km is approximately −50°C, very much lower than typical surface temperatures of 20°C.

Incandescent bulbs

Illuminating thermal expansion

D. C. Agrawal and V. J. Menon



Art by Sergey Ivanov

MOST OF YOU have heard about thermal expansion, a property which must be taken into account in many engineering projects. The most common textbook example is railroad tracks—gaps are left between the rails to accommodate the rails' expansion in hot weather. Thermostats in homes and appliances depend on the difference in expansion of the two metals of a bimetallic strip. Another household item that exhibits thermal expansion is the lightbulb.

If your physics teacher asks, "What is the hottest thing in your home?" you should answer, "The filaments in the incandescent bulbs." Indeed, the filament in a typical 100-W bulb operating at 120 V has a temperature as high as 2900 K, which is about half the temperature of the solar photosphere. This raises a number of questions. For example, how do we know the filament is so hot? Also, the filament must have expanded noticeably in both length and diameter because room temperature (~300 K) is much lower. What are the physical effects of this expansion? Let's try to answer these questions using some basic physical arguments and some numerical estimates.

Filaments in 100-W bulbs are made of extremely thin, smooth tungsten wire with length $L_0 = 0.475$ m and radius $r_0 = 3.05 \cdot 10^{-5}$ m at room temperature $T_0 = 300$ K. To accommodate such a long wire in a small glass housing, the filament is wound in the form of a coil or perhaps a coiled coil. At room temperature the resistance is fairly low,

$$R_0 = \frac{\rho_0 L_0}{\pi r_0^2} = 9.18\ \Omega,$$

where $\rho_0 = 5.65 \cdot 10^{-8}\ \Omega\cdot\text{m}$ is the resistivity of tungsten.

When such a bulb is turned on, a relatively large current begins to flow,

$$I_0 = \frac{V}{R_0} = 13.1\ \text{A},$$

where V is the constant voltage for a direct current or the rms value for an alternating current. As you would expect, this causes a large amount of joule heating, and the temperature of the filament quickly rises. Within about 0.1 s, the temperature achieves its normal operating value T, and the bulb glows with its full brightness.

The physical properties of metals are usually temperature dependent. For instance, the resistivity of tungsten is experimentally known to obey a power law,

$$\rho = \rho_0\left(\frac{T}{T_0}\right)^{1.215}.$$

Furthermore, the temperature dependence of the length is given by

$$L = L_0(1 + \sigma(T)),$$

where

$$\sigma(T) = \frac{L - L_0}{L_0}$$

is the effective coefficient for thermal expansion of tungsten at temperature T. Metallurgists have measured this coefficient and state its value as a percentage change in the length compared to the length measured at the standard temperature 293 K. The value of the coefficient varies over a wide range, from 0.003% at 300 K to 2.209% at 3655 K. In many textbooks, $\sigma(T)$ is approximated by $\alpha(T - T_0)$, where α is considered a constant. However, over the large temperature range considered here, α for tungsten varies significantly.

When we apply this thermal expansion to both the length and the radius of the filament, we find that the resistance of the filament at a temperature T becomes

$$R = \frac{\rho L}{\pi r^2} = \rho_0\left(\frac{T}{T_0}\right)^{1.215}\left[\frac{L_0(1+\sigma(T))}{\pi r_0^2(1+\sigma(T))^2}\right] = R_0\left(\frac{T}{T_0}\right)^{1.215}\frac{1}{1+\sigma(T)}.$$

As you may recall, the rated power P of a bulb in normal operation is given by the rate of production of joule heat,

$$P = \frac{V^2}{R} = \frac{V^2(1+\sigma(T))}{R_0}\left(\frac{T_0}{T}\right)^{1.215}. \quad (1)$$

In principle this equation can be solved for T. This may appear complicated because of the presence of $\sigma(T)$, for which we only know the tabular form from the literature. Luckily, the saving feature of the equation is that $\sigma(T)$ is numerically small compared to unity throughout the range $300\text{ K} \le T \le 3655$ K. Therefore, we can resort to a simple approximation technique to find the solution.

For this purpose, let's introduce a model temperature T' that satisfies equation (1) when $\sigma(T) = 0$. Therefore,

$$T' = T_0\left(\frac{V^2}{R_0 P}\right)^{\frac{1}{1.215}}. \quad (2)$$

The physical significance of T' is that it represents the temperature of a hypothetical hot filament that does not expand thermally. Because we can expect T' to be reasonably close to the unknown temperature T, we can, to a first approximation, replace $\sigma(T)$ by $\sigma(T')$, a constant, and resolve equation (1) to obtain

$$T \cong T'(1+\sigma(T'))^{\frac{1}{1.215}}.$$

Using the binomial theorem we can further approximate this expression to read

$$T \cong T'\left(1+\frac{\sigma(T')}{1.215}\right). \quad (3)$$

This is the desired temperature of the filament. The difference between T and T' is due to the thermal expansion.

When we insert typical parameters for the 100-W bulb into equation (2), we find $T' = 2890$ K. We now look in the *CRC Handbook of Physics and Chemistry* to locate the pair of temperatures surrounding T' and deduce the value of $\sigma(T')$ by linear interpolation. Thus we find $\sigma(2890\text{ K}) = 1.553\%$ and

$$T \cong (2890\text{ K})\left(1+\frac{0.016}{1.215}\right) = 2927\text{ K}.$$

The data and results for other bulbs manufactured by General Electric are shown in table 1.

Although the coefficient of thermal expansion $\sigma(T)$ is small, the difference $T - T' \cong 37$ K is substantial. The reason for this significant effect is that the temperature T' in equation (3) is large compared to room temperature. This is why we are justified in saying that incandescent bulbs provide a good illustration of the effects of thermal expansion in metals.

Table 1

P (W)	L_0 (m)	r_0 (m)	T' (K)	$\sigma(T')$ (%)	T (K)	$T - T'$ (K)
6	0.371	$5.71 \cdot 10^{-6}$	2276	1.092	2296	20
60	0.533	$2.29 \cdot 10^{-5}$	2497	1.251	2523	26
100	0.475	$3.05 \cdot 10^{-5}$	2890	1.553	2927	37
500	0.874	$9.02 \cdot 10^{-5}$	2773	1.458	2806	33
1000	0.973	$1.45 \cdot 10^{-4}$	3134	1.752	3179	45
5000	1.128	$3.68 \cdot 10^{-4}$	3418	1.996	3474	56
10000	1.384	$5.84 \cdot 10^{-4}$	3492	2.061	3551	59

D. C. Agrawal *is a reader in the Department of Agricultural Engineering and* **V. J. Menon** *is a UGC research scientist in the Department of Physics at Banaras Hindu University, Varanasi, India. E-mail: vjmenon@banaras.ernet.in*

Variations on a theme

The Arithmetic Mean–Geometric Mean inequality

by Mark Saul and Titu Andreescu

THE THEME OF THIS COLumn is a classic inequality called the Arithmetic Mean-Geometric Mean (AM-GM) inequality. But before we introduce this theme, we will use a procedure pioneered by Ludwig van Beethoven in his "Eroica" Variations for piano, op. 35: We will present some preliminary variations before stating the theme. Please try them before reading on.

Variation –3: What is the smallest value that the square of a real number can take on?

Variation –2: Figure 1 shows a semicircle with center O. Its diameter has been divided into two segments of lengths a and b. Which is larger, OP or XY?

Variation –1: In trapezoid $ABCD$, segment MN connects the midpoints of legs AD and BC. Segment XY divides the trapezoid into two smaller trapezoids similar to each other. Figure 2 shows XY closer to the smaller base than to the larger base and therefore smaller than MN. Is this correct?

THEME: The arithmetic mean of any two positive real numbers is greater than or equal to their geometric mean. The two means are equal if and only if the two numbers are equal.

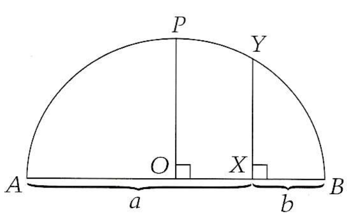


Figure 1

In other words, if a, $b > 0$, then

$$\sqrt{ab} \le \frac{a+b}{2},$$

and

$$\sqrt{ab} = \frac{a+b}{2}$$

if and only if $a = b$.

Proof: The square of a real number cannot be negative. Therefore $(\sqrt{a} - \sqrt{b})^2 \ge 0$. But this means that

$$\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab} \ge 0,$$

or

$$a + b \ge 2\sqrt{ab},$$

which gives us the required result. Equality holds only when $(\sqrt{a} - \sqrt{b})^2 = 0$, or when $a = b$.

Now can you see the meaning of variations –3, –2, and –1? You will find more exercise in the following variations. Often the AM-GM inequality is used to compare a product to a sum or to transform the one into the other. Watch how this theme unfolds.

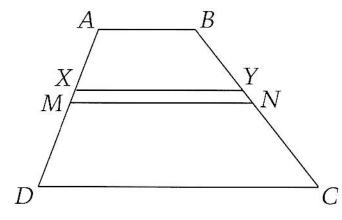


Figure 2

Variation 1: A rectangle has perimeter 20. What is its largest possible area?

Variation 2: A rectangle has area 100. What is its smallest possible perimeter?

Variation 3: Generalize the solutions to variations 1 and 2 to show that (a) if the sum of two positive numbers is constant, then their product is maximal when they are equal, and (b) if the product of two positive numbers is constant, then their sum is minimal when they are equal.

Variation 4: If x is a real number, find the smallest possible value of the expression $x + 1/x$.

Variation 5: If x is a positive real number, show that $2\sqrt{x} - x \le 1$.

Variation 6: If x is a real number, find the largest possible value of the expression $(x + 4)(6 - x)$.

Variation 7: If $0 < x < \pi/2$, find the smallest possible value of $\tan x + \cot x$.

Variation 8: For any real number x, find the largest possible value of $(\sin^2 x)(\cos^2 x)$.

Variation 9: If x is a real number, find the minimum value of the expression $2^x + 2^{-x}$.

Variation 10: If x, y, and z are nonnegative real numbers, show that

$$x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \le xy + yz + zx.$$

Variation 11: If a, b, c, and d are positive real numbers, show that

$$\sqrt{(a+c)(b+c)} \ge \sqrt{ab} + \sqrt{cd}.$$

Variation 12: Point D is chosen in the interior of angle ABC. A variable line passes through D, intersecting ray BA at M and ray BC at N. Find the position of line MN that gives the smallest possible area for triangle MBN. (For a hint to this rather difficult problem, glance at the diagram in the solution without reading the details.)

Variation 13: We start with n positive numbers $x_1, x_2, \ldots, x_n$ whose product is 1. Show that if we add 1 to each number, the product of the new numbers must be greater than or equal to 2^n.

For our last variation, we again follow Beethoven's lead. Like his, our last variation is something of an extended fugue, and it uses a rather advanced generalization of the AM-GM inequality: For n positive numbers $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

(That is, the arithmetic mean of n numbers is not less than their geometric mean.) Equality holds just when $a_1 = a_2 = \ldots = a_n$. For a beautiful proof of this generalized inequality, due to Cauchy, see for example *An Introduction to Inequalities* by Edwin Beckenback and Richard Bellman (Washington, D.C.: Mathematical Association of America, 1961).

Variation 14: (This problem was posed by one of the authors at the Romanian National Olympiad, final round, 1984.) Take n numbers $x_1, x_2, \ldots, x_n$ in the open interval $(1/4, 1)$. Find the minimal value of the expression

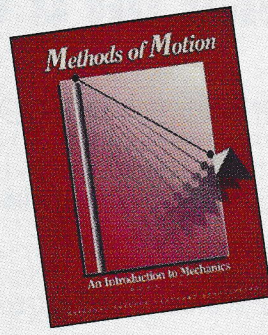
$$\log_{x_1}\left(x_2 - \frac{1}{4}\right) + \log_{x_2}\left(x_3 - \frac{1}{4}\right) + \log_{x_3}\left(x_4 - \frac{1}{4}\right) + \ldots + \log_{x_n}\left(x_{n+1} - \frac{1}{4}\right).$$

For what values of $x_1, x_2, \ldots, x_n$ does this minimum occur? Hint: As your first step, use the AM-GM inequality to find a lower bound for the expression $x_k^2 + 1/4$.

