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QUANTUM

THE STUDENT MAGAZINE OF MATH AND SCIENCE

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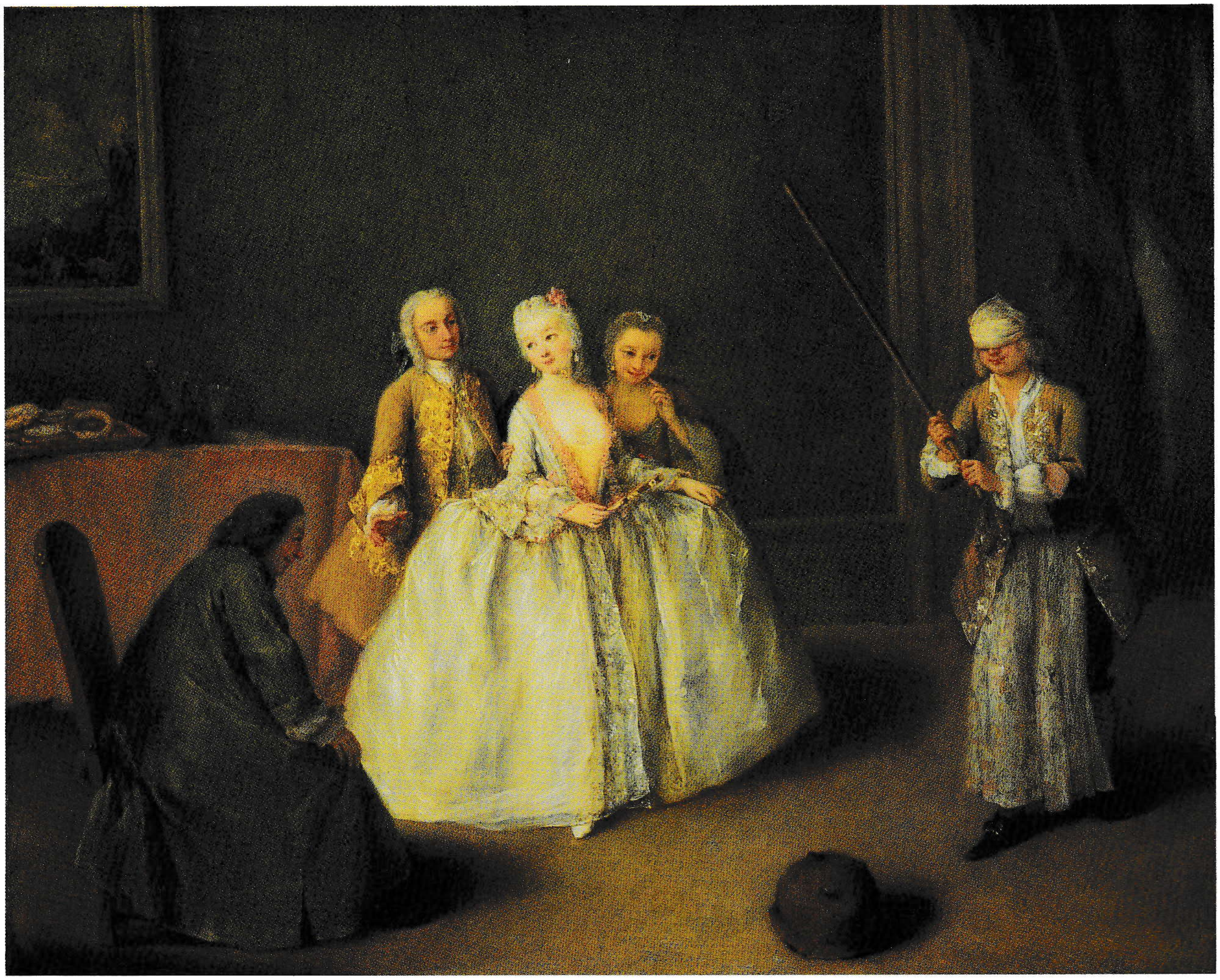

National Gallery of Art, Washington (Collection of Samuel H. Kress) © 1993 NGA

Blindman's Buff (ca. 1745) by Pietro Longhi

IF YOU WANDER AROUND THE NATIONAL GALLERY of Art long enough, you'll eventually come across this scene of Venetian life in the mid-eighteenth century. The painting seems misnamed: the Oxford English Dictionary defines blindman's buff as "a game in which one player is blindfolded and tries to catch and identify any one of the others, who, on their part, push him about and make sport with him." (The "buff" comes from "buffet.") The activity depicted by Pietro Longhi (1702–1785) seems more akin to the festive Latin-American custom of breaking the piñata. Nowadays we usually see papier-mâché piñatas in the shapes of animals, but they used to be painted pottery jars. Then, as now, they were filled with candy and gifts. The object on the floor certainly looks like an upturned piece of crockery, and one suspects that a treat is hiding underneath.

So, let's trust our eyes and leave the title aside. In so doing we mimic the abrupt change of direction in Longhi's early career. He began painting under the tutelage of a history painter, but his one important work of this sort, a monumental ceiling entitled "Fall of the Giants," was a failure. Longhi then turned to genre painting—that is, the depiction of everyday scenes. Rather than paint the imaginary acts of mythical persons, he recorded life as it was actually lived by his contemporaries, sometimes with a touch of irony. A painting like "Exhibition of a Rhinoceros at Venice" (1751) provides documentation of a society curious about novelties from abroad but also curious about itself.

"Blindman's Buff" raises some interesting questions. For instance, how do we explain the attire of the boy wielding the stick? And who is the person faintly visible in the drapery? (Do you see a resemblance with the girl in the middle of the picture? We suspect that Longhi moved her there and painted over her original image at the last minute, not realizing that the drapes would "fade.") One thing we know for certain: sooner or later, the boy will find the pot on the floor (mathematically, the probability is 1). But what's the probability that he'll "intersect" it after, say, 10 random steps (forward, backward, left, right)? That's a bit trickier.

We could simplify the problem. Say we place the pot and the boy on a line and give his sister a coin. When she flips heads, he takes a step forward; tails, he steps back. He would then be enacting a *random walk*. Several articles in this issue deal with this topic—you can begin your tour with "Randomly Seeking Cipollino" on page 20.

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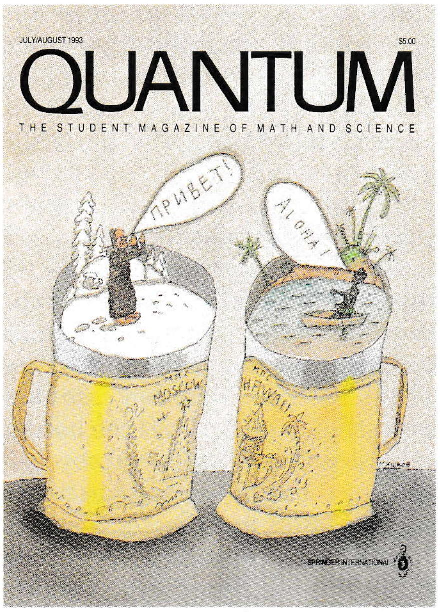

Cover art by Leonid Tishkov

The two figures greeting each other on our cover (the Russian word is pronounced prih-VYET) live in completely different worlds—or should we say, tea glasses? The glass holders bear the initials of the Ministry of Railways (МПС) and were a familiar sight to the millions of Soviet citizens who traveled by train over the course of seventy years. Many of them were headed to the resorts on the Black Sea, and while some may have dreamed of the far-off tropical paradise of Hawaii, few managed to get there—and none by rail.

One clever guy, though, figured out how to turn the frigid tundra into the sunny seashore. Well, actually, he devised a way of turning iced tea into hot (*boiling* hot) tea by bringing it into contact with boiling hot water (but not mixing the two liquids). Well, theoretically, at least. He hasn't managed to make it work with real glasses of tea and water.

Maybe that's because the guy's a mathematician. But even physicists will want to look at his unexpected results. They are reported in "Superheated by Equations," which begins on page 4.

FEATURES

DEPARTMENTS

PUBLISHER'S PAGE

A circuitous route

The road to "relevance" in science education

WHEN I WAS IN THE TENTH grade, my family moved from Kansas City to Shreveport, Louisiana. After spending one day in the new high school, I decided it wasn't for me. I hitchhiked back to Kansas (a dangerous and stupid thing to do) and lived with a friend of mine and his family. They had a place with several acres and a couple of horses, which were put under my care. In the afternoons I'd saddle up one of those beautiful quarter horses and ride back to school to show off. Not only had I returned to my old stomping grounds, I had returned in style!

It's curious that I felt such an attachment to that school, since I hadn't come across anything in school that I thought was worth learning. I was drifting, doing enough to pass but little else.

Then Charlie, my friend's older brother, began to talk with me about electronics. He was as an electronics engineer and an amateur radio operator. I wouldn't leave Charlie alone until he explained everything to me in ways that I could understand. By the time I was a junior I could explain any electric circuit. I spent every spare moment searching the local junkyards for old radios and electronic instruments that I could cannibalize for parts.

But then I ran into a serious problem. Even though I knew exactly how something worked and could explain it in terms of electron flow through or into every component, when I would apply my knowledge to make a circuit, guess what happened? It wouldn't work! All I got was smoking insulation, burned parts, blown fuses, sometimes a little weak noise, and not much else.

To my chagrin, I discovered that it isn't enough to understand *descriptively* (in words) how a circuit works. I found that you have to know the *size* of each component—the resistors, capacitors, and so on. And to know what that means, you have to know more about the components and about electricity. I had to understand them *quantitatively*—I had to know mathematics, which I had never bothered with! I needed to learn equations, which gave numerical values in units such as ohms, amps, and so on.

My experience with chemistry was similar. I'd mix up all kinds of chemicals indiscriminately (another stupid and dangerous thing to do!) and usually get some harmless goop that had to be thrown out. Even if I got some sort of reaction, I had no idea what it was. Descriptive chemistry was of little use—even if I had the right reactants, I didn't know how much to use or under what conditions. I needed quantitative chemistry, with its atomic mass units, moles, equilibrium constants, and so on. And a knowledge of pressure, temperature, and energy needed or released would come in handy (especially in avoiding an early demise).

Along the way I had seen that electronics is connected with chemistry, and that all the sciences are closely bound up with mathematics. I also realized that I was most interested in underlying principles, which led me to the most fundamental of the sciences—physics. So throughout my undergraduate and graduate education, I maintained an almost even balance between physics and mathematics.

Now, what's the point of all of this? I find myself engaged in a national debate about how best to help young people like you learn science. There are those who don't think a high school kid like I was can handle science or math at the level that you—*Quantum* readers—are learning it. But you and I both know better. Sure, we know there are some who really can't do it. But the vast majority just won't work hard enough. They have no interest in it. You know the type, and you know that many of your fellow students are not really that different from you. They could do just as well, and some could do even better.

Well, these folks with whom I disagree so strongly want to offer something else. It goes under several names: "Science, Technology, and Society"; "A Thematic Approach"; "Practical Applications"; "Physics, Chemistry, or Biology in the Community." They all take the same tack: kids can't learn real science, so let's give them something they can use in their daily lives to make personal and societal decisions. Along the way they'll pick up the right words for things, and so they'll gain "scientific literacy."

But my question is, how can young people learn science by using problems like these as a starting point? What basic knowledge can they bring to bear on such problems? How can they hold anything constant or carry out any experiment that might lead to an understanding of a complex phenomenon like acid rain, the ozone hole, or global warming?

Relevance played an important role in motivating me to learn science and math. But that isn't what sustained my interest. And that isn't what sustains your interest, is it?

The process leading to an understanding of some basic scientific principle or law, and the comprehension that it is applicable *everywhere* in our universe, are awesome and deeply satisfying experiences. Working one's way through the creation of a theory to account for an entire array of seemingly unrelated scientific principles or laws opens a whole new world of thought. The experience of learning a basic part of science for yourself can be a powerful, long-term source of motivation. The history of science is filled with examples of a small, apparently insignificant learning experience that was sufficient to sustain the person until an exponentially increasing number of such experiences produced major scientific achievements. How many young people have never had even one such experience?

These are not privileged experiences open only to the elite few who will become research scientists. They should be open to all students.

I'll always remember the time I came closest to the profound understanding of physics as it existed when I was in graduate school. It was as if I had pulled myself up to a high window that opened onto a beautiful and complex world I had never experienced. But to stay up there and enter that world would have required a level of effort and, yes, ability that I simply didn't have. I figuratively lowered myself from that window and pursued those things for which I was fully capable. I became a teacher, but I retained a profound admiration for those few who could remain and work in that different world.

Many *Quantum* readers will undoubtedly not only glimpse that world, but will become part of it. The rest of us can have a few glimpses, and we can use what we see there in important ways. *That* is the way to learn science, and *that* makes science relevant for all.

What do you think about this? Send us a letter or e-mail message (72030.3162@compuserve.com).

—Bill G. Aldridge

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Тишков 1993
МПС
Живу в Москве, пью чай горячий, мечтаю о Гавайях

Superheated by equations

A mathematician turns iced tea into hot tea

by Dmitry Fomin

THERE'S A WELL-KNOWN joke among scientists that shows the difference between mathematical and physical modes of thinking. A mathematician and a physicist are asked: "How do you boil water if you are in a kitchen with a gas stove, a kettle, a water faucet, and a box of matches?" The answer is obvious from both of them: "Fill the kettle with water, take a match, light the burner, and put the kettle on the stove."

Then a second question is posed: "Imagine that your kettle is already full. How do you achieve the desired goal now?"

The physicist answers: "Take a match (*another* match, I should say!), light the burner, and put the kettle on the stove again." But this shortest solution can be simplified further—here's how a mathematically minded person deals with this second problem: "Pour out the water. Now we have arrived at the previous problem, which has already been solved."

Take this anecdote as a kind of epigraph to the article that follows. And now the real story begins.

It's the story of one apparently simple physical question that a classmate proposed to me in 1980. He was good in physics, but he asked me to solve the problem because it had a large mathematical component.

Here's the problem. Imagine you have two identical cans of liquid, one of them hot, the other cold—for instance, a can of iced tea and a can of boiling water. But you want to drink some hot tea, and you have to heat the tea using only the hot water in the other can. How hot do you think you can make your tea? Would you believe me if I say that the whole can of tea can be heated—at least, theoretically—to a temperature as close to that of boiling water as you wish? I don't suppose you would. Nobody believes me—at first. Well, then your astonishment will be all the greater by the time you reach the end of my story.

It will be convenient to assume that the temperature of the cold liquid (tea) is 0° and that of the hot liquid (water) is 1°. These might look like pretty strange readings, but a temperature scale is just a matter of convention. And since we won't mix the two liquids (you don't want diluted tea, do you?), the heat must be transferred by conduction, which occurs when certain volumes of the two liquids are brought into direct contact. This will help us keep track of the temperature of each liquid.

At left: "I live in Moscow, drink hot tea, and daydream about Hawaii . . ."

Art by Leonid Tishkov

Types of heat transfer

Method 1. *Can-to-can.* The first and most obvious method of heating the tea is to bring the cans into contact. After thermal equilibrium is established, both cans will have the same temperature: (1/2)°. So it seems that this temperature is the maximum we can obtain for the tea by heat conduction. And this is what everyone usually thinks.

Method 2. *Hot jet.* If you've taken the time to think things over, you might have come up with the idea of dividing the hot water into several portions. To begin with, split the water into two equal halves and bring them into contact with the can of tea one after the other (each time waiting until thermal equilibrium is established). Then after the first heat exchange the common temperature of the tea and the first half of the water is

$$\frac{1\cdot 0^\circ + \frac{1}{2}\cdot 1^\circ}{1+\frac{1}{2}} = \left(\frac{1}{3}\right)^\circ.$$

And after the interaction between this heated tea and the remaining half of the water, their temperature becomes equal to

$$\frac{1\cdot\left(\frac{1}{3}\right)^\circ + \left(\frac{1}{2}\right)\cdot 1^\circ}{1+\frac{1}{2}} = \left(\frac{5}{9}\right)^\circ = 0.555\ldots^\circ.$$

That's it! The result is greater than (1/2)°! This encourages us to keep going. Let's divide the water into n equal portions and spray the tea with this hot jet. To calculate the final temperature of the tea, denote its temperature after the interaction with the kth "drop" of water by t_k. Certainly, $t_0 = 0°$. Then

$$t_{k+1} = \frac{1}{n+1}\left(nt_k + 1\cdot 1\right),$$

or, equivalently,

$$1-t_{k+1}=1-\frac{nt_k+1}{n+1}=\frac{n(1-t_k)}{n+1}.$$

Wonderful! Now it's easy to see that

$$1-t_n=\left(\frac{n}{n+1}\right)^n,$$

or

$$t_n=1-\left(\frac{n}{n+1}\right)^n.$$

As you probably know,[1] the sequence $a_n = (1 + 1/n)^n$ increases and approaches $e = 2.71828...$ with the growth of n. Therefore, the sequence $t_n = 1 - 1/a_n$ also increases and approaches $1 - 1/t \cong 0.632$.

In particular, this means that the temperature of the water can be made arbitrarily close to $1/e \cong 0.388$—that is, much lower than that of the tea![2] This method of heat transfer for $n = 3$ is graphically illustrated in figure 1: every single act of heat transfer is represented there by a pair of arrows joining the initial temperatures of the tea and that of the water droplet (which is always 1°) to their equilibrium temperature.

Method 3. *Cold jet.* Maybe you're sophisticated enough to have come up with the opposite idea: spray the water with a cold jet of tea! Let's investigate this process—perhaps we'll get a better result.

As before, we denote by t_k the temperature of the water after interacting with the kth drop of tea ($t_0 = 1°$). Then, as we can see in figure 2,

$$t_{k+1}=\frac{1}{n+1}(nt_k+1\cdot 0)=\frac{n}{n+1}t_k.$$

So

$$t_n=\left(\frac{n}{n+1}\right)^n\to\frac{1}{e}$$

as n approaches infinity, and we have exactly the same result as before: the temperature of the water is about $(1/e)°$, which means that the tea has been heated to $(1 - 1/e)°$. Unfortunately, this adds nothing to what we've gained with the previous method. But don't lose heart! I have one more idea up my sleeve.

Method 4. *Jet-to-jet.* We've already studied the interaction of the entire can of cold tea or hot water with a jet of small portions of the other liquid. If you're consistent, you'll invent the last (but not least!) method of heat transfer. Although it's rather complicated from the computational point of view, we can investigate mathematically the interaction of two jets: one hot, one cold. What do I mean? Imagine that we've divided both liquids into n equal portions. Let's bring the first portion of water into contact with all the portions of tea one by one; then let's do the same with the second portion of water; and so on. So, step by step, the portions of tea will be heated by the portions of water. It can be described like this: the drops of the water jet travel along the tea jet, exchanging heat as they go. This method calls for closer study.

$$\begin{array}{r|ccc}
 & & & \text{iced tea} \\
 & & & 0 \\
 & & & \Downarrow \\
 & 1 & \to & 1/4 \\
 & & & \Downarrow \\
\text{hot water} & 1 & \to & 7/16 \\
 & & & \Downarrow \\
 & 1 & \to & 37/64
\end{array}$$

Figure 1

n equal portions of tea

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 \\
 & & \Downarrow & & \Downarrow & & & & \Downarrow \\
1 & \to & \dfrac{n}{n+1} & \to & \left(\dfrac{n}{n+1}\right)^2 & \to & \ldots & \to & \left(\dfrac{n}{n+1}\right)^n
\end{array}$$

Figure 2

Generous heat exchange

The jet-to-jet process of heat exchange is illustrated in figure 3. The horizontal rows describe the path of the water droplets, while the vertical columns record the path of the tea droplets. Since the masses of all the portions of the two liquids are the same, the temperature of any two portions after their interaction is the arithmetic mean of their initial temperatures. It follows that the temperature t_{ij} of the ith portion of water and jth portion of tea immediately after their interaction does not depend on the number of portions. Formally, the numbers t_{ij} for $i \geq 1$, $j \geq 1$ are found from the recursive relation

$$t_{ij}=\frac{1}{2}\left(t_{i,j-1}+t_{i-1,j}\right), \tag{1}$$

and the "initial conditions" for the temperatures before any interactions are $t_{0j} = 0$, $t_{i0} = 1$. Even a cursory glance at figure 3 yields several observations that will be helpful later. First, we notice that $t_{nn} = 1/2$ for all $n \geq 1$. Actually, this is a particular case of a more general relation that expresses a certain kind of symmetry in our diagram:

$$t_{ij} + t_{ji} = 1. \tag{2}$$

To prove this equality for all $i \geq 0$, $j \geq 0$ (except $(i, j) = (0, 0)$, of course), put $s_{ij} = t_{ij} + t_{ji}$. Then the numbers s_{ij} obviously satisfy the same half-sum recursive law: $s_{ij} = (s_{i-1,j} + s_{i,j-1})/2$; but they obey different initial conditions: $s_{0j} = s_{i0} = 1$. Clearly, if we start to fill out a diagram similar to that in figure 3 but having ones instead of zeros in its upper row, we'll certainly obtain ones everywhere, so $s_{ij} = 1$ for all i, j. This completes the proof.

Another observation is that the numbers t_{ij} taken along any "diagonal" $i - j$ = constant—that is, the numbers $t_{n,n-k}$ for any fixed k—form a monotonic sequence. That is, the sequence is always decreasing for $k > 0$, increasing for $k < 0$, and constant for $k = 0$ ($t_{nn} = 1/2$).

[1]See, for instance, "[Getting to Know] The Natural Logarithm" by Bill Aldridge in the November/December 1991 issue of *Quantum*.—*Ed.*

[2]It goes without saying that in these considerations the inevitable heat losses are ignored. It's hardly possible, though, to conceive of a real hot-jet experiment for a sufficiently large number n of small portions of water in which the losses would indeed be negligible. This remark applies to the rest of the article as well.—*Ed.*

The proof is similar to the one above. Let $d_{ij} = t_{i+1,j+1} - t_{ij}$ (it's the difference between two consecutive numbers along a diagonal). Then the numbers d_{ij} again obey the half-sum law, and in addition $d_{0j} = t_{1,j+1} - 0 > 0$, $d_{nn} = 1/2 - 1/2 = 0$, $d_{i1} = t_{i+1,2} - 1 < 0$. It follows that $d_{ij} > 0$ above the "main diagonal" (for $i < j$), and $d_{ij} < 0$ below the main diagonal, which means that the sequence $\{t_{n,n-k}\}$ increases as n grows for any fixed $k < 0$ and decreases for $k > 0$.

Now we're in a position to evaluate the average temperature T_n of the tea after all n^2 heat exchanges have occurred. For any row k of our figure 3, let $S_k = t_{k1} + t_{k2} + \ldots + t_{kk}$. In particular, $S_1 = t_{11}$. Let $S_0 = 0$ (a reasonable choice for t_{00}). Note that $S_n = nT_n$.

Write out equations (1) multiplied by 2 for $i = n$ and all $j = 1, 2, \ldots, n$—

$$\begin{aligned} 2t_{k1} &= t_{k-1,1} + 1, \\ 2t_{k2} &= t_{k-1,2} + t_{k1}, \\ &\vdots \\ 2t_{k,k-1} &= t_{k-1,k-1} + t_{k,k-2}, \\ 2t_{kk} &= 1 \end{aligned}$$

—and add them up:

$$2S_k = S_{k-1} + 2 + S_k - t_{k,k-1} - t_{k,k},$$

or

$$S_k = S_{k-1} + (3/2 - t_{k,k-1}).$$

(The reader should check this for the special cases $k = 1$ and $k = 2$.)

$i \backslash j$	1	2	3	…
	0	0	0	
	⇓	⇓	⇓	
1	1 → 1/2 →	1/4 →	1/8 →	…
	⇓	⇓	⇓	
2	1 → 3/4 →	1/2 →	5/16 →	…
	⇓	⇓	⇓	
3	1 → 7/8 →	11/16 →	1/2 →	…
	⇓	⇓	⇓	
⋮	⋮	⋮	⋮	

Figure 3

Now we add these equations for $k = 1, 2, \ldots, n$. The left-hand side "telescopes": adjacent terms cancel out, leaving only the first and last. This gives us

$$\begin{aligned} S_n - S_0 = S_n &= (3/2 - t_{n,n-1}) + (3/2 - t_{n-1,n-2}) \\ &\quad + \ldots + (3/2 - t_{21}) + (3/2 - t_{10}), \end{aligned}$$

or

$$\begin{aligned} T_n &= \frac{1}{n}\left[\left(\frac{3}{2} - t_{10}\right) + \left(\frac{3}{2} - t_{21}\right) + \ldots \right. \\ &\quad \left. + \left(\frac{3}{2} - t_{n,n-1}\right)\right] \\ &= \frac{3}{2} - \frac{1}{n}\left(t_{10} + t_{21} + \ldots t_{n,n-1}\right). \quad (3) \end{aligned}$$

Let's show that $t_{n,n-1}$ approaches 1/2 as $n \to \infty$. We know that this is a decreasing sequence; it's also bounded ($0 \le t_{n,n-1} \le 1$); therefore, it has some limit a_1. Similarly, for every k the sequence $t_{n,n-k}$ has some limit a_k. Taking the limit of both sides of the equation

$$t_{n,n-k} = \frac{1}{2}\left(t_{n-1,n-k} + t_{n,n-k-1}\right),$$

we obtain

$$a_k = \frac{1}{2}\left(a_{k-1} + a_{k+1}\right),$$

or $a_k - a_{k-1} = a_{k+1} - a_k$ for all k. This means that a_k is an arithmetic sequence, and since it's bounded ($0 \le a_k \le 1$), it must be constant. Thus, $a_k = a_0 = 1/2$ for all k, and, in particular, $t_{n,n-1} \to a_1 = 1/2$. Now we can apply the following lemma.

Cesaro's lemma. *If a sequence x_n has a limit, and $s_n = x_1 + x_2 + \ldots + x_n$, then the sequence s_n/n has the same limit.*

(Roughly speaking, the proof starts with the observation that all the numbers x_n with a sufficiently large n are approximately equal to their limit within an accuracy as high as we wish. Therefore, the same is true for the arithmetic mean of the first n terms of the sequence. A strict proof is left as an exercise for readers familiar with the theory of limits.)