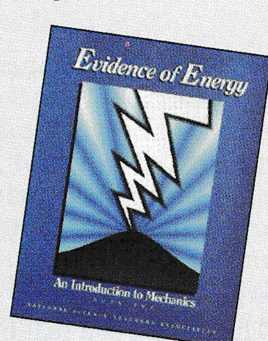
ANSWERS, HINTS & SOLUTIONS ON PAGE 48

Physics Phluency

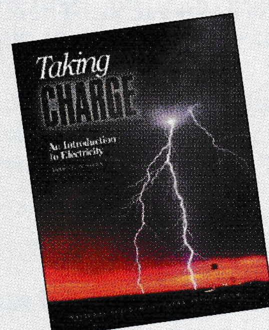
Let NSTA help you speak the language of physics



Methods of Motion
An Introduction to Mechanics, Book One
Twenty-seven teacher-created activities aim to simplify the daunting world of Newtonian mechanics for students and teachers.
(grades 6–10, 1992 revised ed., 168 pp.)
#PB039X $18.50



Evidence of Energy
An Introduction to Mechanics, Book Two
The informal hands-on activities in this book use a variety of techniques to combat common misconceptions about mechanics.
(grades 6–10, 1990, 200 pp.)
#PB080X $17.95



Taking Charge
An Introduction to Electricity
Spark student interest in electricity with 25 hands-on, teacher-tested activities using readily available materials.
(grades 5–10, 1992, 160 pp.)
#PB096X $18.95

Energy Sources and Natural Fuels
Explore energy, photosynthesis, fossil fuels, and more in this collaboration between NSTA and the Russian Academy of Science. (A teacher's guide and classroom sets also are available.)
(grades 9–10, 1993, 80 pp.)
#PB104X $12.95

To Order, Call:
(800) 722–NSTA
MasterCard, VISA, Discover, and Purchase Orders
Please make all orders payable in U.S. currency

QJF95

The lunes of Hippocrates

An early attempt to square the circle

by V. N. Berezin

IN THE FIFTH CENTURY B.C. in Greece, there lived a scientist whose name was Hippocrates of Chios. It was Hippocrates who undertook the first recorded attempt to write down the basic principles of geometry. Unfortunately, his work did not survive until our days.

A legend says that Hippocrates was an unlucky merchant who came to Athens seeking justice after he had been robbed by pirates. There he met wise men who spent most their time solving geometry problems. So, when he failed to prosecute the robbers, he took comfort in besting the most skilled of these wise men in the study of geometry.

Hippocrates was trying to "square the circle"—that is, to construct, using only a compass and straightedge, a circle whose area would equal the area of a given square. It's not difficult to "square" a triangle or a trapezoid, so Hippocrates might have thought that his major purpose, squaring a circle, was at hand. However, he couldn't do it (since it is impossible, as Carl Louis Ferdinand von Lindemann proved at the end of the last century), yet he managed to construct several figures that look like planar domains bounded by circular arcs, their area still equal to the area of a square.

In the history of geometry, Hippocrates' name is associated with figures of a special shape, the so-called "lunes." We can define the lune in the following way: If we draw a semicircle in the exterior of a chord of a circle so that its ends coincide with those of the chord, the figure bounded by the smaller arc of the circle and the semicircle is the lune.

We see four yellow lunes in figure 1. Hippocrates noticed that the sum of their areas equals the area of the blue square. In fact, the reader is invited to check that the sum of the areas of the semicircles drawn on the sides of the square equals the area of the circle circumscribed about it. If we delete the violet segments from the semicircles, we obtain four lunes, and if we delete them from the large circle, we obtain the square. This shows that the areas of the four yellow lunes add up to the area of the square.

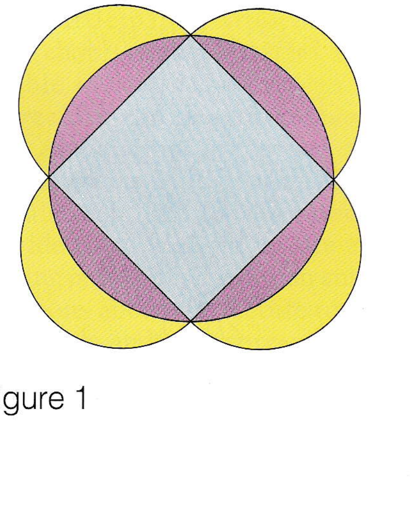


Figure 1

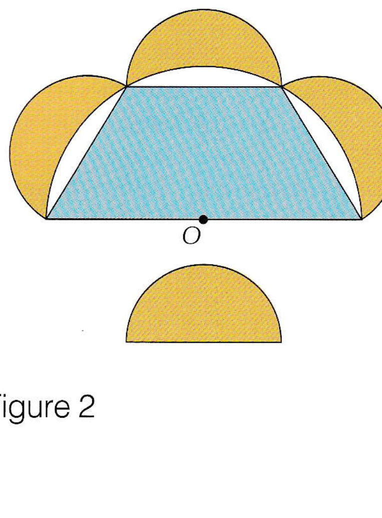


Figure 2

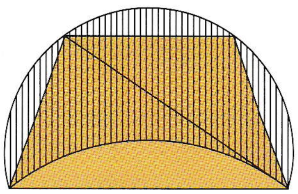


Figure 3

Figure 2 illustrates another theorem of Hippocrates. The figure shows a trapezoid whose larger base is a diameter of the large semicircle and whose other three sides are equal in length to the radius of the semicircle. It turns out that the area of such a trapezoid equals the sum of the areas of the oranges lunes and the area of the orange semicircle. (These lunes are congruent by construction, and the semicircle is equal to those that form the outer boundary of the lunes.) The proof of this theorem is similar to that of the theorem in the preceding paragraph.

The configuration in figure 3 was also suggested by Hippocrates. Both legs and the smaller base of the trapezoid have lengths equal to 1, and its larger base has a length of $\sqrt{3}$. The lower arc bounding the domain

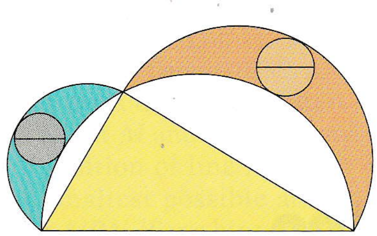


Figure 4

shaded by black lines in figure 3 (we can call this domain "the generalized lune") is tangent to the diagonal of the trapezoid. It turns out that the area of the shaded lune equals the area of the trapezoid (to prove it, we can check that the sum of the areas of the two lunes on the legs of the trapezoid and the lune on the smaller base equals the area of the lune on the larger base).

Figure 4 represents a configuration whose properties Hippocrates apparently did not know. The area of the right triangle in figure 4 equals the sum of areas of the lunes. The proof of this statement can easily be derived from the Pythagorean theorem. By the way, this configuration has another wonderful property: The lunes in it have equal width. More accurately, the diameters of the greatest circles that can be inscribed in the lunes are equal to half the difference of the sum of the legs of the right triangle and its hypotenuse.

The configuration in figure 5 was suggested by Archimedes. At least, so said the famous Arabian mathematician of the ninth century

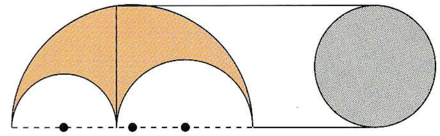


Figure 5

Thabit ibn Qurrah, who was an expert in the works of Archimedes. Archimedes' books, unfortunately, do not survive. This figure is usually called an *arbelos*, or "cobbler's knife."

Thabit ibn Qurrah demonstrated that the area of the arbelos equals the area of the circle whose size and construction are shown in figure 5 (it seems likely that he followed Archimedes' reasoning). We invite the reader to find a proof of this statement.

Corrections

November/December 1997

See the table below for a list of corrections. The following paragraph replaces the last paragraph on page 21:

We can easily check that $a_n^2 < b_n^2$. Indeed, this would mean that $2n + 1 + 2\sqrt{n(n+1)} < 4n + 2$. A short computation will show that this inequality is equivalent to the inequality $4n^2 + 4n < 4n^2 + 4n + 1$, which is certainly true. Thus $a_n^2 < b_n^2$, and (since both are positive) $a_n < b_n$. Next we show that $a_n^2 > 4n + 1$. Again, the inequality $2n + 1 + 2\sqrt{n(n+1)} > 4n + 1$ is equivalent to $\sqrt{n(n+1)} > n = \sqrt{n \cdot n}$, which is certainly true. Also, the number $b_n^2 = 4n + 2$ gives a remainder of 2 when divided by 4, and thus it cannot be the square of an integer (the reader is invited to check this directly). Therefore, the square of the integer $[b_n]$ is not greater than $4n + 1$.

Page	Column	Line	Error	Correction
12	3	12	is $2k$	is $3k$
21	2	14	$\sqrt{3}-\sqrt{2}=n^2\left(\sqrt{3}-\sqrt{2}\right)$	$\sqrt{3}-\sqrt{2}\le n^2\left(\sqrt{3}-\sqrt{2}\right)$
22	1	17	$\sqrt{4n+2+\left(\sqrt{n+\sqrt{n+1}}\right)}$	$\sqrt{4n+2}+\left(\sqrt{n}+\sqrt{n+1}\right)$
23	2	7 from bottom	e_m	e_{2n}
23	2	6 from bottom	$2a_{n-1}$	$2c_{n-1}$
24	2	12	$e_n \cong (2+\sqrt{2})^n/\sqrt{2}$	$e_n \cong (2+\sqrt{2})^{n-1}/\sqrt{2}$
24	2	13	e_{n+1}/e_n	$e_{2(n+1)}/e_{2n}$
24	2	4 from bottom	$\bar{\lambda}+\bar{\mu}=\bar{\lambda}+\bar{\mu};\ \lambda\mu=\bar{\lambda}\bar{\mu}$	$\overline{\lambda+\mu}=\bar{\lambda}+\bar{\mu};\ \overline{\lambda\mu}=\bar{\lambda}\bar{\mu}$
25	1	8	$\left(\sqrt{2}\to\sqrt{2},\sqrt{3}\to\sqrt{3}\right)$	$\left(\sqrt{2}\to\sqrt{2},\sqrt{3}\to-\sqrt{3}\right)$
39	2	25	$MF + MA$	$MF = MA$
40	1	21	$\lvert AM - MK\rvert$	$\lvert AM - AK\rvert$
41	1	9	3 4 7 11 17	3 4 7 11 18
48	2	32	So $\angle MP_1A$ is	So $\angle MP_1D$ is
48	3	1	$\angle DCQ$	$\angle DCQ_1$
49	1	28	(new) cubic function	(old) cubic function
53	1	3	$\angle AGM$	$\angle AGD$

Bulletin Board

Electron eloquence

With his 1897 discovery of an electron, British physicist J. J. Thomson changed the world considerably. Flicking the light switch, turning on the radio, and using cellular phones are miracles that are taken for granted today. However, causing these devices to operate would not be so simple without this technological breakthrough.

On February 3, 1998, the U.S.-based Institute of Electrical and Electronics Engineers, Inc. (IEEE), will downlink throughout North America a live broadcast of this year's Faraday lecture produced in Great Britain by the Institution of Electrical Engineers. The lecture, free of charge to participating organizations, is entitled "Bright Sparks of the Universe—An Electron's Eye View." It will document the discovery of the electron and how it affected our views of the world.

Professors Frank Close and Bryson Gore will deliver the presentation in conjunction with the Science Museum of London. This engaging program is one hour long and includes colorful background, demonstrations, and experiments using audience participants. Students from high schools and middle schools will be able to easily comprehend and appreciate the program through clear explanations of complex technical concepts. Participating schools can also ask IEEE local sections to provide them with an on-site engineering professional who can answer questions about the lecture topics.

Schools and other organizations can register for the Faraday lecture at no cost. For more information on how to receive this broadcast, contact: IEEE, 445 Hoes Lane, PO Box 1331, Piscataway, NJ 08855-1331. Phone: (732) 562-5595; Fax: (732) 981-1686; E-mail: t.garnys@ieee.org.

Calling all contestants

Looking to enter a physics bowl? Want to pit your robotic creation against all challengers? Think you can design a futuristic city that will outshine all others? If these types of competitions interest you, you should check out Scott Pendleton's new book, *The Ultimate Guide to Student Contests Grades 7–12*, published by the Walker Publishing Company ($15.95, 384 pp., ISBN 0-8027-7512-8).

This volume provides information on more than 250 contests that students can enter in the fields of science, engineering, computing, mathematics, and many other disciplines. The contests are divided into three sections. Part I, Unveil Your Talents, focuses on contests that require skills such as photography, essay writing, video production, and other artistic talents. Part II, Show What You Know, includes contests that test your knowledge in specific content areas. Part III, See and Be Seen, explains how to compete for various recognition awards, join honor societies, and bring your work to the attention of academic talent searches.

Sample questions for many contests are provided along with contact information, including addresses, fax and phone numbers, and web sites. Entry deadlines are provided in most cases. If you're up to the challenge, pick up a copy of this book to find a contest that will test your skills.

How not to be a "chipskate"

This month's CyberTeaser was not exactly brain surgery, but if you've ever had a hard time carving up the check among you and your friends at a restaurant, it may have brought back memories. Happily, in our scenario (brainteaser B221 in this issue), each person ended up paying his fair share for the potato chips he ate.

Here are the first ten contestants who sent in a correct answer to our crunchy question:

Judith-Hana Tovshteyn (Brooklyn, New York)
Matthew Wong (Edmonton, Alberta)
Clarissa Lee (Perak, Malaysia)
Jim Grady (Branchburg, New Jersey)
Cobus de Waal (Windhoek, Namibia)
Leonid Borovskiy (Brooklyn, New York)
Theo Koupelis (Wausau, Wisconsin)
Jim Paris (Doylestown, Pennsylvania)
Oleg Shpyrko (Somerville, Massachusetts)
Hayden Huang (Cambridge, Massachusetts)

Congratulations! Each of our winners will receive a *Quantum* button and a copy of the January/February issue. Everyone who submitted a correct answer in the time allotted was entered in a drawing for a copy of *Quantum Quandaries*, our collection of the first 100 *Quantum* brainteasers.