By this lemma, the second term in the right side of equation (3) has a limit equal to the limit of $t_{n,n-1}$—that is, 1/2. And, finally, we find that

$$T_n \to \frac{3}{2} - \frac{1}{2} = 1.$$

Incredible! Our calculation has led us to an astounding result: we can heat our cold tea almost to the unit temperature, while the hot water not only becomes colder, its temperature falls pretty close to zero.

This paradox shows that common sense sometimes deceives us, and reality can be more complicated than one would think.

More on the two-jet method

The diagram in figure 3 has two very interesting properties. You can try to prove them on your own.[3]

1. It turns out that there exists an explicit formula for t_{ij}:

$$t_{ij} = \frac{1}{2^{i+j-1}} \sum_{k=0}^{i-1} \binom{i+j-1}{k},$$

where $\binom{n}{k}$ is the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This expression opens another approach to the proof that $a_1 = 1/2$, but it involves a rather complicated calculation using the so-called *Stirling formula*:

$$n! \cong \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

2. There exists a remarkable probabilistic interpretation of the numbers t_{ij}. Imagine a person who walks randomly through our diagram, moving only upwards or to the left (against the arrows)

[3]Or you can look into "Counting Random Paths" on page 39.—*Ed.*

with equal probabilities. Then t_{ij} equals the probability that, starting at the point with coordinates (i, j), the walker will leave the diagram through its left edge rather than through the top.

Another remarkable arithmetic property of our diagram emerges when the two volumes of liquid are divided into unequal portions. Assume that we have divided the hot water into portions of masses $p_1, p_2, \ldots, p_n$, and the cold tea into portions of masses $q_1, q_2, \ldots, q_m$, and that we have carried out the process described in the previous section. We can certainly construct a similar array of the numbers t_{ij} defined by the relations

$$t_{ij} = \frac{1}{p_i + q_j}\left(p_i t_{i,j-1} + q_j t_{i-1,j}\right).$$

This formula for the equilibrium temperature of the portions p_i and q_j after their heat exchange follows from the law of conservation of energy (in our particular situation, energy equals the product of mass, temperature, and a factor that depends only on the specific properties of water). An example is shown in figure 4. In fact, we can forget the physical sense of what's going on and study the situation from the purely mathematical point of view.

$$\begin{array}{c|ccccc}
 & & & p_1 = 1/3 & & p_2 = 2/3 \\
\hline
 & & & 0 & & 0 \\
 & & & \Downarrow & & \Downarrow \\
q_1 = 3/4 & 1 & \to & 9/13 & \to & 81/221 \\
 & & & \Downarrow & & \Downarrow \\
q_2 = 1/4 & 1 & \to & 75/91 & \to & 8361/17017
\end{array}$$

Figure 4

Let $H(p_1, \ldots, p_n; q_1, \ldots, q_m)$ be the full energy of the "tea" after the end of the process; we can assume that the units of measurement are chosen so that H is simply the average temperature of the tea. Then, for the example in figure 4,

$$H(p_1, \ldots, p_n; q_1, \ldots, q_m) = \frac{75}{91} \cdot \frac{1}{3} + \frac{8361}{17017} \cdot \frac{2}{3} \cong 0.602.$$

It can be proven (though the verification is not obvious) that this function is *semisymmetric*—that is, its value doesn't depend on the order of p_i, nor on the order of q_j! In other words, *the final result of the heat exchange process is the same, no matter what order of the portions in both jets is chosen!*

I'll leave you with some exercises to mull over.

Exercises

1. Try to show that (1) $H(p_{i_1}, \ldots, p_{i_n}; q_{j_1}, \ldots, q_{j_m}) = H(p_1, \ldots, p_n; q_1, \ldots, q_m)$, where $p_{i_1}, \ldots, p_{i_n}$ is any permutation of $\{p_i\}$, $q_{j_1}, \ldots, q_{j_m}$ is any permutation of $\{q_j\}$; (2) $H(q_1, \ldots, q_m; p_1, \ldots, p_n) = H(p_1, \ldots, p_n; q_1, \ldots, q_m)$.

2. Try to prove the invariance of the final result of heat exchanges using more physics—for example, the notion of entropy.

The omnipresent and omnipotent neutrino

This particular lightweight just may be the champion of the universe

by Chris Waltham

THE MOST REMARKABLE member of our inventory of elementary particles is the neutrino. It's probably the most numerous particle in the universe (together with the photon) and has a density of several hundred per cubic centimeter *everywhere*, including the interior of the Earth, the Sun, and your own body. This can be compared to the average density of the universe, which is inferred to be the equivalent of one hydrogen atom in every 100,000 cm^3. Even though neutrinos are so numerous, we have only "seen" about a million of them in the 35 years since they were discovered. By way of comparison, we see that many photons by looking at the bright star Sirius for a few seconds with the naked eye.

The neutrino is an extremely light particle. If it has a mass at all, it's less than the smallest mass we're capable of measuring, which is about a factor of 100,000 less than the mass of the electron.[1] And yet the neutrino may dominate the mass of the universe.

The neutrino is extremely small. It may even exist as a mathematical point. If it has any extent at all, we can't see it on the smallest distance scale we can observe, which is of the order of 10^{-18} m. And yet the neutrino may dominate the structure of the universe.

Since they have no charge, neutrinos interact with matter only very weakly. It's possible for them to pass through a light-year of lead. Yet the neutrino participates in the universe's most energetic and violent events, from the day-to-day energy production in our Sun to exploding supernovas (and potentially apocalyptic devices on Earth like nuclear reactors and bombs).

How can this be? How can we infer such great things about an obscure and apparently ineffectual particle? In this article I'll sketch how we came to believe all this, and I'll describe an ambitious experiment to clarify our picture of the neutrino and its place in the universe.

A brief chronology

1914: James Chadwick, an Englishman working in Berlin, found hints of missing energy in some radioactive decays—the so-called beta-decays. Many heavy atomic nuclei were known to be unstable, and some decayed by the emission of an electron (then called a beta-ray). If this was the whole story, the electron would move in the direction opposite to the recoiling nucleus, and each particle would have a well-defined kinetic energy. However, Chadwick showed this was not the case. The electron energy was variable, even though the decaying nuclei were identical.

1930: Wolfgang Pauli, an Austrian working in Zurich, reasoned that Chadwick's observations were due to kinetic energy being shared randomly between the decay products. This energy sharing can be random only if there are more than two decay products. So there must be a third—unseen—particle in addition to the electron and the recoiling nucleus. He postulated the existence of a weakly interacting *neutrino* and its antiparticle, the *antineutrino* (denoted by the Greek letters ν and $\bar{\nu}$, respectively), which would carry away energy without being observed. This was a radical idea—the great Neils Bohr was initially more inclined to abandon the law of energy conservation than to accept the "invention" of a new particle.

Art by Sergey Ivanov

[1]The electron is the lightest known particle, with the exception of the photon (which is believed from indirect arguments to be absolutely massless).


p + p → d + e⁺ + ν + energy

1933: Enrico Fermi, in Rome, produced a detailed mathematical description of the interactions involving neutrinos—the so-called weak interactions. This description has survived, with a few modifications, to the present day. He was also responsible for naming the neutrino—it means "little neutral one" in Italian.

1938: Hans Bethe, a German émigré at Cornell University, developed the first working model for the energy generation in the Sun. The fundamental reaction is the fusion of two protons (p) to form a deuteron (d), a positron (e^+), and a neutrino (ν), all of which carry kinetic energy. The proton is the nucleus of hydrogen, the deuteron is the nucleus of "heavy hydrogen" (a proton and neutron bound together), and the positron is a positively charged electron (the normal electron's antiparticle). This reaction can be written

$$p + p \rightarrow d + e^+ + \nu + \text{energy}.$$

For this and further contributions to the field of nuclear physics, Bethe was awarded the 1967 Nobel Prize for physics.

1956: Two physicists from the Los Alamos National Laboratory in New Mexico, Cowan and Reines, observed neutrino interactions for the first time at the Savannah River reactor in Georgia. The flux of antineutrinos from beta-decays of fission products in a reactor is huge—typically 10^{13} cm^{-2} s^{-1}. But even so, the detector had to be the size of a small room to see just a handful of neutrino interactions in many weeks of observations.

1962: Leon Lederman, Mel Schwartz, and Jack Steinberger, working at the Brookhaven National Laboratory on Long Island, showed that neutrinos can be produced by particle accelerators and that there are at least two kinds: ν_e and ν_μ. One kind (ν_e) can be seen as the chargeless relative of the electron, the other (ν_μ) as the relative of the muon (μ). The muon was first detected in cosmic rays (which at the Earth's surface are mostly muons), and it behaves exactly like the electron except that it's 207 times heavier. Nobody knows why it exists or understands the relationship between electrons, muons, and their neutrinos. However, for uncovering this mystery, Lederman, Schwartz, and Steinberger received the 1988 Nobel Prize for physics.

1975: Martin Perl and his coworkers at Stanford found a third electronlike particle, the heavy tau (τ). There is also evidence (missing energy in tau-decays, in the same manner as in Chadwick's experiment) that it has a neutrino partner, the ν_τ. In fact, all known particles come in groups of three. Nature is obviously trying to tell us something, but as yet no one has figured out what.

1980s: More and more elaborate attempts to weigh the three neutrino types continued (and still continue) to fail to detect any mass at all. The ν_e must therefore have a mass of less than 1/100,000 that of the electron, which is the limit of our sensitivity. For the other neutrinos we know less—the ν_τ mass could be anywhere from zero to 40 times the mass of the electron.

By now the "big bang" theory of creation had made sense of many observations of the universe as a whole. It also predicted that the universe is filled everywhere with a sea of totally invisible neutrinos. These neutrinos would have no effect on anything except, if they have any mass at all, via the gravitational interaction. There are so many of them that they could determine the gross structure of the universe. They would be a type of "dark matter," totally invisible but able to slow up the expansion of the universe by its gravitational attraction.

"Seeing" neutrinos

Neutrinos are now routinely made and observed in reactors and accelerator experiments, and detecting them from astronomical bodies is becoming possible. In each case, enormous numbers are made, but the chance of one interacting with an atom in a detector is tiny. So the detectors have to be huge—the largest are several thousand tons (that is, on the order of 10^{33} atoms). A way of seeing the interactions directly is by "encouraging" the part of the atom struck by the neutrino to give off a few photons of light. These can readily be detected by photomultiplier tubes (PMTs) placed around the detector. The PMT is like a light bulb in reverse: it converts light into electricity. One photon striking a PMT produces a small pulse of electricity (typically a few millivolts for several nanoseconds), which is easily recordable. There are also indirect ways of detecting neutrinos, but I don't have the space to describe them here.

A conceptually simple type of detector is a large tank of water surrounded by thousands of PMTs. When a neutrino strikes an electron, the electron recoils faster than the local speed of light ($0.75c$ in water). This electron then emits the light equivalent of a sonic boom in the form of a conical shower of blue and ultraviolet photons, named Cherenkov light after their Russian discoverer (who won the Nobel Prize in 1958). These photons are detected by the PMTs, and thus we "see" the neutrino. In practice these neutrino detectors are often placed deep underground. This is to provide a heavy rock shield to block cosmic-ray muons, which can cause large flashes of light much more frequently than the faint ones from neutrinos.

Solar neutrinos

The only part of the Sun we can observe directly is the surface. It appears to be a sphere of radius 700,000 km, radiating at a surface temperature of 6,000 K. Yet in the only plausible model of the Sun we have (based on Bethe's work), the energy production has to be deep inside the core, hidden from direct view. One of the few testable predictions of this model is the flux of neutrinos—the neutrinos can leave the core of the Sun without any scattering. By contrast, the heat and light generated in the core make a

10,000-year random walk to the surface.

We expect a huge flux of low-energy neutrinos and a smaller flux of neutrinos with higher energies. The only problem is that two experiments designed to see the high-energy neutrinos (a water-Cherenkov detector in Japan and one of another type in South Dakota) see only a small fraction of what's expected (about a half and a third, respectively). And two experiments sensitive to the lower energy neutrinos (called *pp*—see problem 1 below) are also reporting numbers lower than expected.

This state of affairs has come to be known as the solar neutrino problem. Solar theorists claim to understand the Sun so well from so many different types of observation (surface temperature and composition, oscillations—yes, the Sun wobbles very informatively) that bending their models to suit the low neutrino flux will destroy good agreement in another area.

So, some particle physicists have tried to understand the situation in terms of a model where the different types of neutrino have different masses. These masses would have to be so tiny as to be unmeasurable in the lab. If this model is correct, interference would arise in which some ν_e would change to ν_μ or ν_τ as they traveled toward our detectors. Existing detectors are not very sensitive to neutrinos other than ν_e, so the changed neutrinos would go largely unrecorded. This is the MSW effect, named after the two Russians and an American who dreamed it up: Stanislav Mikheyev, Alexey Smirnov, and Lincoln Wolfenstein.

Even a tiny neutrino mass may be of cosmological significance when multiplied by the expected neutrino density throughout the entire universe. Needless to say, the cosmologists, in their perennial search for their Holy Grail of dark matter, are watching this situation closely. This may be a way of taking the measure of that sea of unobservable neutrinos.

Supernova 1987A

It's rather bold of us to infer what is going on inside the Sun. So it's even more audacious to claim we know how a star *explodes*. Yet by the end of 1986, astrophysicists were claiming that if a star of a certain type exploded at a certain distance, they could tell us how many neutrinos would be observed in existing detectors. In the immense pressures caused by the collapse of the stellar core, hydrogen atoms get squeezed so tightly that the electron and proton coalesce into a neutron—and a neutrino, which rapidly escapes. In fact neutrinos—and as

yet unobserved gravity waves—are the only things that can get out during the initial collapse. Because you get one neutrino per atom in the entire core, the number of neutrinos emitted is immense, on the order of 10^{57} for an average supernova. After the core collapses to an enormous density, it bounces back, causing an outgoing shock wave that blows the star apart. This is what we see optically, and during the fireworks more neutrinos are thought to be released.

On February 23, 1987, a supernova was observed visually in the Large Magellanic Cloud (170,000 light-years away) by Ian Shelton of the University of Toronto. In two large water detectors (both of which at that time were designed for other purposes) in mines in Ohio and Japan, twenty tracks appeared within the space of a few seconds, all pointing back to the Large Magellanic Cloud. This, as you will see from the second problem, was almost precisely the expected number. Thus, the new science of neutrino astronomy was born.

The Sudbury Neutrino Observatory

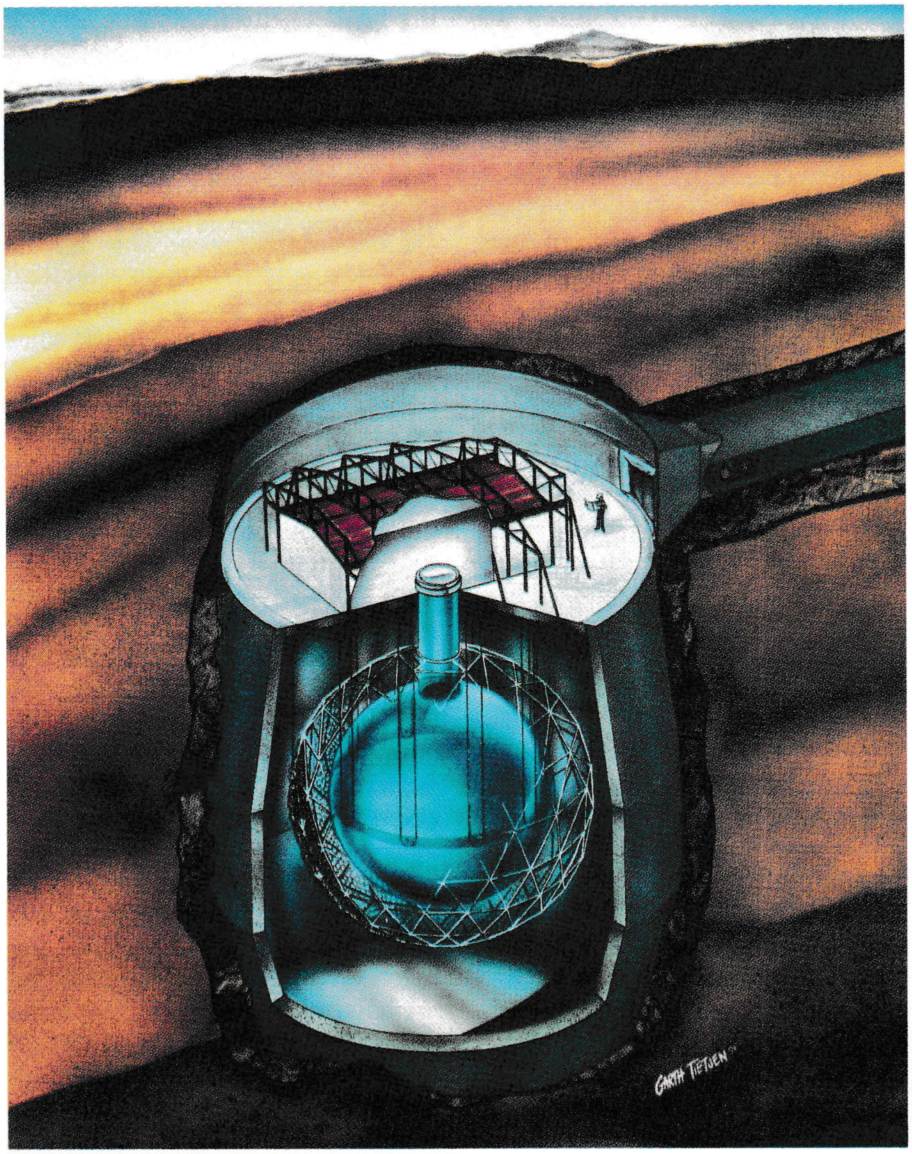
A new science needs a facility dedicated to it. The first job of this new science is to clear up the solar neutrino problem, which means designing a detector that will have a big event rate so the results are statistically significant, and one that can distinguish between neutrino types so the MSW hypothesis can be proven right or wrong. The second job will be to observe any astronomical event that generates neutrinos.

Water makes a fine detector, but it has been known for a long time that heavy water works even better. (Heavy water is chemically like ordinary water except that the hydrogen nuclei are replaced by deuterons. It therefore has a molecular weight of 20 rather than 18 and so is 10% more dense.) This occurs because the neutrinos interact with the neutron in the heavy hydrogen nucleus, and the probability of this is 100 times greater than interactions with the electrons. It also interacts with all types of neutrino in such a way that the types can be distinguished. A gift to neutrino physicists? Maybe, but at $300 a liter, the necessary 1,000 tons looked hopelessly expensive.

Now, it just so happens that the nuclear power industry in Canada has a large reserve of heavy water.[2] Back in 1984 some deft negotiations by George Ewan, a physicist at Queen's University in Ontario, and the late Herb Chen at the University of California, Irvine, ensured not only the long-term loan of 1,000 tons of heavy water but also a 2-kilometer-deep mine shaft in Sudbury, Ontario, in which to place it.

Nine years later, the $60 million Sudbury Neutrino Observatory (SNO), designed by 55 physicists

[2]Canada's CANDU reactors use heavy water as a neutron moderator.



SNO Information Office, Laurentian University, Sudbury, Ontario

Figure 1
The Sudbury Neutrino Observatory (artist's rendering).

(and innumerable engineers, technicians, and students) from Canada, the United States, and Britain,[3] is two years away from taking data. Figure 1 shows an artist's rendering of what SNO will look like underground. The central sphere (blue) is made of 5-centimeter-thick acrylic and will hold the 1,000 tons of heavy water. Outside this is 6,000 tons of light water for support and a shield against natural radioactivity from the rock. In the light water are ten thousand 20-centimeter-diameter PMTs (shown in yellow) facing inward. The cavity is 30 m high and 22 m across.

Figure 2 shows a video display from the SNO simulation program, which replicates what we expect from the actual detector. In the bottom left is a picture of a neutrino event. The PMTs are mounted on the 17-meter-diameter geodesic frame shown in figure 2. Those registering a photon "hit" are picked out in yellow. The hits form a ragged circle as the Cherenkov cone, scattered by the intervening water, intersects the sphere of PMTs. The number of PMT hits gives the neutrino energy, and the position of the circle indicates the direction of its origin.

When SNO finally stops taking data in August 2000, according to the current schedule, we expect to have

- Solved the solar neutrino problem;
- Measured the neutrino's mass and confirmed or disproved the MSW hypothesis (adding fuel to the "missing mass" debate while learning something of the relationship between the triplets of fundamental particles);
- Gained a picture of the nuclear processes in the core of the Sun (taking the temperature of the Sun's core, in effect).

Note that we have deliberately called our detector an "observatory." We expect to see something unexpected. Don't worry, we won't have wrapped everything up. There will still be loose ends to guide you, the next generation of scientists, into labyrinths of untold mystery . . .

In token of which I leave you with two problems to work through.

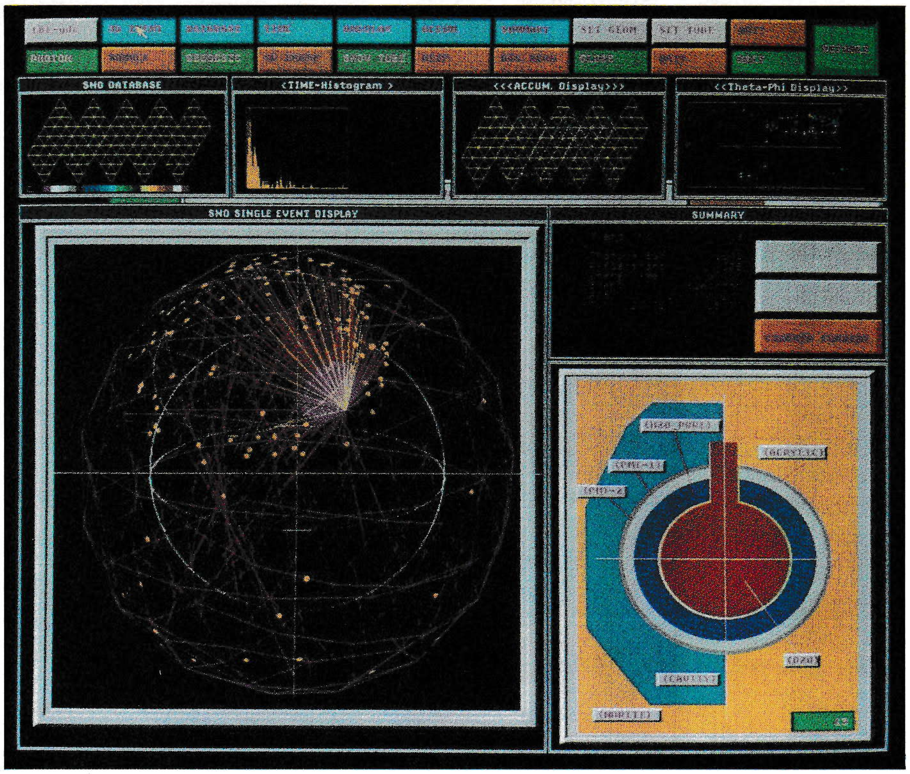
Problem 1

Solar neutrinos in your body. The primary energy- and neutrino-producing reaction in the Sun is hydrogen fusion:

$$4\,p \rightarrow 1\ \mathrm{He} + 2\,e^{+} + 2\,\nu_{e} + \text{energy}.$$

Note that this is just double the reaction I gave above. "He" is the nucleus of the helium atom containing 2 protons and 2 neutrons. On average, the helium and the 2 e^{+} carry away 26.3 MeV of kinetic energy, which is responsible for heating the Sun (and us). The neutrinos carry away 0.2 MeV of energy each, on average, but they interact so infrequently with matter that they

[3]The SNO Collaboration consists of Queen's University, Kingston, Ontario; Chalk River Nuclear Laboratories, Chalk River, Ontario; the University of Guelph, Guelph, Ontario; Laurentian University, Sudbury, Ontario; Carleton University, Ottawa, Ontario; the University of British Columbia, Vancouver, BC; Princeton University, Princeton, NJ; the University of Pennsylvania, Philadelphia, PA; Los Alamos National Laboratory, Los Alamos, NM; the Lawrence Berkeley Laboratory, Berkeley, CA; and Oxford University, Oxford, UK.



Yuendat Chan, SNO group, Lawrence Berkeley Laboratory

Figure 2
Video display from SNO simulation program.

leave the Sun without contributing anything to its heat. The solar constant—the Sun's power per unit area at the top of the Earth's atmosphere—is 1,377 W/m^2. An MeV—a million electron volts—is the standard unit of energy in subatomic physics and is equal to $1.6 \cdot 10^{-13}$ J.

(a) What is the flux of these *pp* neutrinos at the Earth's surface? Don't be put off by the enormity of your answer! (b) Estimate how many neutrinos are in your body at any given time. You can assume that the neutrinos move at the speed of light; if they have a tiny mass, then this assumption is still good enough.

Unfortunately, these neutrinos are too low in energy to be seen by SNO. However, two experiments have been designed to see them. One is operating now in Italy, the other in Russia. These large and complex devices should see 1.2 neutrinos per day. The first sees 0.8 ± 0.2, the other sees even fewer. So—solar neutrinos are still a problem, but let's hope not for long.

Problem 2

Neutrinos from a supernova. When the explosion from Supernova 1987A was observed on Earth in February 1987, it was the first time that neutrinos were seen from such an event. A total of 20 were seen in two underground water detectors in Ohio and Japan, whose combined volume was 4,000 m^3. Now, the chance of any one neutrino being detected is extremely small, because its mean free path (MFP) in water is approximately one light-year. That is to say, the probability of detection equals the path length in the water divided by the mean free path. Given that SN1987A was d = 170,000 light-years from Earth, how many neutrinos were emitted in the explosion?

Hint: if you're not comfortable with mean free paths, mentally reconfigure the detector as a tube whose length is one mean free path, pointing to the supernova, while preserving the volume of the detector. Now any neutrino entering it has, on average, a unit probability of interacting and being seen. This gives the right answer, because in reality the tiny chance of interaction depends only on the number of atoms in the detector, not on its shape.