Why not try your hand at the latest CyberTeaser? You'll find it at **www.nsta.org/quantum/contest.** Remember, even if you're not among the first ten, if you submit a correct answer you might win a prize!

Constructing quadratic solutions

A novel use for compass and straightedge

by A. A. Presman

ONE WAY TO EXPLORE THE solution of a quadratic equation is to graph the corresponding parabola. This is easily done with software but can be tedious when working by hand. In geometry we often use the classical tools of straightedge (unmarked ruler) and compass to perform constructions.

One day one of my students asked, "Why don't we solve quadratic equations using a compass and straightedge when it is so easy to draw a circle?!" he asked. Thus the following problem arose during our math lesson.

Problem: Given three real numbers a, b, and c—which are coefficients of the quadratic equation $ax^2 + bx + c = 0$—find the radius and coordinates of the center of a circle that meets the x-axis in points whose x-coordinates are the roots of this equation (we assume throughout that $a \neq 0$).

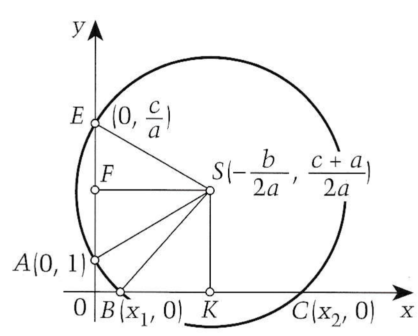


Figure 1

From the start it is clear that there is more than one such circle. Therefore, we can suppose that it passes through points $B(x_1, 0)$ and $C(x_2, 0)$, where x_1 and x_2 are the roots of the quadratic equation, *and* (for definiteness) through the point $A(0, 1)$ (see fig. 1). Then, according to the two-secant theorem,[1]

$$OC \cdot OB = OE \cdot OA,$$

from which we get

$$OE = \frac{OB \cdot OC}{OA} = \frac{x_1 x_2}{1} = \frac{c}{a},$$

the product of the roots.

The center S of the circle lies in the intersection of perpendiculars SF and SK drawn to the midpoints of chords AE and BC, thus

$$OK = \frac{x_1 + x_2}{2} = \frac{\frac{-b}{a}}{2} = \frac{-b}{2a} \text{ and}$$

$$OF = \frac{1 + \frac{c}{a}}{2} = \frac{a + c}{2a}.$$

We can now find the roots of the quadratic equation $ax^2 + bx + c = 0$ using a compass and straightedge.

[1]This theorem says that if we draw two secants to a circle from a point outside, the products of each whole secant and its external segment are equal.

We assume that, given line segments of lengths p and q, the reader knows how to construct, with straightedge and compass, segments of lengths $p + q$, $p - q$, kp and p/k (for a natural number k), and given a segment of unit length, p/q and pq. We begin by constructing points

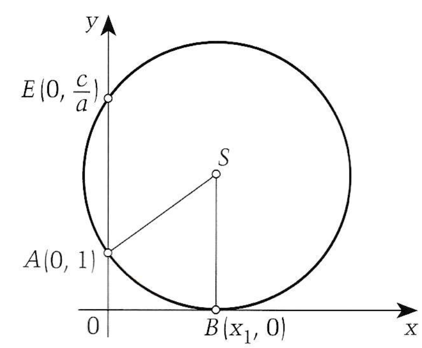


Figure 2

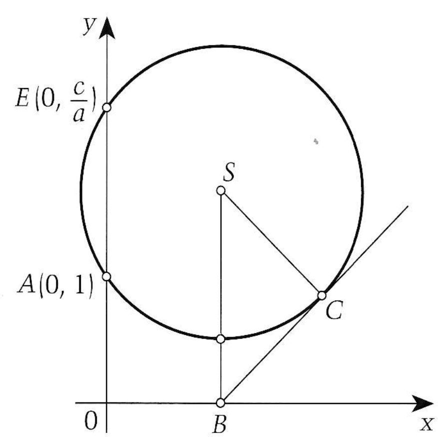


Figure 3

$S(-b/2a, (c + a)/2a)$ (the center of the circle) and $A(0, 1)$. After this, we draw a circle with radius SA. The x-coordinates of the points where the circle meets the x-axis are the roots of the quadratic equation.

We invite the reader to prove the correctness of this construction. We'll point out three special cases that should be distinguished and a diagrammatic way to find *complex* roots of the equation.

(1) The radius of the circle is greater than the y-coordinate of the center. The circle meets the x-axis at points $B(x_1, 0)$ and $C(x_2, 0)$ (see fig. 1). In this case the equation has two real roots.

(2) The radius of the circle is equal to the x-coordinate of the center. The circle touches the x-axis at point $B(x_1, 0)$(see fig. 2).

In this case, the equation has two equal real roots. The x-coordinate of the contact point is

$$x_{1,2} = \frac{-b}{2a}$$

(3) The radius of the circle is less than the y-coordinate of the center. The circle does not meet the x-axis (see fig. 3).

In this case the equation has two complex conjugated roots:

$$x_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i.$$

The real part of the complex root is expressed by the length of $BC = -b/2a$—the x-coordinate of the center. The absolute value of the imaginary part is equal to the length of tangent BC.

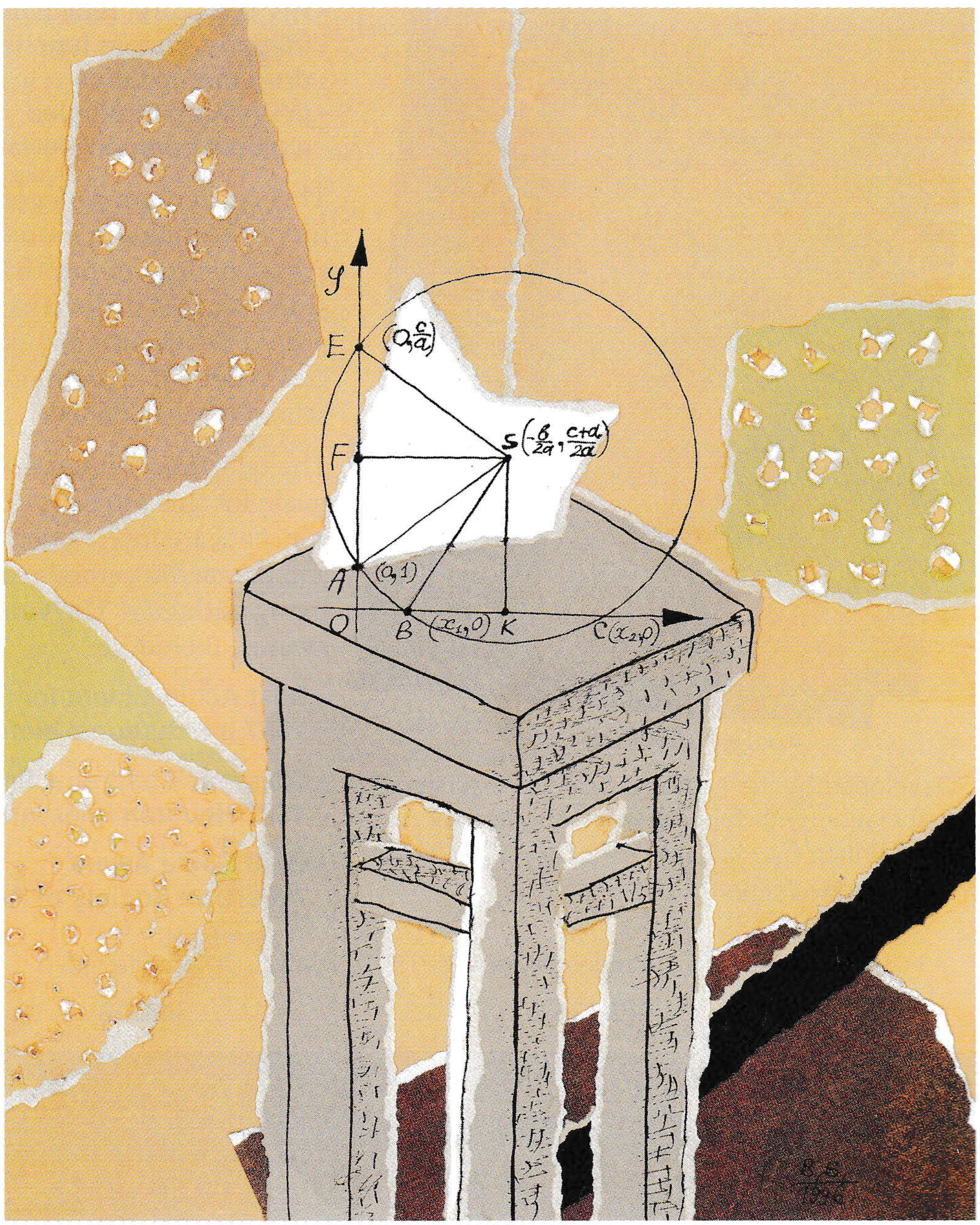
Using this construction, it's easy to investigate properties of the roots, including formulas for their sum and product. But it would be unfair to use them when we justify our construction. So we'll conclude the article with an algebraic interpretation of the suggested solution of a quadratic equation.

Instead of using the equation $ax^2 + bx + c = 0$, we'll consider an equivalent system:

$$\begin{cases} ax^2 + bx + c + ay^2 - (c + a)y = 0, \\ y = 0. \end{cases}$$

Each root x_1 of the equation corresponds to the solution $x = x_1$, $y = 0$ of the system and vice versa. The first equation of the system defines a circle (we invite the reader to check this), and the second the x-axis. Their intersections are points $(x_i, 0)$, where x_i are the roots.

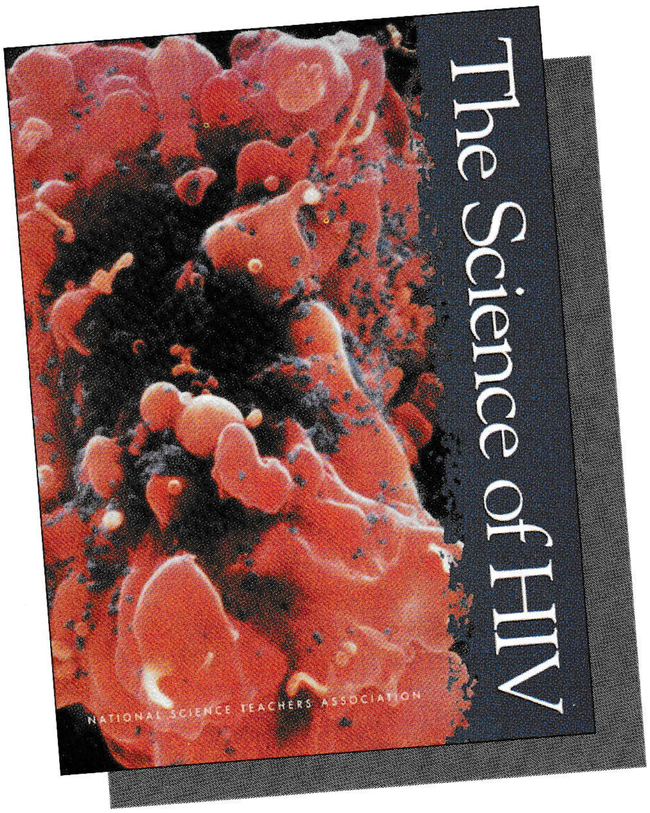
The solution of quadratic equations using straightedge and compass appears as early as Euclid's *Elements*. But the solution of equations of higher degree (such as most cubic equations) turns out to be impossible by the same methods. This fact is connected with two of the three great construction problems of antiquity: trisecting the general angle and duplicating the cube. The reader is invited to research the interesting and well-documented unfolding of this story through history. The third problem, squaring the circle, is touched on in the article "The lunes of Hippocrates" on page 37.



Art by Vasily Vlasov

A Community Resource
To Understand and Prevent AIDS

The Science of HIV Curriculum Package



Developed by the National Science Teachers Association with funding from Abbott Laboratories. Written by Michael DiSpezio. Video by Summer Productions.

NSTA's new science-based resource guide is different from most "AIDS books"—its activities and readings focus on biological concepts relating to HIV. Activities cover the following subjects:

- **selected topics in cell biology**
- **basic virology**
- **HIV structure, replication, and genetics**
- **immune system function and HIV infection**
- **drug therapeutics**
- **prevention strategies**
- **a global perspective on the AIDS pandemic**

This curriculum package can be used as a community educational resource or to expand upon a high school biology or health curriculum. Reproducible student pages make lesson plans flexible; educator pages provide background and presentation strategies. Material appropriate for anyone at the high school level and above.