Suggestions for further reading

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ANSWERS ON PAGE 59

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M86

Number building. You're allowed to perform two operations with numbers: "doubling" and "increasing by 1." If you start from zero, what is the smallest number of operations needed to build up (a) the number 100; (b) an arbitrary positive integer n? (M. Sapir)

M87

Splitting areas and sides. Each of the three midlines of a convex hexagon (the lines that join the midpoints of its opposite sides) divides its area in half. Prove that they meet at one point. (V. Proizvolov)

M88

Roots to logs. For any positive integer n, prove the equality

$$[\sqrt{n}]+[\sqrt[3]{n}]+\ldots+[\sqrt[n]{n}]$$
$$=[\log_2 n]+[\log_3 n]+\ldots+[\log_n n],$$

where $[a]$ is the largest integer not exceeding a. (V. Kisil)

M89

Zeros everywhere. Several points are plotted on the plane and a number is written near every point. For any line that passes through two or more of the given points, the sum of numbers written along this line is zero. Prove that if the points do not lie on one line, then all the numbers are equal to zero. (F. Vainshtein)

M90

Regulated shuffling. A stack of $2n + 1$ cards can be rearranged in two ways: by operation A, in which a portion of the stack can be taken from the top and inserted underneath the rest of the stack without changing the order of cards; or by operation B, in which the top n cards can be inserted (in the same order) into the n spaces between the bottom $n + 1$ cards. Prove that, following these rules, one can obtain no more than $2n(2n + 1)$ different rearrangements of the given stack. (D. Fomin)

Physics

P86

Parabolic ride—sideways. A smooth wire is curved in the horizontal plane in the form of a parabola $Y = AX^2$. A bead of mass M slides along the wire (fig. 1). What force does the bead exert on the wire as it passes the vertex of the parabola with a velocity V_0? (A. Zilberman)

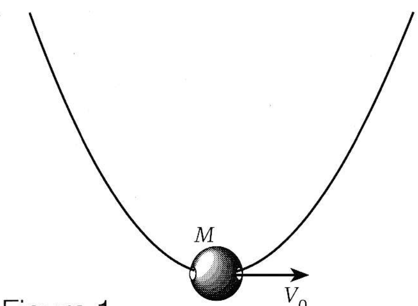

Figure 1

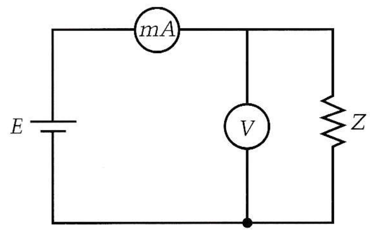

Figure 2

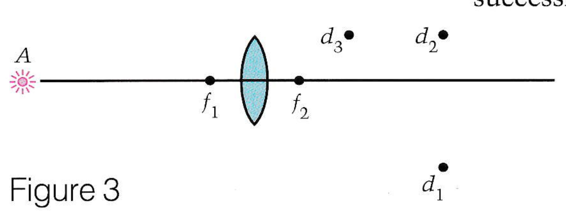

Figure 3

P87

Irresistible. To measure electrical resistance, a circuit is created out of an electric battery, an ammeter, a voltmeter, and a resistor (fig. 2). The voltmeter shows a voltage of 2.9 V and the ammeter shows a current of 3 mA. When we change the circuit by removing the resistor and putting it in parallel with the ammeter, the reading drops to 1 mA. Assuming the battery's internal resistance to be negligible, find the resistance of the resistor. (A. Zilberman)

P88

What's cookin'? When we take a hot pan off the stove, we all use a cloth pot holder. If the pot holder is wet, are we more likely or less likely to get burned? (S. Krotov)

P89

Plates on a sheet. Two metal plates with areas S_1 and S_2 are brought up parallel to a large flat metal sheet. The corresponding distances from the metal sheet are d_1 and d_2. What are the electrical capacitances obtained by connecting any two conductors by a wire? The distance between the two plates is very large, and the distances d_1 and d_2 are much smaller than the sizes of the plates.

P90

"I spy something bright." Figure 3 shows a converging lens with foci at points f_1 and f_2. There is a point light source A on the major optical axis. What will you see when your eye is successively at points d_1, d_2, and d_3? The relative size of the lens is shown in the figure. (A. Zilberman)

ANSWERS, HINTS & SOLUTIONS ON PAGE 56

NEW OPTIONS IN MATHEMATICS

College of Arts and Sciences, Northwestern University

MENU

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Northwestern University announces the formation of an expanded set of programs for selected students interested in mathematics and its related fields. MENU will be offered for the first time to entering students in the fall of 1993. If you have strong intellectual curiosity in this direction and seek a major university and the opportunity to work closely in a small and personalized setting with other students and professors, read on.

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In 1976, Northwestern inaugurated the Integrated Science Program (ISP), which is a successful, one-of-a-kind effort to bring together mathematics and the sciences at the undergraduate level. This was followed by the creation of a parallel program in Mathematical Methods in the Social Sciences (MMSS). Enrollment in these programs is limited to approximately 30 students per year.

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Applications will be accepted by the director of MENU during the Freshman Year from students who are enrolled in Mathematics B90. The specific MENU programs will begin with Sophomore Year. It is expected that MENU students will take some calculus in high school, including the BC and/or AB examinations administered by the CEEB Advanced Placement Program. Other specific questions can be administered by the Director by writing to: Mark Pinsky, Director of MENU, Kresge 324, Northwestern University, Evanston, IL 60208-2206

C A S

* High School students interested in entering ISP or MMSS must complete the relevant application concurrently with the Northwestern application in the last year of high school.

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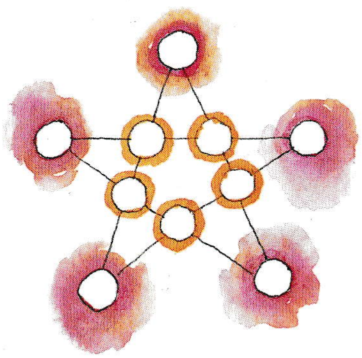
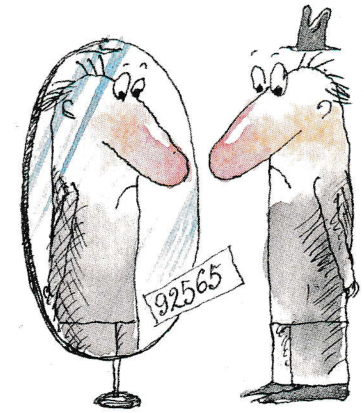


B86

Odd sums on a star. Is it possible to inscribe ten different integers in the circles on the star shown at left so that the sum of the four numbers along each of the five lines is odd? (A. Domashenko)



B87

Mirror numbers. Two numbers are called mirror numbers if one is obtained from the other by reversing the order of digits—for example, 123 and 321. Find two mirror numbers whose product is 92,565. (A. Vasin)

B88

Two scales, one log. The ends of a log are placed on two scales. The first scale shows 200 kg, the second only 100 kg. What is the log's mass? Where is its center of gravity? (V. Vigun)

B89

Seen and unseen. One magazine lies on top of another one as shown in the figure at right. Is the part of the bottom magazine that we see bigger or smaller (in area) than the covered part? (A. Domashenko)

B90

Clever tactics. Prince Ivan made up his mind to fight the three-headed, three-tailed dragon. So he obtained a magic sword that could, in one stroke, chop off either one head, two heads, one tail, or two tails. A witch revealed the dragon's secret to him: if one head is chopped off, a new head grows; in place of one tail, two new tails grow; in place of two tails, one new head grows; and if two heads are chopped off, nothing grows. What is the smallest number of strokes Prince Ivan needs to chop off all the dragon's heads and tails? (V. Rusanov)

Art by Pavel Chernusky

ANSWERS, HINTS & SOLUTIONS ON PAGE 59


O
K
N

Randomly seeking Cipollino

Adventures of a probabilistic detective

by S. Sobolev

THE BOOK *CIPOLLINO AND His Friends* by the Italian writer Gianni Rodari is very popular with little children in Russia. It's about a country inhabited by vegetables. Mr. Carrot, a detective, is searching for the main character, an onion-boy by the name of Cipollino. Mr. Carrot's method takes the form of a rhyme:

One step forward, one step back,
There's no escape for the thieving brat.

This "search principle" could be regarded as the author's little joke if its effectiveness didn't stem from an actual theorem in probability theory. I'll talk about this very theorem and related problems.

Random walk in a line

Let **Z** be the set of all integer points on the number axis. Put a chip at point $a \in \mathbf{Z}$ and toss a coin. If it turns up heads, move the chip a unit distance to the right; if it's tails, move it the same distance to the left. Toss the coin again, and follow the same procedure. If the coin is fair, we always move the chip to the left or to the right with the same probability of 1/2.[1] It's convenient to assume that the process begins at time $t = 0$ and that every step takes a unit time. Then the movement of the chip is called the *symmetric random walk* in the set **Z** of integer points of the line starting at point a. The term "symmetric" reflects the assumption that both directions—to the right and to the left—are equally probable (that is, the coin is fair).

So from the mathematical point of view, Detective Carrot in his search for Cipollino executes the symmetric random walk in the set **Z**.

Exercise 1. Take a coin and a chip. Place the chip at point 0. Toss the coin, move the chip, and write down its location. Repeat this about 10 times. In the end you should obtain a line of consecutive positions of the chip.

A specific walk can be graphically pictured by drawing its path. For every $t = 0, 1, 2, \ldots$, we plot the point M_t with coordinates (t, k), where k is the location of the chip at time t ($k \in \mathbf{Z}$). Joining every M_t, $t = 0, 1, 2, \ldots$, to M_{t+1} with an arrow (fig. 1), we get the graph of the chip's path in tk-coordinates.

For example, if the tossing results in the sequence H, H, T, H, T (H = heads, T = tails) and the initial location of the chip is at the origin, its consecutive locations are as follows:

time	0	1	2	3	4	5
location	0	1	2	1	2	1

The corresponding path is shown in figure 1.

Exercise 2. Draw the path of the chip from exercise 1.

The probability of hitting a given point

Let the walk of our chip begin at the origin. It's interesting and important to find the probability that at time t the chip arrives at location k. In other words, what is the probability that a path with t segments ends at the point (t, k)? Denote this probability by $P_t(k)$. The answer to the question depends upon the number $N_t(k)$ of paths leading from point (0, 0) to point (t, k).

Let's write the number $N_t(k)$ near every point (t, k) that is the endpoint of at least one path for $t = 0, 1, 2, 3$ (fig. 2).

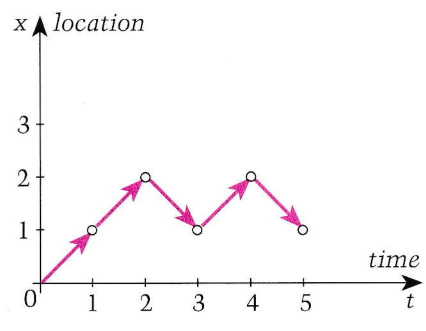


Figure 1

[1]The simplest "common sense" definition of probability—which is, however, enough for you to understand this article—and some elementary basic facts can be found in "Combinatorics–polynomials–probability" (March/April) and "Geometric Probabilities" (May/June). See also "Counting Random Paths" in this issue for the "classical" definition of probability and for more information on random walk.—*Ed.*

Art by Dmitry Krymov

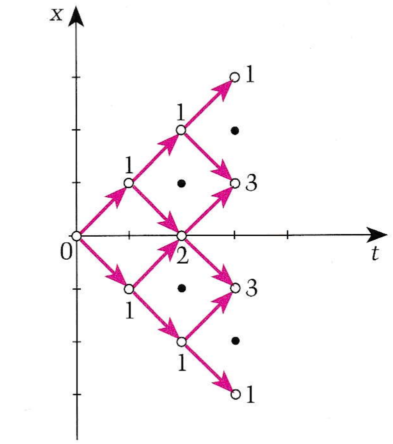


Figure 2

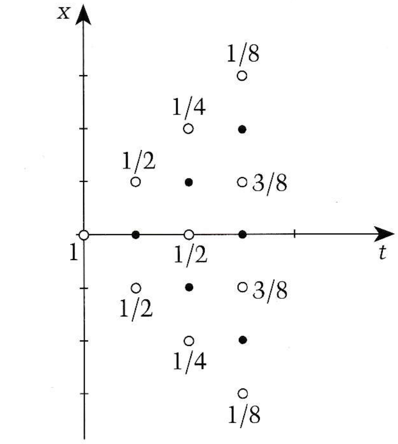


Figure 3

Exercise 3. Extend figure 2 for t = 4, 5, 6.

You see that a path can arrive at point (t, k) only via $(t-1, k-1)$ or $(t-1, k+1)$. Therefore,

$$N_t(k) = N_{t-1}(k-1) + N_{t-1}(k+1). \quad (1)$$

The numbers $N_t(k)$ arranged in this way form the so-called *Pascal's triangle*.[2] It is generated, column after column, by the conditions $N_0(k) = 0$ for $k \neq 0$, $N_0(0) = 1$, and equation (1). For instance,

$$\begin{aligned}
N_1(-1) &= N_0(-2) + N_0(0) = 0 + 1 = 1,\\
N_1(1) &= N_0(0) + N_0(2) = 1 + 0 = 1,\\
N_2(-2) &= N_1(-3) + N_1(-1) = 0 + 1 = 1,\\
N_2(0) &= N_1(-1) + N_1(1) = 1 + 1 = 2,\\
N_2(2) &= N_1(1) + N_1(3) = 1 + 0 = 1,
\end{aligned}$$

and so on.