The text is coordinated with an original video made for this project. Animations of complex concepts are interwoven with scientist interviews and compelling stories of adolescents who are living with HIV. The video has won numerous awards, including:

- **Best Achievement for Children's Programming**
  1997 International Monitor Awards
- **Silver for Children's Programming**
  1997 Houston International Film Festival
- **Gold Circle Award**
  American Society of Association Executives

Grades 9–College, 1997, 184 pp, 30-minute video

#PB136X **$45.00**

To Order Call 800-722-NSTA

Visit Our Web Site www.nsta.org

ANSWERS, HINTS & SOLUTIONS

Math

M221

Consider a progression in which the first term is $b_1 = 2^{20}$ and the common ratio is $q = 5/4$. Since $b_1 > 10^6$ and $b_{11} = 5^{10} < 10^7$ and the first through eleventh terms are natural numbers, we conclude that the first eleven terms of this progression are seven-digit numbers. Now let's prove that eleven is the best possible result. Suppose a progression exists that contains twelve or more seven-digit numbers. Let $q = m/n$ be the common ratio of this progression (where m/n is a fraction in lowest terms and $q > 1$) and b_1 its first term. Then $m \le 4$. Indeed, if $m > 4$, then

$$b_{12} = b_1 q^{11} = \frac{b_1 m^{11}}{n^{11}} \ge m^{11}$$

(because b_{12} must be an integer, and thus b_1 is divisible by n^{11}), and $m^{11} \ge 5^{11} > 10^7$. If $m < 5$, then the least possible value of q is 4/3. But $(4/3)^{11} > 10$, and thus $b_{12} > 10^7$.

M222

Suppose the center of the circle inscribed in ABC is J, and AJ meets KL at P (fig. 1). Triangle JKA is a right triangle (because JK is a radius of the inscribed circle). We have $AJ \perp KL$ since $AK = AL$ and AJ bisects angle KAL. Also, KP is the altitude to the hypotenuse of right triangle KAJ. Thus $JK^2 = JP \times JA$. But $JM = JK$. Therefore $JM^2 = JP \times JA$, or

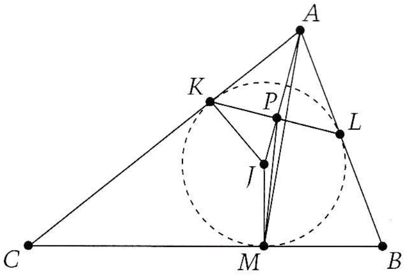


Figure 1

$$\frac{JM}{JP} = \frac{JA}{JM}.$$

This equality means triangles JPM and JMA are similar (because $\angle PJM = \angle AJM$). Thus

$$\frac{PM}{AM} = \frac{JM}{JA} = \frac{JK}{JA} = \sin\frac{\alpha}{2}. \quad (1)$$

Furthermore, $\angle JPM = \angle JMA$, and therefore

$$\begin{aligned}\angle MPL &= 90^\circ - \angle JPM \\ &= 90^\circ - \angle JMA = \angle AMB. \quad (2)\end{aligned}$$

Now we can find the desired ratio:

$$\begin{aligned}MM_1 &= PM \cdot \sin \angle MPL, \\ AA_1 &= AM \cdot \sin \angle AMB.\end{aligned}$$

From (1) and (2) we conclude that

$$\frac{PM \sin \angle MPL}{AM \sin \angle AMB} = \frac{PM}{AM} = \sin\frac{\alpha}{2}.$$

M223

We multiply both sides of the given equation by cos $(3\pi/11)$ and square both sides to obtain

$$\left(\sin\frac{3\pi}{11} + 4\sin\frac{2\pi}{11}\cos\frac{3\pi}{11}\right)^2 = \left(\sqrt{11}\cos\frac{3\pi}{11}\right)^2,$$

or

$$\begin{aligned}&\sin^2\left(\frac{3\pi}{11}\right) \\ &+ 8\sin\left(\frac{3\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right) + \\ &+ 16\sin^2\left(\frac{2\pi}{11}\right)\cos^2\left(\frac{2\pi}{11}\right) \\ &= 11\cos^2\left(\frac{3\pi}{11}\right). \quad (1)\end{aligned}$$

We can now work with each term separately. Our goal will be to express each as a sum of cosines of certain angles.

Using the identity

$$\sin^2 A = \left(\frac{1}{2}\right)(1 - \cos 2A),$$

we have

$$\sin^2\frac{3\pi}{11} = \left(\frac{1}{2}\right)\left(1 - \cos\frac{6\pi}{11}\right).$$

Using the identities

$$\sin 2A = 2\sin A \cos A$$

and

$$\cos(A - B) - \cos(A + B) = 2\sin A \sin B,$$

we have

$$\begin{aligned}8\sin\frac{3\pi}{11}\sin\frac{2\pi}{11}\cos\frac{3\pi}{11} &= 4\sin\frac{6\pi}{11}\sin\frac{2\pi}{11} \\ &= 2\cos\frac{4\pi}{11} - 2\cos\frac{8\pi}{11},\end{aligned}$$

Similarly, we have

$$\begin{aligned}&16\sin^2\frac{2\pi}{11}\cos^2\frac{3\pi}{11} \\ &= 4\left(1 - \cos\frac{4\pi}{11}\right)\left(1 + \cos\frac{6\pi}{11}\right) \\ &= 4 - 4\cos\frac{4\pi}{11} + 4\cos\frac{6\pi}{11} \\ &\quad -2\cos\frac{2\pi}{11} - 2\cos\frac{10\pi}{11},\end{aligned}$$

and finally,

$$11\cos^2\frac{3\pi}{11} = \frac{11}{2} + \frac{11}{2}\cos\frac{6\pi}{11}.$$

Now we go back to (1) and replace each term with a sum of cosines. If we move all these cosines to the left and all the constants to the right, we obtain

$$2\cos\frac{2\pi}{11}+2\cos\frac{4\pi}{11}+2\cos\frac{6\pi}{11}+2\cos\frac{8\pi}{11}+2\cos\frac{10\pi}{11}=-1.$$

We now have, on the left, a sum of the cosines of angles in the arithmetic progression, and can use a clever but standard trick. We multiply both sides by $\sin(\pi/11)$, which is the sine of the common difference of the arithmetic progression, to get

$$2\sin\frac{\pi}{11}\cos\frac{2\pi}{11}+2\sin\frac{\pi}{11}\cos\frac{4\pi}{11}+2\sin\frac{\pi}{11}\cos\frac{6\pi}{11}+2\sin\frac{\pi}{11}\cos\frac{8\pi}{11}+2\sin\frac{\pi}{11}\cos\frac{10\pi}{11}=-\sin\frac{\pi}{11}.$$

We now use the identity

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)=\sin(A+B)-\sin(B-A)$$

to transform the products on the left to sums. We get

$$\sin\frac{3\pi}{11}-\sin\frac{\pi}{11}+\sin\frac{5\pi}{11}-\sin\frac{3\pi}{11}+\sin\frac{7\pi}{11}-\sin\frac{5\pi}{11}+\sin\frac{9\pi}{11}-\sin\frac{7\pi}{11}+\sin\frac{11\pi}{11}-\sin\frac{9\pi}{11}$$
$$=\sin\frac{11\pi}{11}-\sin\frac{\pi}{11}=-\sin\frac{\pi}{11},$$

which verifies the identity.

M224

First we introduce the new variables u and v:

$$x=\frac{u-1}{u+1},\ y=\frac{v-1}{v+1}.$$

We then transform each expression in the first equation. We have

$$\frac{x+y}{1+xy}=\frac{(u-1)(v+1)+(v-1)(u+1)}{(u+1)(v+1)+(u-1)(v-1)}=\frac{2uv-2}{2uv+2}=\frac{uv-1}{uv+1},$$

and

$$\frac{2-x}{1-2x}=\frac{2(u+1)-(u-1)}{u+1-2(u-1)}=\frac{u+3}{-u+3}.$$

So we have

$$\frac{uv-1}{uv+1}=\frac{u+3}{-u+3},\ u^2v=-3.$$

Similarly, from the second equation, we obtain $u=-2v^2$. Substituting this expression for u in the first equality, we compute

$$v^5=-\frac{3}{4},\ \text{so } v=-\sqrt[5]{\frac{3}{4}}.$$

Also,

$$u=-2v^2=-\sqrt[5]{18}.$$

Answer:

$$x=\frac{\sqrt[5]{18}+1}{\sqrt[5]{18}-1},\ y=\frac{\sqrt[5]{3}+\sqrt[5]{4}}{\sqrt[5]{3}-\sqrt[5]{4}}.$$

M225

Not necessarily. An arithmetic progression of natural numbers can be uniquely given by a pair (a, d), where a is the first member and d is the difference. Therefore, all such progressions can be numbered. (The pairs (a, d) are divided into two groups with constant sum $a + d$ and are then numbered within each group—say, by the growth of a. This method of numbering will give the following sequence of pairs: (1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), ...) Let A_i, $i = 1, 2, \ldots$, be the ith progression in our numbering. Beginning with any number p_1 from A_1, we will choose the numbers p_2 from A_2, p_3 from A_3, and so on, such that p_{i+1} is greater than $2p_i$. Then the set $P = \{p_1, p_2, \ldots,\}$ will not contain three-member arithmetic progressions, since $p_k - p_j > p_j > p_j - p_i$ for any $k > j > i$. At the same time, the set of natural numbers that are not members of P will not contain a single infinite progression, since at least one element of any such progression is a member of P.

Physics

P221

The distance L from the rod's center to the wall will be covered during time $\tau = L/v = 5$ s. To make the parallel collision possible, the rod must make a half-integral number of turns during time τ. Therefore,

$$\omega_n\tau = n\pi,$$

where $n = 1, 2, 3, \ldots$, from which we get

$$\omega_n = n\frac{\pi}{\tau}. \qquad (3)$$

Is this the final solution? Not at all! When ω is large enough, the rod will strike the wall with one end before the parallel collision occurs. To find the angular velocity necessary for a parallel collision, note that at the moment of collision the velocity of end A of the rod (fig. 4) cannot be directed away from the

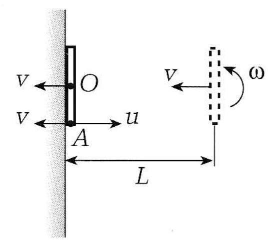


Figure 4

wall. Indeed, if it was directed away from the wall, this would mean that just before the collision the rod was inside the wall, which, of course, is impossible. The velocity of point A is composed of the velocity of the rod's center and the linear velocity of the rod's end about its center. Therefore,

$$v_A = u - v = \frac{\omega l}{2} - v > 0.$$

from which the condition for ω follows:

$$\omega < \frac{2v}{l} = 2\ \text{s}^{-1}.$$

Plugging formula (3) into this condition, we see that only three values of ω_n are solutions to the problem:

$$\omega_1 = 0.63\ \mathrm{s}^{-1},\ \omega_2 = 1.26\ \mathrm{s}^{-1},\ \text{and}\ \omega_1 = 1.89\ \mathrm{s}^{-1}.$$

P222

The air in the tube moves slowly, so the force of viscous friction can be considered proportional to the air velocity. The extra pressure in the bubble is small relative to the atmospheric pressure, so we can neglect changes in air density in the bubble.

At any moment the force of viscous friction is almost precisely counterbalanced by the force of the extra pressure, and this pressure ΔP is inversely proportional to the bubble's radius. Therefore, the air velocity

$$v \propto \Delta P \propto \frac{1}{r}.$$

The rate of change in the bubble's voume is proportinal to the air velocity,

$$\frac{\Delta V}{\Delta t} \propto v \propto \frac{1}{r}.$$

Therefore,

$$\frac{\Delta V}{V} \propto \frac{\Delta t}{r^4}.$$

If we have two bubbles with radii r_1 and r_2, the ratio of the periods necessary to decrease their volumes by the same small fraction is given by

$$\frac{\Delta t_1}{\Delta t_2} = \frac{r_2^4}{r_1^4}. \qquad (4)$$

Now we split the continuous process of the collapse of a bubble into steps in which every bubble decreases its volume by the same fraction. During the entire process, the ratio of the radii is constant and equal to their initial ratio. Therefore, equation (4) is valid not only for differential values Δt but also for integral values of the bubbles' lifetimes. Thus,

$$t_2 = 16t_1.$$

P223

There are two answers to this problem. If the internal resistance of the voltmeter is large, then a current of 0.2 mA flows through it. In this case, the voltage drop across the voltmeter is between 4.5 and 4.8 volts, so

$$4.8 - V = 2.2R \text{ and } V - 4.5 = 2R.$$

Here the internal resistance of the ammeters R is measured in kilohms. The solution of this equation is

$$V = 4.64\ \mathrm{V} \text{ and } R = 0.071\ \mathrm{k\Omega}.$$

In another (and quite possible) case, when the internal resistance of the voltmeter is low, its voltage drop is smaller than the voltage of either battery. Therefore, a total current of 4.2 mA flows through the voltmeter. In this case we have another system of equations:

$$4.8 - V = 2.2R \text{ and } 4.5 - V = 2R.$$

The respective values are $V = 1.5$ V and $R = 1.5$ kΩ. The voltmeter's internal resistance is 1.5 V/4.2 mA = 0.36 kΩ, which is far from being a good characteristic of a voltmeter!

P224

According to the conditions of the problem, only a small fraction of the neon atoms are ionized. This means that most atom-electron collisions can be considered elastic. Energy and momentum conservation give us the value of an electron's loss in kinetic energy during a frontal collision with a neon atom at rest:

$$\Delta KE_1 = \frac{4m}{M}\cdot\frac{mv^2}{2},$$

where v is the electron's velocity before the collision (we took into account that $m << M$). In a typical collision, the energy loss will be of the same order of magnitude.

Let's suppose that between two collisions an electron moves with uniform acceleration $a = eE/m$ during time $\tau = l/v$. After a collision, the electron's velocity can be directed anywhere, but on average the kinetic energy an electron gains between collisions is on the order of

$$\Delta KE_2 = eE\frac{a\tau^2}{2} = \frac{(eEl)^2}{2mv^2}.$$

The velocity (and therefore the kinetic energy) of an electron will not vary when $\Delta KE_1 = \Delta KE_2$, so

$$\frac{mv^2}{2} = \frac{eEl}{4}\sqrt{\frac{M}{m}}.$$

The corresponding "electron temperature" is

$$T = \frac{mv^2}{3k} = \frac{eEl}{6k}\sqrt{\frac{M}{m}} \sim 4\cdot 10^4\ \mathrm{K}.$$

P225

A simple plot (see fig. 5) of the light paths provides the position of the screen S corresponding to the

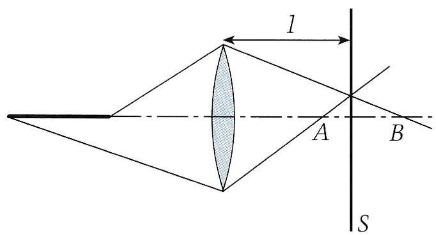


Figure 5

smallest light spot. The positions of points A and B can be found from the lens formula, and the size of the spot from similar triangles. The distance of point A from the lens is 20/3 cm, and that of point B is 10 cm. Therefore,

$$\frac{10 - l}{10} = \frac{l - \frac{20}{3}}{\frac{20}{3}}.$$

Thus, $l = 8$ cm, and the spot's diameter d is 0.2 of the lens's diameter, which yields $d = 2$ mm.

Brainteasers

B221

John should be paid \$1.60 because Jim owes him 20¢, too. In fact, each boy ate 7/3 of a bag of popcorn. Thus,

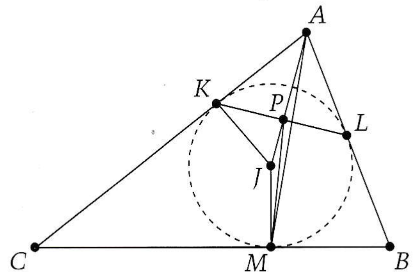


Figure 6

1/3 of a bag is 20¢, and Jim ate 1/3 of a bag more than he bought.

B222

Suppose $AD = a$ and $BC = b$. If we set AB equal to x (see fig. 6),then $CD = a - b - x$. The area of the figure is

$$\frac{\pi}{8}\left(AC^2 - AB^2 + BD^2 - CD^2\right)$$
$$= \frac{\pi}{8}\left[(x+b)^2 - x^2 + (a-x)^2 - (a-b-x)^2\right]$$
$$= \frac{\pi}{8}\left[b(b+2x) + (2a-b-2x)b\right] = ab\frac{\pi}{4}.$$

Thus, this is enough information to calculate the area of the figure.

B223

Consider three squares situated like those in figure 7. If the number of different paths to the left square is a and the number of paths to the lower square is b, then the number of paths to the third square is $a + b$. Now, we can simply start from the lower left square and write in each square the number of different paths that go to this square (see fig. 8).

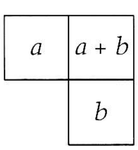


Figure 7

1	5	9	17	34
1	4	4	8	17
1	3		4	9
1	2	3	4	5
1	1	1	1	1

Figure 8

B224

The broken line $AKLMBB_1A_1$ (fig. 9) passes through all the vertices of the cube $ABCDA_1B_1C_1D_1$. (Here D_1 belongs to AK, C_1 to KL, and D to MB.)

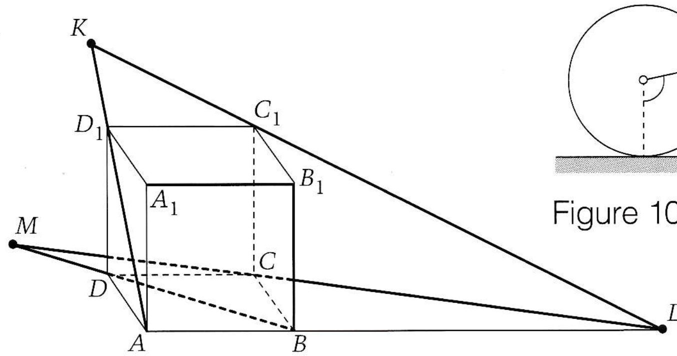


Figure 9

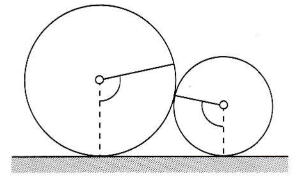


Figure 10

B225

Look at the angles between the vertical radius of each circle when they intersect and the radius that was originally vertical. It is given that the arcs of these two central angles are equal, so the angles themselves must be equal. But this is impossible if the radii are not equal (figure 10).

Kaleidoscope

1. In a mixture of water and alcohol, the molecules are more tightly packed than in either of the component fluids, which have marked gaps between the molecules.

2. Due to the pressure difference between the inside and outside of the ball, the air molecules diffuse through the ball's rubber wall, so its pressure drops.

3. The mean kinetic energy of any molecules in the gas mixture is the same (it is determined by the gas's temperature), so the light molecules have higher velocities than the heavy molecules. Therefore, the light molecules diffuse through the wall more rapidly than the heavy molecules.

4. The slowest ions are those with the largest mass, so in the race to the cathode, the ^3H isotope lags behind its brethren.

5. Due to inelastic collisions between copper and chlorine molecules, the resulting pressure on the copper-covered end is about half the pressure that affects the opposite end.

6. The hammer strokes press the two pieces of iron together. At white-hot temperatures the iron molecules from each piece diffuse deeply into the other, forming a very strong weld.

7. The equal volumes contain the same number of moles, provided the pressure and temperature of both gases are identical. The average molar mass of air is larger than that of an air-water mixture. Therefore, the vessel with wet air is lighter than that with dry air.

8. Air behaves more like an ideal gas in the stratosphere because it is more rarefied.

9. The radius of the path of a charged particle in a uniform magnetic field is proportional to the speed of the particle. The path of the particle in the Wilson cloud chamber is made visible by ionizing atoms along its path. This ionization causes the particle's kinetic energy to decrease. Consequently, the particle slows down and the curvature of the path continaully decreases.

Microexperiment

Water vapor itself is invisible. When the burner is on, rising jets of hot air fly around the kettle and warm the water vapor. When the burner is turned off, the water vapor cools and condenses. Therefore, we observe a cloud consisting of the tiny water droplets.

Gradus

Variation –3: The answer is, of course, 0. This need not be taken as an axiom describing the relation "greater than." Rather, it follows from the axioms of the mathematical structure called an *ordered field.*

Variation –2: Segment XY is half of a certain chord, and OP is half of a diameter. Since a diameter of a circle is its longest chord, $OP > XY$. Analytically, this follows from (or can be considered a proof of) the main theorem, since $OP = (a + b)/2$ and $XY = \sqrt{ab}$.

Variation –1: Yes, it is, but this time it's easiest to appeal to the theorem of the theme. Let $AB = a$ and $CD = b$. Then a standard theorem tells us that $MN = (a + b)/2$. If trapezoids $ABYX$ and $XYCD$ are similar, then $AB{:}XY = XY{:}CD$, which leads to $XY = \sqrt{ab}$. Our theme tells us that $XY < MN$ and is thus closer to the smaller base.

Variation 1: If the length and width of the rectangle are denoted by a and b respectively, then we have $a + b = 10$, and we must find the maximum of ab. But the AM-GM inequality says $2\sqrt{ab} \le a + b = 10$, so that

$$ab \le \left(\frac{a+b}{2}\right)^2 = 25.$$

A quick check will show that if $a = b = 5$, then the maximum is achieved. In this case the rectangle is a square.

Variation 2: If the length and width of the rectangle are denoted by a and b respectively, then we have $ab = 100$, and we must find the minimum of $2a + 2b$, or, equivalently, the minimum of $a + b$. Again, the AM-GM inequality says $a + b \ge 2\sqrt{ab} = 20$, with equality if and only if $a = b = 10$. The shape of this minimal rectangle is (again) a square, and its perimeter is 40.

Variation 3: The generalizations are immediate. (a) If $a + b$ is constant, then $[(a + b)/2]^2$ is also constant, and is an upper bound for ab. The two expressions are equal if and only if $a = b$. (b) If ab is constant, then $2\sqrt{ab}$ is constant, and this is a lower bound for $a + b$, achieved also when $a = b$.

Variation 4: Since the product of x and $1/x$ is constant (it is 1), their sum is minimal when they are equal, which is when $x = 1/x = 1$. The smallest possible value of the expression is 2.

Variation 5: This looks different from the previous variation. But we can make it look the same if we rewrite it so that it compares a sum (rather than a difference) to a product, which is what the AM-GM inequality does for us. Here, we can write $1 + x \ge 2\sqrt{x}$. Then, letting $a = 1$ and $b = x$ in the AM-GM inequality, we have our result.

Variation 6: One could, of course, multiply this out, get a quadratic function in x, and use some standard techniques for finding the maximum. However, we can also note that $(x + 4) + (6 - x) = 10$, a constant, so the product of the two numbers is maximal when they are equal. This is when $x = 1$, and the largest possible value of the expression is 5.

Note that we have not (yet) violated the condition of the AM-GM inequality that requires both numbers to be positive. However, one might ask if we could get a still larger product if either term were negative, a situation not covered by the AM-GM inequality. But of course in this case the product is negative, and our maximum value is larger. The reader is invited to explore the situation for expressions of the form $(x - a)(b - x)$ for various values of a and b.

Variation 7: On the domain indicated, and for any real number x for which $\tan x$ and $\cot x$ are defined, $(\tan x)(\cot x) = 1$. Thus their sum is minimal when $\tan x = \cot x$, which is when $\tan x = 1$. The required minimum value is 2.

Variation 8: The sum $\sin^2 x + \cos^2 x$ is constant (it equals 1). So the largest value of the given product occurs when they are equal, for example when $x = \pi/4$. This largest value is 1/4. Note that this implies, for $0 < x < \pi/2$, that the largest value of $\sin x \cos x$ is 1/2. This result leads to another solution when we note that $\sin 2x = 2 \sin x \cos x$, so $\sin x \cos x = (1/2)\sin 2x$, whose maximal value is 1/2.

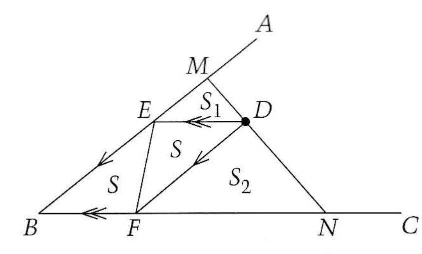


Figure 11

Variation 9: The product $(2^x)(2^{-x})$ is constant (it is 1), so the expression is minimal when $2^x = 2^{-x}$, which is when $x = 0$. The minimal value is 2.

If we consider the related expression $(e^x + e^{-x})/2$ (where the number e is the base of the natural logarithms), then we are studying the function $y = \cosh x$ (the hyperbolic cosine of x), which is of importance in engineering and theoretical work. Its minimal value over the real numbers is also 2.

Variation 10: The square roots on the left side of the given inequality are an open invitation to apply the AM-GM inequality. We have

$$\begin{aligned} &x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \\ &\le x\frac{y+z}{2} + y\frac{x+z}{2} + z\frac{x+y}{2} \\ &= xy + yz + xy. \end{aligned}$$

If only all our estimates would fall out so neatly! The two expressions are certainly equal when $x = y = z$. But are there any other possibilities for equality?

Variation 11: You can try writing the square roots as sums immediately, but it probably won't work. In this case, it is easier to square both sides and work with the equivalent inequality

$$(a + c)(b + d) \ge ab + cd + 2\sqrt{abcd}.$$

This simplifies to $ac + bd \ge 2\sqrt{abcd}$, and now we can use the AM-GM inequality directly.

Variation 12: As shown in figure 11, we draw line DE parallel to BC (with E on ray BA), and line DF parallel to AB (with F on ray BC). Then we draw segment EF. Let S_1 denote the area of triangle MED, S_2 the area of triangle DFN, and S the area of triangle DEF. Then, because diagonal EF bisects parallelogram $EDFB$, the area of triangle EFB is also S. So the area to minimize is $S_1 + S_2 + 2S$.

Notice that the positions of lines DE and DF, and thus the value of S, does not depend on the position of line MN, so we only need to minimize the sum $S_1 + S_2$. Now the triangles MED and EBF have equal altitudes from points D and F, so the ratio of their areas is just the ratios of their bases, or $S_1/S = ME/EB$. Similarly, the ratio of the areas of triangles DFN and EFB is equal to the ratio of their bases, or $S_2/S = FN/BF$. Finally, the ratio $ME/EB = MD/DN$, and $FN/BF = DN/MD$, because parallel lines intercept proportional segments on any transversal. Therefore

$$\frac{S_1}{S}\frac{S_2}{S} - \frac{MD}{DN}\frac{DN}{MD} = 1.$$

Thus $S_1 S_2 = S^2$, a constant, and $S_1 + S_2$ is minimal when $S_1 = S_2 = S$. This happens when MN is parallel to EF.