Of course, there are twice as many t-segment paths as $(t-1)$-segment paths, because every $(t-1)$-segment path generates two t-segment paths. That's why there are two one-segment paths, four two-segment paths, and so on. The number of t-segment paths is 2^t.

Since heads and tails are equally likely to turn up, all the 2^t paths with t segments are also equally probable; and $N_t(k)$ of them end at point (t, k). So the probability $P_t(k)$ that a t-segment path arrives at (t, k) is equal to $N_t(k)/2^t$. Writing the probability $P_t(k)$ near every point (t, k) that lies on at least one path, we get a number array called the *triangle of probabilities*. It is generated, starting with $P_0(0) = 1$, by the *half-sum law*

$$P_t(k) = \frac{1}{2}P_{t-1}(k-1) + \frac{1}{2}P_{t-1}(k+1), \quad (2)$$

which is obtained simply by dividing equation (1) by 2^t (see figure 3).

Exercise 4. Extend the triangle of probabilities to the right for t = 4, 5, 6.

Alternatively, equation (2) can be derived in a more "probabilistic" way. The chip can arrive at point k *either* from $k-1$ or from $k+1$. So $P_t(k)$ is the sum of the probability that it arrives at point $k-1$ at the moment $t-1$ and then heads turns up (so the chip moves to the right, from $k-1$ to k) and the probability that it arrives at $k+1$ at the moment $t-1$ and then tails turns up. Either one is the probability of the intersection of two events: (1) arriving at a certain location and (2) getting a certain side of the coin. Since these two events are independent of each other (the first event depends on the results of the first $t-1$ tosses, whereas the second event depends on the tth toss), we can apply the Multiplication Rule (see, for instance, "Geometric Probabilities" in the last issue), according to which the first probability equals $P_{t-1}(k-1)\cdot \frac{1}{2}$ and the second equals $P_{t-1}(k+1)\cdot \frac{1}{2}$.

The next two exercises are intended for readers familiar with the definition and formula for the number of combinations of n things taken m at a time $\binom{n}{m}$ and with the Binomial Theorem.[3]

Exercise 5. Prove that

$$P_t(k) = \frac{\binom{t}{(t+k)/2}}{2^n}.$$

Exercise 6. Verify that $P_t(k)$ is the coefficient of x^k in the expansion of the polynomial $\frac{1}{2^t}\left(x + \frac{1}{x}\right)^t$ in terms of the powers of x (a) for t = 1, 2, 3, 4; (b) for any positive integer t.

Will Cipollino be captured?

Assume that Cipollino is hiding at the origin of the number axis, while Mr. Carrot is at point k. The detective, in search of Cipollino, executes the symmetric random walk starting at time zero. What is the probability that Mr. Carrot ever finds Cipollino—that is, arrives at the origin?

Denote this probability by $P(k)$. As with any probability, $P(k)$ is a fraction of unity: $0 \leq P(k) \leq 1$. Notice that $P(0) = 1$, because if $k = 0$, then the detective and Cipollino are at the same point from the very beginning, so capture is certain and has a probability of one. If $k \neq 0$, we can apply an argument similar to the second method used to derive equation (2) above. There are two ways for the detective to get to the origin: either (1) make the first move from point k to point $k+1$ and then a series of moves leading to zero, or (2) move to $k-1$ first and then to zero. The probabilities of attaining the origin by starting at $k+1$ and $k-1$ are $P(k+1)$ and $P(k-1)$, respectively. The two possible first moves have the same probability 1/2 and are independent of where the irresolute Mr. Carrot decides to head thereafter. So,

[2]You'll recall this from "Formulas for Cos nx and Sin nx" in the last issue of *Quantum*.—*Ed.*

[3]See, for instance, "Combinatorics–polynomials–probability," where the number of combinations that coincide with a binomial coefficient was denoted by $C(n, m)$ rather than $\binom{n}{m}$.—*Ed.*

using the Multiplication (and the Addition) Rule, we get, for $k \neq 0$,

$$P(k) = \frac{1}{2}P(k-1) + \frac{1}{2}P(k+1). \quad (3)$$

Imagine now that we've plotted the points $M_k(k, P(k))$ on the xy-plane for all integers k. Since the midpoint of the segment joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, every point M_k $(k \neq 0)$ is the midpoint of segment $M_{k-1}M_{k+1}$. Therefore, the points M_k, where $k \geq 0$, all lie on some half-line issuing from $M_0(0, 1)$, and points M_k, where $k \leq 0$, lie on another half-line. And both these half-lines must be part of the strip $0 \leq y \leq 1$, which is shaded in figure 4. This is possible only when together they make one whole line $y = 1$. So all the points M_k lie on this line and $P(k) = 1$ for all k.

Thus, the detective will sooner or later find Cipollino, no matter where he starts his "random search"!

"But what if our coin toss always results in heads, and Mr. Carrot starts from some positive point k?" you may ask. "Won't the detective go off along the line to the right and never find Cipollino?" This is logically possible, of course, but the *probability* of such a thing occurring, or, in general, that the path of the walk never passes through the origin, is equal to zero. And that's exactly what our theorem says. As to the problem of its applicability to reality, that's a subtle matter, and we won't get into it here.

The random walk on the plane and in space

Consider a triangular grid on the plane. Take a chip and a fair die. Label the six directions of the grid lines 1, 2, 3, 4, 5, 6 (fig. 5). Place the chip at one of the grid's nodes and throw the die. Move the chip to the neighboring node according to the number that turns up. Then throw it again and do likewise. This process is called the symmetric random walk in the planar triangular grid.

Imagine that Cipollino is hiding at one of the grid's nodes, and that Mr. Carrot is "randomly walking" around it. Will he find Cipollino? It turns out that in this case the answer is exactly the same as for the line: *the detective catches his victim with a unit probability*. This is also a theorem of probability theory, sometimes worded metaphorically as "all roads lead to Rome." Its proof is somewhat more difficult, and I won't give it here. The same result is valid for the planar square grid (in this case, a "four-way" random generator will be needed—one could use, for instance, a die in the form of a regular tetrahedron).

The result is quite different, however, for the symmetric random walk in the three-dimensional cubic grid (here we can use an ordinary die again, since such a grid has six edges issuing from each vertex). If Cipollino is hiding a sufficiently great distance from Mr. Carrot, the probability of capture becomes insignificantly small. This is a really difficult theorem, and I won't touch on its proof at all.

A few words about applications

Don't be misled by the playful origin of this article. The theory of random walk is an important and very useful part of probability. It's usefully employed in studying various processes in physics, chemistry, and even economics—especially nowadays, when random processes can be so easily simulated on computers. Also, it's effectively applied to the numerical solution of certain differential equations by means of the so-called Monte Carlo method. But that's a topic for another article.

To conclude this one, here are two more exercises.

Exercise 7. Suppose Cipollino is hiding at the origin of the axis, there is a chasm at point N, and Detective Carrot is located at point k, $0 < k < N$. If the detective gets to the origin, he'll find Cipollino, but if he comes to point N first, he'll fall into the chasm (see the illustration at the beginning of the article). What is the probability that, walking in the same random manner as above, Mr. Carrot will capture Cipollino? What is the probability that he'll fall into the chasm first?

Hint: prove the half-sum law (3) for the probability $P(k)$ defined as above, $1 \leq k \leq N-1$, and check the conditions $P(0) = 1$, $P(N) = 0$. What can you say about the arrangement of points $(k, P(k))$? To answer the second question of the problem, switch the initial conditions for $P(k)$.

Exercise 8. Figure 6 shows four possible configurations of twigs tied together. In each case, points A and B are smeared with glue. A caterpillar crawls along the twigs. When it arrives at a point where m twigs are tied together, it either gets stuck (if it's point A or B) or chooses one of the twigs (with the probability $1/m$ each), crawls along it up to the other end, and continues on. What is the probability that the caterpillar, starting at point k ($k = 1, 2, 3$), will stick at point A? at point B? at any one of these points?

ANSWERS, HINTS & SOLUTIONS ON PAGE 60

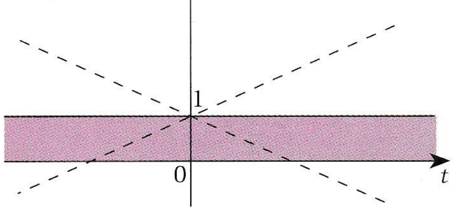


Figure 4

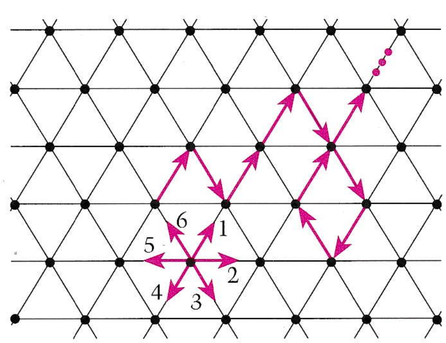


Figure 5

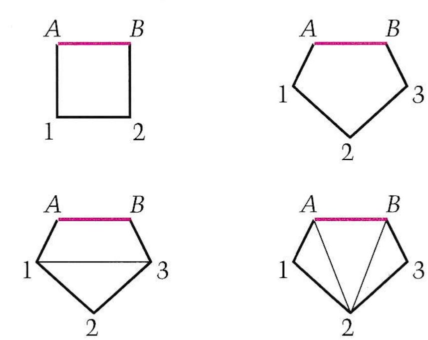


Figure 6

Lunar ironies

"After all, the Moon is usually made in Hamburg, and poorly made at that."
—Nikolay Gogol, "Notes of a Madman"

by M. A. Koretz and Z. L. Ponizovsky

LILLIPUTIAN SCIENTISTS INFORMED GULLIver, as you may recall from Jonathan Swift's account of his travels, that Mars has two satellites. In addition, they told him how far the satellites are from the red planet. It was another hundred years before nonfictional astronomers discovered these satellites and actually confirmed many details of the Lilliputian scholarship. So how did Swift know about Martian satellites?

For a long time this was considered a mere coincidence. Then it became fashionable to attribute such "wonders" to visitors from outer space. Some people believe that extraterrestrials imparted astronomical knowledge to ancient peoples in a kind of code. This information has come down to us in the form of songs, myths, and fairy tales.

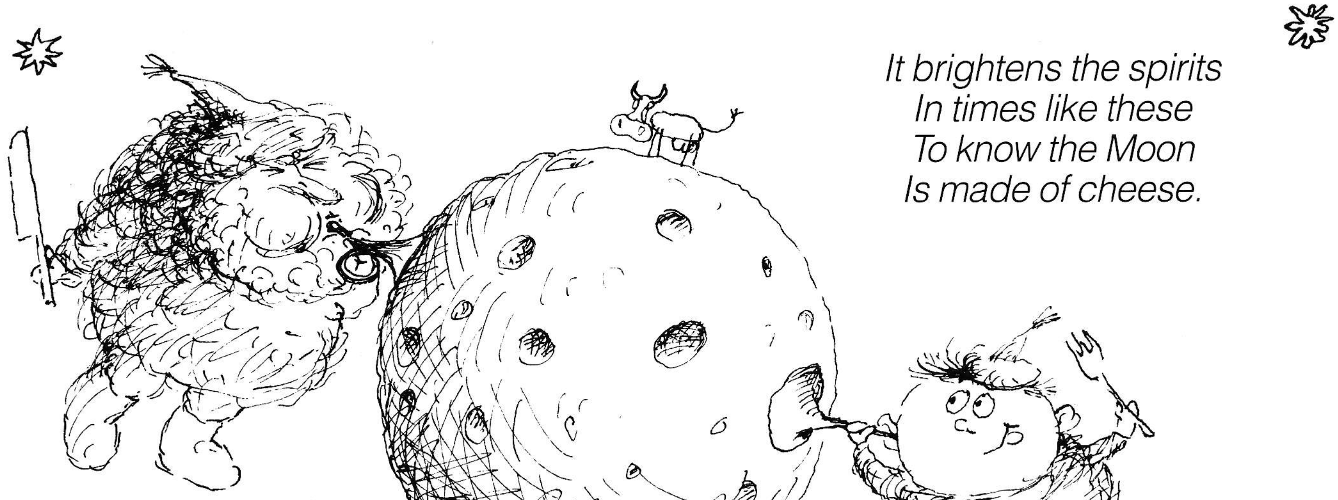
There's a children's rhyme in Russian that goes like this:

> «Почему Луна не из чугуна?»
> «Потому что на Луну не хватило б чугуну.»[1]

> "Why isn't the Moon made of iron?"
> "There's not enough to go around."

No doubt this is one of the bits of information about the Moon that our ancestors were given by aliens. Sooner or

[1]Pronounced something like: Puh-chi-MOO loo-NAH nyi iz choo-goo-NAH? Puh-tah-MOO shtuh na loo-NOO nyi hvah-TYEE-lub choo-goo-NOO.—*Ed.*



It brightens the spirits
In times like these
To know the Moon
Is made of cheese.