Variation 13: We know $x_1, x_2, \ldots, x_n = 1$, and we want to show that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \ge 2^n.$$

We can use the AM-GM inequality to transform each sum on the left to a product. We have

$$1+x_1 \ge 2\sqrt{1\cdot x_1} = 2\sqrt{x_1},$$

and similarly for the other factors. Multiplying, we find that

$$(1+x_1)(1+x_2)\cdots(1+x_n)$$
$$\ge \left(2\sqrt{x_1}\right)\left(2\sqrt{x_2}\right)\cdots\left(2\sqrt{x_n}\right) = 2^n \cdot 1,$$

which is the result we need.

Variation 14: We would like to use the AM-GM inequality on the expression $x_n - 1/4$, but this is not the sum of two positive numbers. Following the hint, we look at $x_k^2 + 1/4$, which is the sum of two positive numbers. By the AM-GM inequality, we have

$$x_k^2 + \frac{1}{4} \ge 2\sqrt{x_k^2 \cdot \frac{1}{4}} = x_k,$$

or

$$x_k^2 \ge x_k - \frac{1}{4}$$

(with similar results when $k = 1$ or $k = n$).Using this estimate, we can find a lower bound for the logarithms in the required sum. We take the logarithm base $k - 1$ of each side of the previous inequality. Remembering that $x_{k-1} < 1$, we are careful to reverse the sense of each inequality, and find that

$$\log_{x_{k-1}} x_k^2 = 2\log_{x_{k-1}} x_k$$
$$\le \log_{x_{k-1}}\left(x_k - \frac{1}{4}\right).$$

Now we can find a lower bound for the required sum. If this sum is S, then

$$S = \log_{x_1}\left(x_2 - \frac{1}{4}\right) + \log_{x_2}\left(x_3 - \frac{1}{4}\right)$$
$$+ \log_{x_3}\left(x_4 - \frac{1}{4}\right) + \cdots + \log_{x_n}\left(x_1 - \frac{1}{4}\right)$$
$$\ge 2\log_{x_1} x_2 + 2\log_{x_2} x_3 + 2\log_{x_3} x_4$$
$$+ \cdots + 2\log_{x_n} x_1.$$

so

$$\frac{S}{2} \ge \log_{x_1} x_2 + \log_{x_2} x_3 + \log_{x_3} x_4$$
$$+ \cdots + \log_{x_n} x_1.$$

This is a sum of positive numbers, since, for each k, both x_k and x_{k-1} are less than 1. Thus we can use the AM-GM inequality once more to find that

$$\frac{S}{2} \ge \log_{x_1} x_2 + \log_{x_2} x_3$$
$$+ \log_{x_3} x_4 + \cdots + \log_{x_n} x_1$$
$$\ge n\sqrt[n]{\left(\log_{x_1} x_2\right)\left(\log_{x_2} x_3\right)\left(\log_{x_3} x_4\right)\left(\log_{x_n} x_1\right)}.$$

By the "chain rule" for logarithms, the radicand is equal to $\log_{x_1} x_1$, which leads to a lower bound for S of $2n$. Equality holds if all the x_k's are equal to 1/2.

The gambler

Problem 1: Let S sum up the simple progression in (2):

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots.$$

Double the fun:

$$2S = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots.$$

Since every term in S is finite, we can subtract S from $2S$:

$$2S - S = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \ldots$$
$$+ \frac{1}{2^n} - \frac{1}{2^n} + \ldots - \ldots.$$

With a bit of pruning, we get

$$S = 1. \qquad (8)$$

Multiply through by the constant \$1 in (2) to get, er, \$1.

Solution 2: We want to find an integer for v such that $2v$ approximates \$67,500,000. Let's equate $2v$ to \$67,500,000 and then round v to the nearest integer. Taking natural logs and solving for v gives us

Index of Advertisers

Bradbury Software Enterprises	31
Burleigh Instruments	27
Harvard University	27
Knowledge Revolution	cover 4
The Learning Team	31
Metrologic Instruments	25
NSTA Special Publications	36, 44, 51, 54

$$v = \frac{\ln 67{,}500{,}000}{\ln 2} = 26.0083841664.$$

So set v equal to 26. Now, 2^{26} = \$67,108,864. To equate $2^{26}(1 + h)$ to \$67.5 million, set h equal to .005828381.

Solution 3: You can think of the expected value of the entire bet as the sum of two expectations: that from the first v tosses and that from the later $N - v$ tosses, whatever N might be. The expected value of the first v tosses is just v. The expected value of the later $N - v$ tosses sums the expectation of each of those tosses. In each case, the expectation is just the probability of getting heads on that toss times the fortune you would receive from that event.

Solution 5: Use the appropriate values for v and h in (4).

Solution 6: Consider that h can range from 0 to 1.

Solution 9: About \$6.

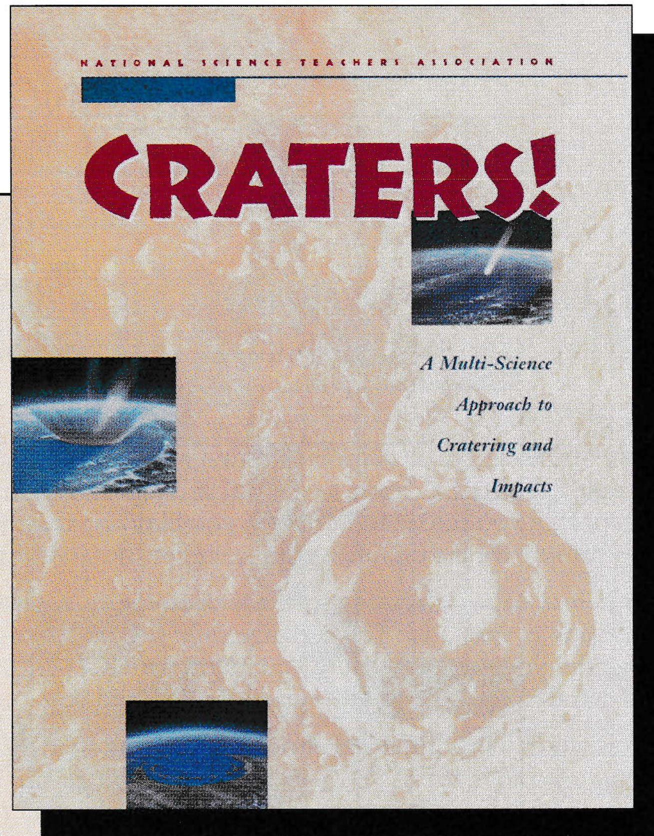
Solution 10: You can check that $U(0) = 0$; $U(1) = 1$; and $U(w) = Z$ as w becomes infinitely large.

How to be a QUANTUM author

Have you written an article that you think belongs in *Quantum*? Do you have an unusual topic that students would find fun and challenging? Do you know of anyone who would make a great *Quantum* author? Write to us and we'll send you the editorial guidelines for prospective *Quantum* contributors. Scientists and teachers in any country are invited to submit material, but it must be written in colloquial English and at a level appropriate for *Quantum*'s target readership of high school and college students.

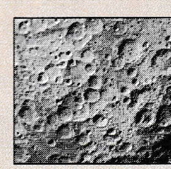
Send your inquiries to:

Managing Editor
Quantum
1840 Wilson Boulevard
Arlington VA 22201-3000



By William K. Hartmann with Joe Cain

Make an impact in your classroom with this interdisciplinary guide to cratering. How do craters form? What can craters tell us about planetary science? How have impacts affected Earth's history and the history of life?
Produced in cooperation with NASA and The Planetary Society, *Craters!* includes 20 ready-to-use, hands-on activities that use cratering to teach key concepts in physics, astronomy, biology, and Earth sciences. Special features include a summary of research on Shoemaker-Levy 9's encounter with Jupiter, and a detailed background section for teachers. The book comes with a Mac/ Windows CD-ROM packed with supplemental images for use with classroom activities. But you don't need a computer to make excellent use of the activities bevause



all of the pages—including beautiful images of the Moon and other cratered surfaces—are photocopy-ready.
Grades 9–12, 1995, 240 pp.
#PB120X $24.95

To order, call 1-800-722-NSTA

Quantum

What I learned in Quantum Land

Have you ever, while standing around at a party, or maybe a businesslike meeting,
Attempted to tell a clever new joke, and ended up feeling a fool?
The people you told it to looked at you blankly, their silence was almost a beating,
You think, "If I only could shrink clean away, as minute as a small molecule!"

But before you go thinking in terms of this sort, please think of just what you are saying;
The Land of the Quantum is dangerous, yes, and you shouldn't go into it careless,
Consider your actions before you go there, it's a perilous game you are playing;
The Land of the Tiny is quite unforgiving—did you know the whole place is airless?

But if you're determined to shrink clean away to avoid the cold glares at the party,
Prepare for adventures you've never conceived, in this Place of the Terribly Small:
Bone up on your Heisenberg, Einstein, and Bohr, and take your last meal; eat hearty!
And then go ahead, shrink away! But please don't overshoot into Nothing At All.

And when you get down there, you'd better keep watch to avoid the high-energy photons:
They'll come up behind you and knock you down flat and be gone ere you come to your senses.
And then if you go on a nucleus tour, say "Hi!" to the neutrons and protons.
(But I doubt you'll be able to make it that far, what with all the strong force's defenses.)

And if you see opposite colors and spins and appearance and charge in a pair,
Well, you can be sure that they soon will be gone in an energy burst of some power.
For one is an everyday particle while the other's an "anti," and there
Is the crux of the problem, for when they combine, you could die in a gamma-ray shower.

And don't even try to make friends with the folks that you meet in the Land That Is Mini,
They'll snub you, reject you, and oust you, and there will be naught you can do that will change it.
There's a Principle all the particles know that will keep you in shamed ignominy,
Some fella named Pauli came up with the rule—I don't think you can rearrange it.

Then, if you are tired and want to sit down, and a nice, empty spot seems to beckon,
Think twice before going and filling that spot, that electronic orbital shell,
At least, be prepared at the drop of a hat (or a photon emission, I reckon),
To vacate the premises pronto! posthaste! or things will not go at all well.

The reason for this is, the place where you've parked is reserved—it's a 'lectron's position.
And while they're away on vacation, you've clearance to stay in their station, rent-free.
But as soon as they've done seeing sights, they will signal that, well, they are now in transition,
So you have an entire femtosecond or two to react, pack your suitcase, and flee.

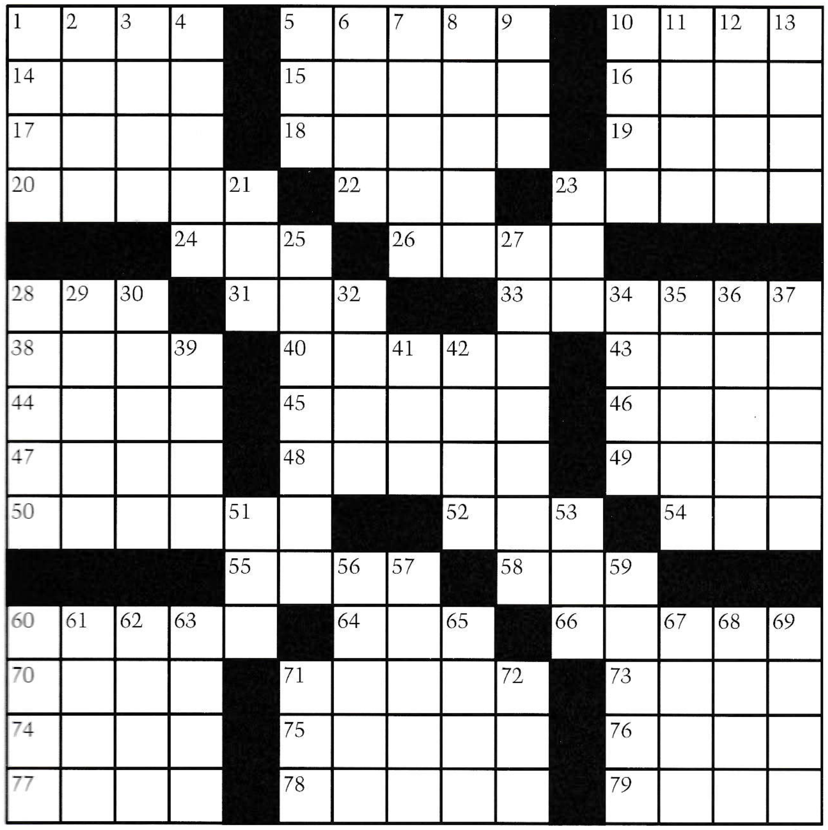
And then, as if *that* weren't enough, you will find that you're getting quite lonely and tired;
This Land of the Quantum is not what you'd planned when you wanted to leave all your friends.
A better idea, perhaps, maybe, is to *prevent* yourself from getting enmired
In hasty, ill-planned, or just silly remarks—it's much better for you in the end.

—David Arns

David Arns *is a graphics software documentation engineer for Hewlett-Packard in Fort Collins, Colorado, and also operates a small business designing and creating web sites. In his spare time he dabbles in poetry on scientific themes.*

Criss × cross science

by David R. Martin



Across

1 Naut. below
5 Soviet writer ___ Babel
10 Rank
14 Archaeologist ___ Kathleen Kenyon
15 Ship part
16 Ye ___ Shoppe
17 Prescription word
18 Polishing material
19 Baseball's Nolan ___
20 Talus
22 Hylobatidae member
23 Sum of diagonal matrix elements
24 ___ value (avg.)
26 Russian chemist Germain Henri ___ (1802–50)
28 Nucleotide
31 Type of transistor
33 Magnetic flux units
38 Pediatrician ___ Schick
40 977,838 (in base 16)
43 Phosphatase unit
44 Nail
45 European flatfish
46 ___ Pavlov
47 Down: pref.
48 Artist's tool
49 Sporogonium stalk
50 Of elements
52 Uncle ___
54 Hawaiian wreath
55 Outer body layer
58 Poise reciprocal
60 Gland: comb. form
64 Queen ___ (Fairy of men's dreams)
66 Planet's path
70 Poet Victor Marie ___
71 Am. Rev. general Horatio ___
73 Unit of heredity
74 "I smell ___."
75 Lab gels
76 Woodwind
77 Acid's opposite
78 Swedish chemist ___ Samuelsson
79 Type of gate

Down

1 Sums
2 Lay
3 Hebrew meas.
4 Unit of magnetic induction
5 ___ wind (of western China)
6 Oceanic crust rocks
7 ___ null
8 Concur
9 Lock opener
10 Unit of pressure
11 Chemist ___ Prigogine
12 60,844 (in base 16)
13 ___ Descartes
21 Potential diff.: abbr.
23 Half a fly
25 __ effect (thermoelectric effect)
27 Relating to stars
28 703,434 (in base 16)
29 Monster: comb. form
30 Greek philosopher
32 10^{12}: pref.
34 Ancient Nile city
35 Surveyor's instrument
36 Decrease
37 Moses's mountain
39 Swedish botanist ___ Afzelius (1750–1837)
41 Doubled: pref.
42 Anthropologist ___ Hrdlicka
51 Same: pref.
53 Reciprocal ohm
56 Real or virtual follower
57 Poet ___ Alterman
59 Quantum of energy
60 Moby Dick character
61 ___ mater (brain membrane)
62 Neurologist Antonio ___ Moniz
63 Musical sound
65 Biochemist Paul ___
67 48,826 (in base 16)
68 ___ ___ the ground floor
69 ___ off (irate)
71 Gossip
72 Fast plane

SOLUTION IN THE NEXT ISSUE

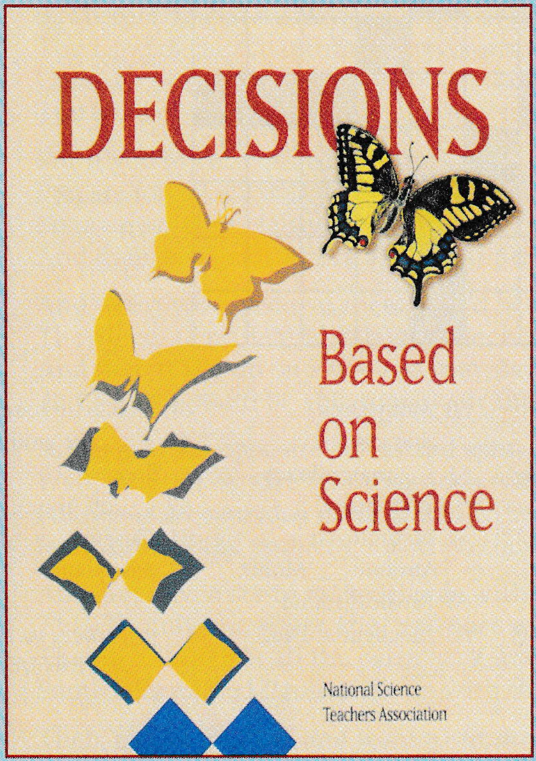
SOLUTION TO THE NOVEMBER/DECEMBER PUZZLE

A	D	A	E	■	R	H	A	B	D	■	L	A	C	■
L	I	M	N	■	E	E	B	A	A	■	A	L	E	E
I	N	I	T	■	S	M	E	L	T	■	M	A	R	Y
T	A	N	H	■	E	E	L	■	A	L	B	I	T	E
■	■	■	A	B	A	■	■	■	B	A	D	■	■	■
G	A	L	L	I	U	M	■	F	A	R	A	D	A	Y
A	D	O	P	T	■	A	M	I	S	S	■	I	R	E
R	A	C	Y	■	E	C	O	L	E	■	B	E	D	A
T	C	A	■	A	L	O	N	E	■	P	I	N	E	S
H	A	L	O	G	E	N	■	S	O	L	V	E	N	T
■	■	■	S	M	C	■	■	■	L	O	A	■	■	■
F	E	R	M	A	T	■	B	O	B	■	L	A	R	K
E	T	U	I	■	R	O	U	G	E	■	E	D	E	A
T	A	B	U	■	O	R	M	E	R	■	N	A	N	O
■	S	Y	M	■	N	A	P	E	S	■	T	M	E	N

DECISIONS

"...Americans are confronted increasingly with questions in their lives that require scientific information and scientific ways of thinking for informed decision making."

—*National Science Education Standards*



Quantum 11/97

Highlights

- Provides full background, introductory activities on important concepts, teaching tips, presentation ideas, and teaching plans
- Includes 10 guided activities with separate student and teacher pages
- Extends learning with 14 independent exercises
- Contains a broad range of science content to fit into existing curricula, including life, physical, Earth, and social sciences
- Enhances writing skills using student journals
- Includes assessment rubrics to evaluate student work

Grades 9-12, 1997, 144 pp.

#PB141X $19.95

Based on Science

Nothing is more important than educating the next generation of decision makers. Developed with support from the American Petroleum Institute, *Decisions* teaches students to use scientific ways of thinking to make everyday choices.

NSTA

To order call 800-722-NSTA or visit our web site at www.nsta.org

Art by Mark Brenneman

COWCULATIONS

Gingerbread man

A spicy byte

by Dr. Mu

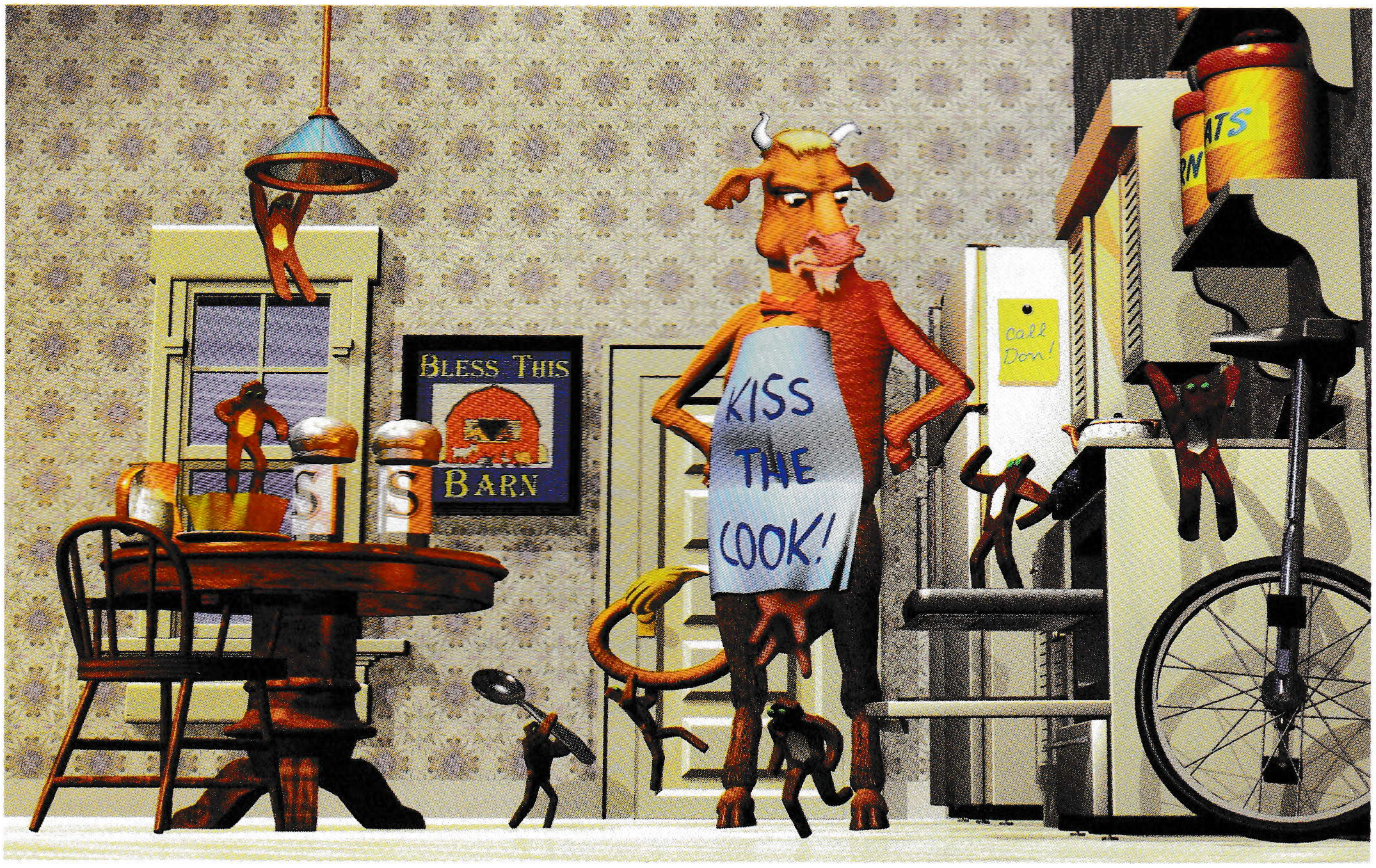
WELCOME BACK TO COWCULATIONS, THE column devoted to problems best solved with a computer algorithm. My neighbor, friend, and dairy farmer John Dough made his annual batch of gingerbread men again this year and gave them as gifts to his friends. He grows his own ginger and swears that his cookies are not only tasty, but "good for what ails ya." Last month, Farmer John gave me some of his fine ginger so I could experiment with making my own spicy bytes.

I recently read an article by Robert Devaney in an old issue of *Algorithm*, published in January 1992. One of the first curious objects he happened upon while experimenting with computer graphics was a gingerbread man. It started with a very simple iteration in the plane, beginning from a single point. By graphing the orbit of the iterations of this point, the image of a gingerbread man magically appeared. This I had to see.

His gingerbread man function defined on points in the plane can be written in *Mathematica* as

```
GBM[{x_, y_}] := {1 - y + Abs[x], x}
```

So, I started with an arbitrary point {1, 2} and applied the GBM function repeatedly. In *Mathematica* the command **NestList** does the job by creating iterations of the initial point a under the function f as in



Art by Mark Brenneman

```
NestList[f, a, 4]
```

```
{a, f[a], f[f[a]], f[f[f[a]]], f[f[f[f[a]]]]}
```

When I applied the GBM function to the point {1, 2}, I got the sequence

```
NestList[GBM, {1, 2}, 6]
```

```
{{1, 2}, {0, 1}, {0, 0}, {1, 0}, {2, 1}, {2, 2},
{1, 2}}
```

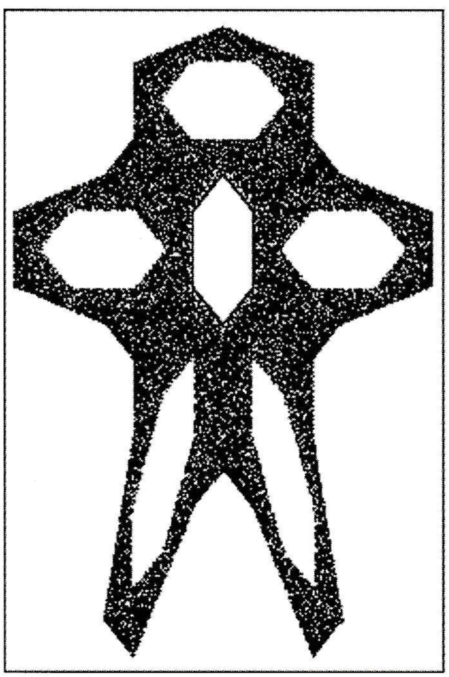
The orbit of the GBM function cycled back to the original point {1, 2} after six transformations. Other points gave different sequences:

```
NestList[GBM, {1, 5}, 19]
```

```
{{1, 5}, {-3, 1}, {3, -3}, {7, 3}, {5, 7},
{-1, 5}, {-3, -1}, {5, -3}, {9, 5}, {5, 9},
{-3, 5}, {-1, -3}, {5, -1}, {7, 5}, {3, 7},
{-3, 3}, {1, -3}, {5, 1}, {5, 5}, {1, 5}}
```

What can we say about the orbits of the GBM function? Some orbits cycle back to the original point. For example, the orbit of {1, 2} cycles back to {1, 2} after six transformations. The orbit of {1, 5} cycles back after 19 transformations. The period of a cycle is the length of the cycle, or how many steps it takes to return to its initial point.