Art by Pavel Chernusky

later science was destined to find support for the idea encapsulated in the rhyme.

Sure enough, lunar research has shown that only 15% of the Moon is iron (compared to 35% for the Earth). Also, the fact that the Moon lacks a magnetic field is due either to a complete absence of a molten, iron-enriched core or a very small one. Our current understanding says the Earth's magnetic field is caused by convective motion in the Earth's molten core. Finally, the density of the Earth is 5.5 g/cm^3, while that of the Moon is only 3.34 g/cm^3, which corresponds to the density of the Earth's crust, consisting primarily of silicates. According to the British professor S. Rancorn, we might suppose that 4.5 billion years ago, when the Earth and her satellite were created, the Moon got less iron. Why? Who knows? Maybe the extraterrestrials knew, but the rest of the rhyme was lost in its endless repetition.

So, if the Moon isn't made of iron, what *is* it made of? We need to look elsewhere for the answer. Do you remember the great Baron Munchausen?[2] He discovered an island on Earth made of cheese. This excellent and trustworthy report was published back in 1785. Since then it has been widely conjectured that the Moon consists of the same material. Bitter disputes have arisen, however, concerning the *type* of cheese that constitutes the Moon.

Not so long ago the esteemed American geophysicist O. Anderson of Columbia University was given the job of measuring the speed of sound in lunar rocks and finding analogous terrestrial materials. His colleagues couldn't wait to see his results.

Well, the end of the year rolled around, and professor Anderson sent New Year's greetings to all his friends. He wrote the message you see printed next to the drawing on the facing page: "It brightens the spirits . . ." And below it he provided a table giving the speed of sound in his samples:

Lunar samples and cheeses	Speed of sound (km/s)
Lunar sample 10017	1.84
Norwegian cheese	1.83
Italian cheese	1.75
Italian cheese (Romano)	1.75
Vermont cheese	1.72
Swiss cheese	1.65
Wisconsin cheese	1.57
Lunar sample 10046	1.25

[2]Some of his adventures have been recounted in these pages—see "The Wolf, the Baron, and Isaac Newton" in the November/December 1991 issue.—*Ed.*

As you can see from the table, according to the parameter selected (the speed of sound), the values for various types of cheese coincide with those for the lunar rock samples. So one can reasonably argue that cheese is the terrestrial equivalent of lunar soil. Unfortunately, from the table it's still not clear exactly which type of cheese plays this role. But, fear not, great scientific minds have solved this puzzle.

Rocket boosters falling to the lunar surface have caused seismic oscillations in the Moon's soil. Contrary to expectations, the oscillations didn't damp in a matter of seconds—they took dozens of minutes to die away. It was hypothesized that this phenomenon was due to the existence of some sort of mysterious cavities of unknown origin. But, as you know, there are some sorts of cheese with cavities! Recalculate their dimensions to the appropriate scale and this obscure phenomenon becomes crystal clear. And that's not all. Since the biggest holes are found in Swiss cheese, the question of the type of cheese is resolved as well.

In conclusion, the Moon is not made of iron but of Swiss cheese. So here's our advice to scientists: take another look at all those folk tales and nursery rhymes. Better yet, let a computer handle it.

MATH INVESTIGATIONS

Digitized multiplication à la Steinhaus

The first of his sto zadan

by George Berzsenyi

IN ADDITION TO OVER 150 research papers and numerous other contributions, the Polish mathematician Hugo Steinhaus (1887–1972) will always be remembered for his two wonderful books *Mathematical Snapshots* and *One Hundred Problems in Elementary Mathematics*, which have done wonders in popularizing mathematics among millions of people on every continent. The first of these books (published in Poland in 1938) appeared in English in 1960 and is probably the better known of the two. Hence I want to call my readers' attention to his second book and to the first of his *Sto Zadan* (as the book is called in Polish).

In this problem, Steinhaus starts with two one-digit numbers, 2 and 3; forms their product (6) as the next member of the sequence; then forms the product of 3 and 6 and writes it down as the next two digits (1 and 8). The next term is 6 (since $6 \cdot 1 = 6$); the next term is 8 (since $1 \cdot 8 = 8$); then come 4 and 8 (since $6 \cdot 8 = 48$); and so on. You should check that the first sixteen terms of the sequence are indeed as given below:

2, 3, 6, 1, 8, 6, 8, 4, 8, 4, 8, 3, 2, 3, 2, 3, ...

My first challenge to you is: **Show that the sequence is infinite**. My second challenge is Steinhaus's problem: **Prove that the numbers 5, 7, and 9 never appear in this sequence**. Your third challenge: **Show that among its members, the sequence will contain a string of 77 consecutive 8's**.

Clearly, if one "seeds" the sequence with initial digits other than 2 and 3 (in that order), the nature of the resulting sequence will be different. Just how different will the two sequences be? For instance: **What is the relationship between the sequences *m*, *n*, ... and *n*, *m*, ..., where *m* and *n* are digits?** One might also ask whether the parity of the seeds might influence the outcome.

To explore such questions, it might be helpful to have the computer generate lots of data. Hence our next challenge is: **Write an efficient algorithm to do the job**. One may also explore the problem in bases other than 10 and generalize it by starting with three seeds and taking the products of three consecutive terms to generate the next ones. The possibilities are truly endless! I look forward to your findings and especially the nice problems you generate.

The Worm Problem

The reactions to "Leo Moser's Worm Problem" have been most gratifying, but thus far nobody has managed to make any *proven* improvement on the results published. Special thanks are due to Art Hovey, a physics teacher at Amity Regional High School in Connecticut, who worked very hard on the problem with his students. Several participants of the USA Mathematical Talent Search are also smitten with the problem, as well as some mathematicians at the National Security Agency. Thus, a breakthrough may be in the offing!

The purpose of this column is to direct the attention of *Quantum*'s readers to interesting problems in the literature that deserve to be generalized and could lead to independent research and/or science projects in mathematics. Students who succeed in unraveling the phenomena presented are encouraged to communicate their results to the author either directly or through *Quantum*, which will distribute among them valuable book prizes and/or free subscriptions.

Art by Sergey Ivanov

AT THE BLACKBOARD

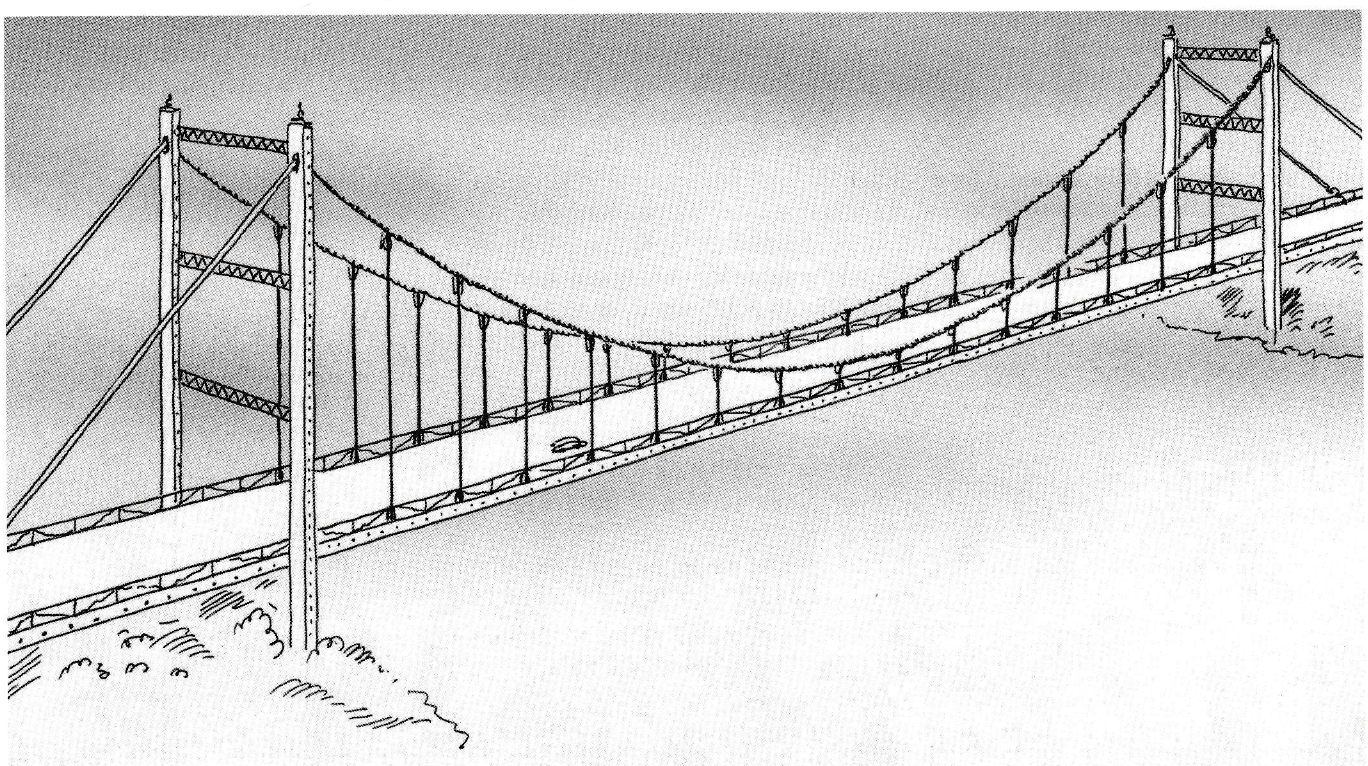
Suspending belief

Name the shape of that incredible curve

by Y. S. Petrov

SUSPENSION BRIDGES . . . IN Russian they're called "hanging bridges," and that's exactly what they are—the roadway seems to hang in midair. These engineering marvels amaze us with the boldness of their underlying idea, the purity and perfection of their lines. Thin steel ropes dropping down from delicately curved, suspended cables somehow hold up many tons of asphalt where trucks, buses, and cars race from one shore to the other. Look at the drawing: what a sense of airiness and lightness! However, a great deal of tension is hidden behind this apparent lightness, tension in all parts of the bridge's suspension. And the majestic immobility of the bridge soaring above the mirrorlike surface of the river is the result of equilibrium among great invisible forces tamed by the designers' intelligence.

Before it emerges as a finished product, constructed out of steel and concrete, a bridge must be planned and painstakingly calculated by a whole team of civil engineers. But the design of any particular bridge has been preceded by the development of the general mathematical theory of such a structure. Many fundamental questions and many particular problems had to be solved. One such question that arises when you look at a suspension bridge is: what must be the curve that defines the shape of the suspension cables such that their equilibrium is ensured? The usual answer is: "Why, it's simply a catenary line. Any flexible, homogeneous, inelastic line whose ends are fixed at two points takes this form.



For instance, a chain or a telephone wire hanging under the influence of gravity looks something like this." But that's the wrong answer! All we need to do is look in any textbook or reference book on bridge building. There we'll find the correct and very simple answer. By the way, in the books where I've found it, the answer is given without any proof, as if it's some kind of dogma. And yet this problem can be solved by using quite elementary methods. We just need to be able to work a little bit with *sliding vectors*. And that's what this little article is about.

The supporting part of a suspension bridge consists of cables and suspension rods. We'll assume that the rods are vertical, located at an equal distance from one another, and are subject to equal loads (see the figure below). We neglect the weight of these rods and that of the cable. Finally, we'll assume that the cable is symmetric relative to some vertical straight line and is absolutely flexible and inelastic. We'll take the vertical plane of the cable to be our coordinate plane. Let the straight line where it intersects the plane of the roadway (assumed to be horizontal) be the x-axis, and let the line of symmetry be the y-axis. Denote the distance between the rods by a and the coordinates of their upper ends M_k by (x_k, y_k) (see the figure). Assume that the tension of the horizontal part of the cable is equal to $\mathbf{T}_0$. Denote the tensions of the next cable portions M_1M_2, M_2M_3, ... by $\mathbf{T}_{12}$, $\mathbf{T}_{23}$, ... and their angles with the horizontal as α_1, α_2, ... The forces $\mathbf{F}_1$, $\mathbf{F}_2$, ... are equal to the load p and act vertically downward on the points M_1, M_2, ... We need to find the values of the ordinates y_k at which the sums of the forces acting on the points M_k are equal to zero.