Devaney discovered that certain points have extremely long periods, and, by plotting the corresponding orbit of these points, a gingerbread man appears. I investigated this by writing a short *Mathematica* program that creates an orbit of 50,000 points, rotates the image 135 degrees (**{x, y} -> {y - x, -y - x}**) and graphs the gingerbread man standing up (the original was lying down).



However, there is one point missing in my program. This is the initial point {*a*, *b*}, which creates the entire gingerbread man. One of your tasks in this issue's "Challenge Outta Wisconsin" is to find this point.

```
Clear[gingerbreadMan]
gingerbreadMan = NestList[GBM, {a, b}, 50000];
ListPlot[gingerbreadMan /. {x_, y_} -> {y - x,
-y - x},
  Frame -> True, FrameTicks -> None,
     AspectRatio -> 1.5, Axes -> None]
```

COW 8. Find an initial point {*a*, *b*} that will generate a gingerbread man. Write a program that will find the period for a point in the plane under the GBM function. Where are the points with short periods and where are those that generate a gingerbread man? Is the period for the gingerbread man finite?

Run, run as fast as you can,
You can't catch me,
I'm the Gingerbread man.

Solution to COW 6

Two issues back, you were asked to consider Farmer Paul's DDS Model for bacteria growth in milk. If we take a series of bacteria measurements in milk at equally spaced times, then the bacteria count changes according to the following Logistic Growth Model. Note: If *now* is the present time period, *now* – 1 is the previous time period, with 1 representing a fixed unit of time.

$$\mathbf{Bacteria[0] = 1;}$$

$$\mathbf{Bacteria[now] = Bacteria[now - 1]} + \frac{\mathbf{(Temperature - 32)}}{\mathbf{200}}\mathbf{Bacteria[now - 1]} \times \left(\mathbf{1} - \frac{\mathbf{Bacteria[now - 1]}}{\mathbf{100}}\right)$$

The length of time it takes the bacteria to reach a reading of 80 is the length of time it takes to sour. The challenge in COW 6 was to find the temperature *T* at which milk soured twice as fast as it does at *T* = 50°F.

First, let's see how long it takes the milk to sour at 50°F.

$$\mathbf{T = 50.; b = 1; i = 0;}$$

$$\mathbf{While[b < 80, b = b +} \frac{\mathbf{(T - 32.)}}{\mathbf{200}}\mathbf{b}\left(\mathbf{1} - \frac{\mathbf{b}}{\mathbf{100}}\right)\mathbf{; i++]; i}$$

```
68
```

Our problem is to find the temperature at which the bacteria level will reach 80°F in half this time, or 34 units. We first create a function **sourTime**, which reads the temperature and returns the time it takes to sour milk.

$$\mathbf{sourTime[T\_] := Module[\{b = 1, i = 0\},}$$

$$\mathbf{While[b < 80, b = b +} \frac{\mathbf{(T - 32.)}}{\mathbf{200}}\mathbf{b}\left(\mathbf{1} - \frac{\mathbf{b}}{\mathbf{100}}\right)\mathbf{; i++]; i]}$$

Now, we close in on the temperature at which milk sours in 34 units of time.

```
T = 50;
```

```
delta = 1.;
Do[While[sourTime[T] > 34, T += delta];
  T = T - delta;
  delta = delta/10, {3}]
Print["The answer is ", T + 10
delta]
```

```
The answer is 68.67
```

Let's check it out.

```
sourTime[68.67]
```

```
34
```

It works as advertised.

Advanced solution

Morton Goldberg submitted a more elaborate cowculation based on the solution of a continuous model (a differential equation) of the discrete relationship.

By making the substitutions Bacteria $\to b$, $n-1 \to t$ and taking Δt as the time unit, the recursion given above can be put into the form

$$\frac{b[t+\Delta t]-b[t]}{\Delta t}=\frac{(T-32)}{200}b[t]\left(1-\frac{b[t]}{100}\right)$$

Taking the limit as $\Delta t \to 0$ gives the differential equation

$$b'[t]=\frac{(T-32)}{200}b[t]\left(1-\frac{b[t]}{100}\right)$$

with the intial value condition $b[0] = 1$. *Mathematica* can find a closed-form solution for this differential equation using **DSolve.**

```
Clear[b, T]
```

$$\texttt{solution = DSolve[\{b'[t]==}\frac{(T-32)}{200}\texttt{b[t]}\left(1-\frac{\texttt{b[t]}}{100}\right)\texttt{,}$$
$$\texttt{b[0]==1\}, b[t], t]}$$

$$\texttt{\{\{b[t]->}\ -\frac{100}{-1+E^{\frac{1}{200}\left(32t-tT+Log[(-11)^{1/100}3^{1/50}]\right)}}$$

This solution can be simplified to

$$\texttt{b[t\_, T\_]}=\frac{100}{1+99\,E^{\left((32-T)\frac{t}{200}\right)}}$$

$$\frac{100}{1+99\,E^{\frac{1}{200}t(32-T)}}$$

Now with $b[t, T]$, which measures the bacteria count at time t with milk at temperature T, we cowculate the **timeToSour.**

```
Clear[timeToSour]
FindRoot[b[timeToSour, 50] == 80.,
{timeToSour, 60}]
```

```
{timeToSour → 66.4602}
```

```
timeToSour = timeToSour /. First[%]
```

```
66.4602
```

Finally, we cowculate the temperature at which it sours twice as fast.

```
FindRoot[b[timeToSour/2, T] == 80.,
  {T, 60}]
```

```
{T -> 68.}
```

This cowculation differs from the discrete solution because the continuous model is approximated by the discrete one.

A New Year

In starting a new year, we are faced, rather abruptly, with a new number—1998. It will take some time to get used to 1998, especially when writing checks. Unlike 1997, 1998 is not a prime year. However, it is between two primes: 1997 and 1999 (called twin primes). That won't happen again for 30 years, as the following cowculation shows.

```
Table[Prime[PrimePi[1997] + i], {i,
0, 6}]
```

```
{1997, 1999, 2003, 2011, 2017, 2027,
2029}
```

In *Mathematica,* **PrimePi[n]** equals the number of primes $\leq n$, and **Prime[j]** equals the jth prime number.

What else is interesting about 1998? Let's look at its divisors.

```
Divisors[1998]
```

```
{1, 2, 3, 6, 9, 18, 27, 37, 54, 74, 111, 222,
333, 666, 999, 1998}
```

Next take the proper divisors of 1998 and add them up.

```
Apply[Plus, Drop[Divisors[1998], -1]]
```

```
2562
```

Since the sum is greater than 1998, we call 1998 an "abundant" number. Thus, I am very encouraged about the abundant year 1998, which is flanked by two prime years. It sounds like a Happy New Year to me, and I hope to you.

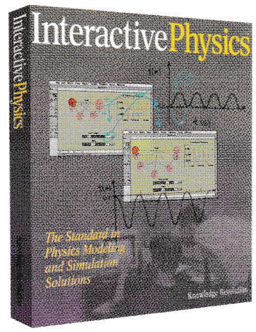
And finally...

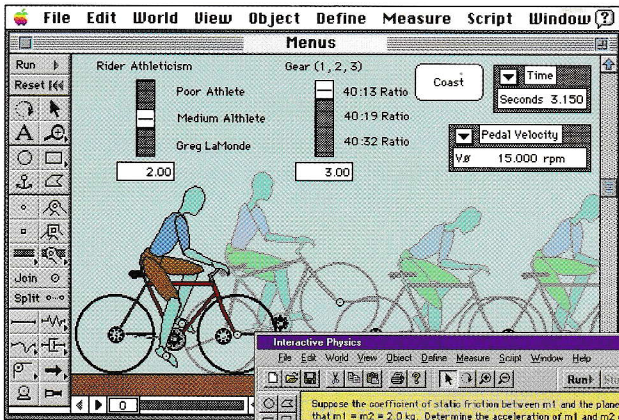
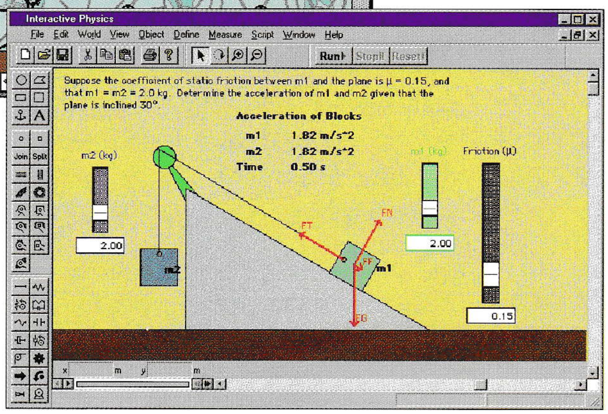
The cowculations sent in on COW 7 will appear in the next issue. Solutions to COW_{n-2} will appear in COW_n. This gives all cowhands more time to ruminate on possible solutions before they e-mail them to me at drmu@uwp.edu. Past solutions are available on the web at usaco.uwp.edu/cowculations.

If competitive computer programming is your interest, stop by the USA Computing Olympiad web site at usaco.uwp.edu. The 1998 USA Computing Olympiad Internet competitions are underway.

Revolutionize Your Curriculum!

Our Integrated Mathematics and Science Solutions Redefine Your Educational Environment



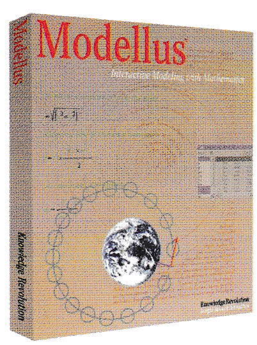



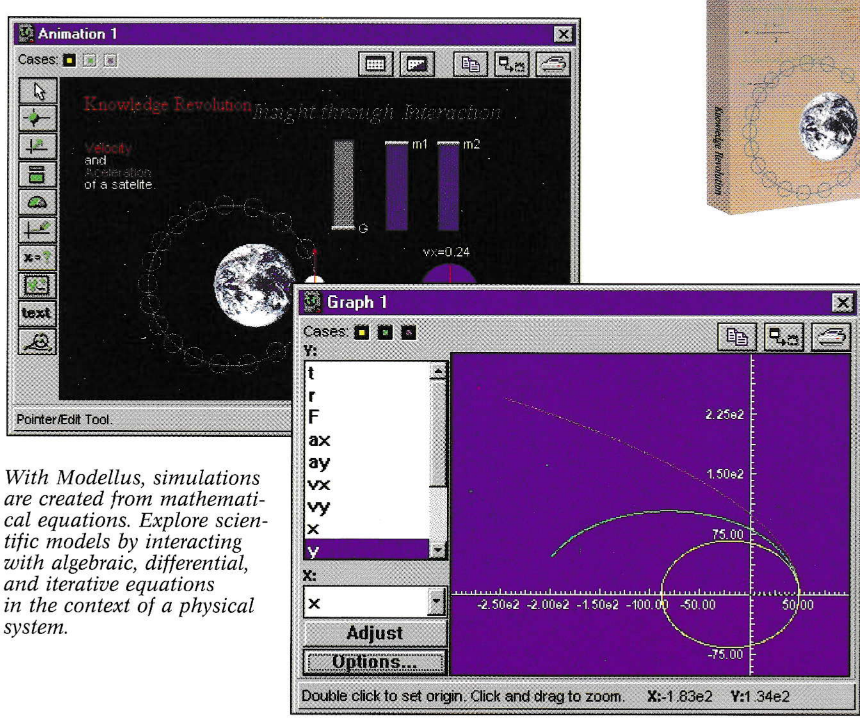
Create simulations by sketching out bodies and attachments with a simple point-and-click user interface. Alter physical properties like friction and mass during the simulation to gain valuable insight into the system.

Interactive Physics™ makes it easy to integrate modeling and simulation into your physics curriculum by offering a complete desktop motion lab. Create models by drawing onscreen with a powerful and easy-to-use graphic interface. Add objects like springs, dampers, ropes and joints. Measure physical quantities like velocity, acceleration, momentum and energy. Interact with your model in real time by changing properties as the simulation runs. Widely adopted by many textbook publishers and with more than half a dozen awards and thousands of educational users, *Interactive Physics* is the standard in physics modeling and simulation solutions.

Interactive Physics: Single User $249
Lab Pack pricing . . . . .Call

NEW





With Modellus, simulations are created from mathematical equations. Explore scientific models by interacting with algebraic, differential, and iterative equations in the context of a physical system.

Modellus™ gives students the power to interact with models in real time to gain powerful insights into the underlying mathematics. Its equation-based approach to modeling offers an educational compliment to *Interactive Physics*. Typical math programs act as calculators, treating the model as a static entity. But *Modellus* lets students manipulate mathematical variables during a simulation to study the effects of their changes on animations, graphs and tables. By supporting multiple cases and multiple data representations, *Modellus* offers students valuable feedback during the problem solving process.

Modellus: Single User $199
Lab Pack pricing . . . . .Call

Knowledge Revolution
Insight through Interaction

66 Bovet Road, Suite 200
San Mateo, California 94402
Fax 650 574 7541
Telephone 650 574 7777
Toll-Free 800 766 6615

Call Now for Demo CD

www.krev.com

Interactive Physics™ and *Modellus*™ are trademarks of Knowledge Revolution. Other trademarks are property of their respective holders. ©1997 Knowledge Revolution.

Circle No. 6 on Reader Service Card