To eliminate any misunderstanding about the directions of the tension, let's denote the tension of the cable portion M_iM_{i+1} in the direction from M_i to M_{i+1} by $T_{i,\,i+1}$ and the same tension in the opposite direction by $T_{i+1,\,i}$. From considerations of symmetry we need to consider only one half of the cable (say, the one with positive x-coordinates). The point M_1 is in equilibrium under the influence of the three forces $\mathbf{T}_0$, $\mathbf{F}_1$, and $\mathbf{T}_{12}$, from which it follows that

$$\tan\alpha_1 = \frac{|\mathbf{F}_1|}{|\mathbf{T}_0|} = \frac{p}{T_0}.$$

The point M_2 is in equilibrium under the influence of the forces $\mathbf{T}_{21}$, $\mathbf{F}_2$, and $\mathbf{T}_{23}$, from which we get

$$\tan\alpha_2 = \frac{2p}{T_0}.$$

Similarly,

$$\tan\alpha_k = \frac{kp}{T_0}.$$

Now we can calculate the coordinates (x_k, y_k) of the point M_k. The coordinates of point M_1 are $(a/2, b)$, where b is the distance from the horizontal part of the cable to the x-axis. The coordinates of the other points can be calculated sequentially according to the formulas

$$x_k = x_{k-1} + a,$$
$$y_k = y_{k-1} + a \tan\alpha_{k-1}.$$

Therefore,

$$x_k = \frac{a}{2} + (k-1)a,$$

$$y_k = b + \frac{ap}{T_0}\left[1 + 2 + \ldots + (k-1)\right]$$

$$= b + \frac{k(k-1)}{2}\frac{ap}{T_0}.$$

There is an unknown tension T_0 in the expressions for x_k and y_k. It can be found if the point of suspension of the last rod M_n is known. Let h be the height of this point. Then

$$h = b + \frac{n(n-1)}{2}\frac{ap}{T_0},$$

from which we get

$$T_0 = \frac{n(n-1)}{2}\frac{ap}{(h-b)}.$$

If k is eliminated from the equations

$$x = \frac{a}{2} + (k-1)a,$$

$$y = b + \frac{k(k-1)}{2}\frac{ap}{T_0},$$

we find that the suspension points M_k lie on the parabola

$$y = \frac{p}{2aT_0}x^2 + \left(b - \frac{ap}{8T_0}\right).$$

And since that is what we were seeking, I'll stop.

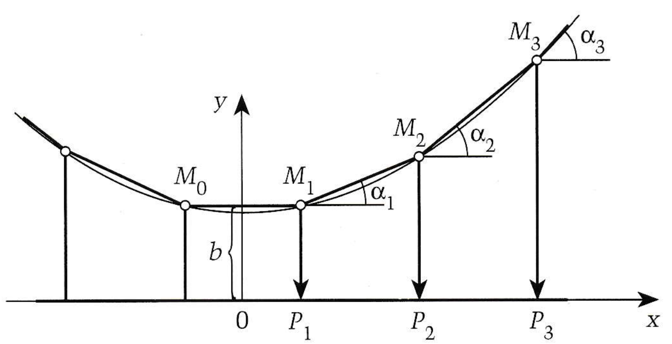


Diagram of the supporting structure of a suspension bridge: the cable is the broken line ...$M_0M_1M_2M_3$..., and the suspension rods are the segments ..., M_1P_1, M_2P_2, M_3P_3, ...

A royal problem

And Alice is caught in the middle

by Martin Gardner and Andy Liu

THE RED QUEEN WAS FURIOUS, as usual. Her current ire was brought on by the absence of the Red King from his Palace. On her rare visits, she expected to see whom she had come to see.

"Bring the old fool back here, or else!" roared the Red Queen, who was related to the Queen of Hearts.

"Or else what?" asked Alice, but only after Her Majesty had swept radiantly out of earshot back to her side of the Palace.

"Off with your head!" Tweedledum said.

"What else?" added Tweedledee rhetorically.

"Oh, dear," said Alice, "this puts a new meaning to ten percent off the top. What shall I do? I don't even know where the Red King is."

The twins brought out a map of the land. It was the familiar 8 × 8 chessboard in figure 1.

"I bet I know where His Majesty is," said Tweedledum.

"On h6!" exclaimed Tweedledee.

"How do you know that?" Alice asked.

"Well," said Tweedledum, "the Red King plays it safe. He never ventures out of his Kingdom into the Borderland."

"He also refuses to cross over to the Queen Side," added Tweedledee.

"So he is confined to twelve squares. That is helpful, but I still don't see how you can be so sure that he is on h6."

"His Majesty likes to be as far away from the Red Queen as he possibly can," Tweedledum said.

"Actually, as far from the Red Queen's Palace as possible," corrected Tweedledee. "He has no control over the whereabouts of Her Majesty."

"There is another problem," said Alice. "If the Red King does not want to come back to e8, how can I persuade him against his wish?"

The twins thought for a while, and fought for a while just to pass the time. Then they both came up with a brilliant idea. Not surprisingly, it was the same idea.

"Are you in mortal fear of the Red Queen?" Tweedledum asked Alice.

"Of course. Who isn't?"

"Of all people, who fears her the most?" asked Tweedledee.

"Hard to say," Alice replied. Then it occurred to her. "The Red King, of course."

"Right!" said Tweedledum. "He could not risk getting caught in a mating situation with the White Queen."

"So if you disguise yourself as that good lady, you can drive His Majesty back here," declared Tweedledee triumphantly.

"It is worth a try," said Alice, somewhat encouraged. "I should not waste any time by venturing outside of those twelve squares either."

"Make sure you don't corner His Majesty on h8," Tweedledum advised Alice.

"Also, do not drive him into the Borderland," said Tweedledee. "His Majesty may find out that it is not as dangerous as he makes it out to be."

"Well, I'd better hurry and bring His Majesty back as soon as I can. The Red Queen's patience is shorter than her temper!"

Problem 1. *On the miniature chessboard in figure 2, White has a lone Queen on e8, and Red has a lone*

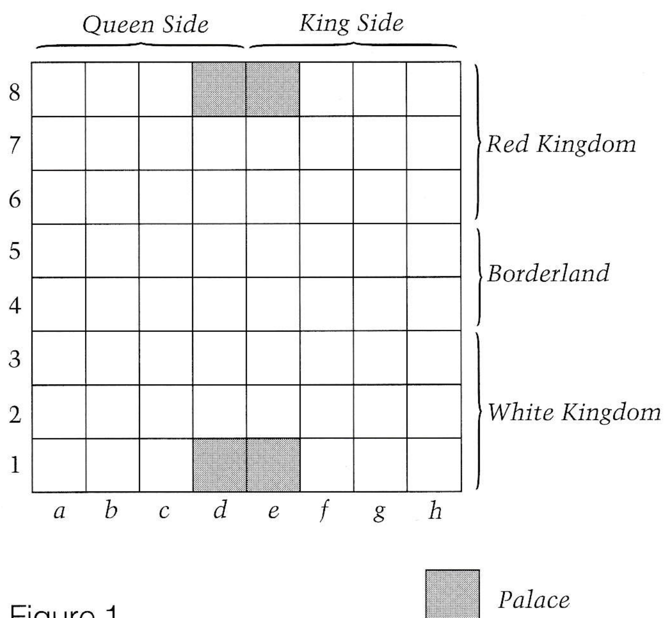


Figure 1

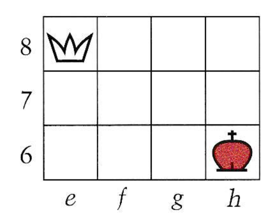


Figure 2

King on h6. White moves first, and wins if the Red King is driven back to e8 within 10 moves. If this is not accomplished, then Red wins. Other than what is noted above, normal chess rules apply. With perfect play, which royalty wins?

Alice was able to accomplish her mission, only to have the Red King slip out again. Humpty Dumpty, in his lofty position on the wall, spotted His Majesty on h4 this time.

Alice correctly deduced that the Red King still harbored no thought of crossing over to the Queen Side. While he had temporarily conquered his fear of the Borderland, he was not yet willing to venture into the White Kingdom.

Having lost much time in accomplishing her first mission, Alice set out immediately to reenact the drama, but on an enlarged stage.

Problem 2. *On the miniature chessboard in figure 3, White has a lone Queen on e8 and Red has a lone King on h4. White moves first, and wins if the Red King is driven back to e8 within 14 moves. If this is not accomplished, then Red wins. Other than what is noted above, normal chess rules apply. With perfect play, which royalty wins?*

Alice drove the Red King back to his Palace just in time.

"Come along," roared the Red Queen. "We have to attend a summit conference with the White Queen and her consort."

"What is the matter this time, dear?" asked the Red King timidly.

"We have been discussing the partition of the Borderland. There is too much goings-on here, especially on h4, or so I hear."

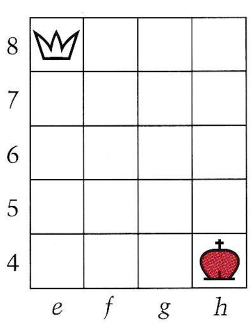


Figure 3

"I can't imagine what," murmured the Red King.

"Anyway, the White Queen and I have agreed to establish our borders between ranks 4 and 5. We just meet to formalize the deal."

"If you say so, dear."

As soon as the new treaty was signed, the Red King headed for h5, the furthest haven within his domain. Alice was dispatched after him a third time.

Problem 3. *On the miniature chessboard in figure 4, White has a lone Queen on e8, and Red has a lone King on h5. Red moves first because the King is already in check. White wins if the Red King is driven back to e8 within 12 moves. If this is not accomplished, then Red wins. Other than what is noted above, normal chess rules apply. With perfect play, which royalty wins?*

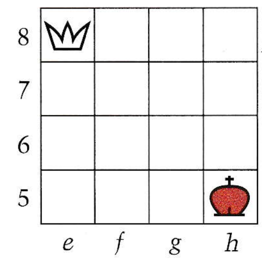


Figure 4

ANSWERS ON PAGE 60

Martin Gardner's *latest books include* Penrose Tiles to Trapdoor Ciphers *and* Fractal Music, Hypercards and More *(both W. H. Freeman).* **Andy Liu** *received the 1993 Rutherford Award for Excellence in Undergraduate Teaching at the University of Alberta.*

Art by Sergey Ivanov

KALEIDOSCOPE

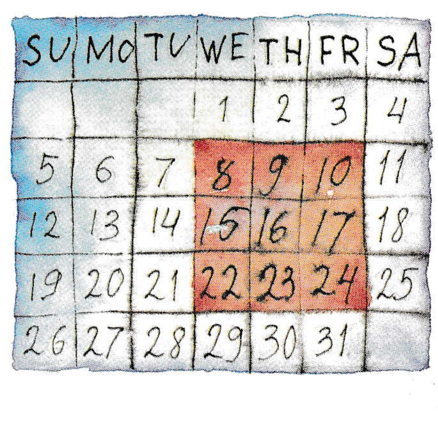
A summer festival of puzzlers

A dozen fun problems from all over the world

Art by Sergey Ivanov

1. *Graceful and amusing.* Three graces, each holding the same quantity of fruit, met nine muses. After each grace had given the same number of pieces of fruit to each muse, all the graces and all the muses had the same amount of fruit. How many pieces of fruit did each grace have before she met the muses if the total number was not greater than 70? (Greece)

2. *Calendar magic.* The 3 × 3 square highlighted in the calendar at right is "almost magic":[1] the sums of

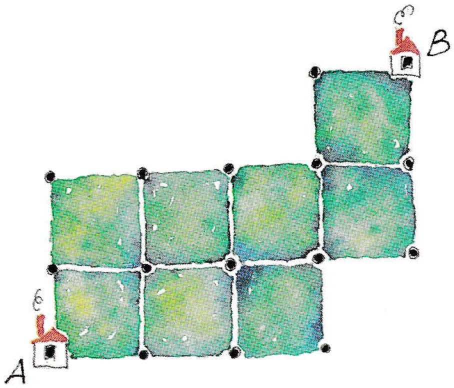


Compiled by Anatoly Savin from national magazines of secondary school mathematics and recreational math.

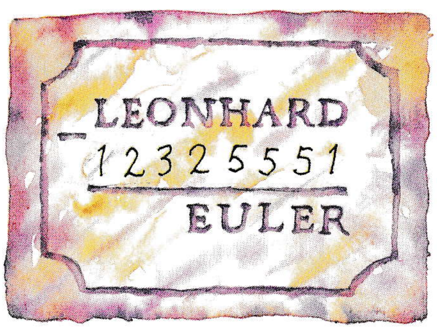
[1]See "Some Mathematical Magic" by John Conway in the March/April 1991 issue of *Quantum* for more on magic squares (and other magical shapes).—*Ed.*

numbers on its diagonals, the middle column, and the middle row are all equal to one another. Prove that this is valid for any 3×3 square selected in any such calendar. (United Kingdom)

3. *Partial withdrawals.* Someone took a fifth of what was kept in a treasury. Another one took a sixth of the remainder and left 150 gold coins in the treasury. How many coins were kept in the treasury initially? (Egypt)



4. *Counting shortest paths.* The figure above shows the map of a village. Janek lives in the house labeled *A*, Marzenka lives in the house labeled *B*. Obviously, to get to Marzenka, Janek must walk at least 7 blocks (all the blocks are the same length). What is the number of such shortest paths from Janek's house to Marzenka's? (Czechoslovakia)



5. *Digital name.* Solve the alphanumeric puzzle above, where EULER is as great as possible. (Germany)

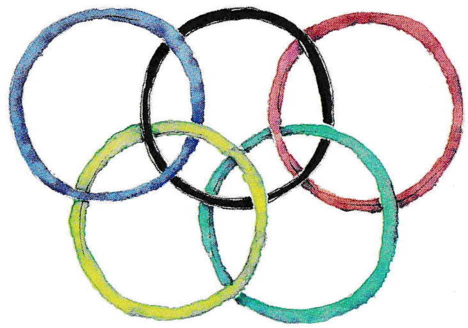
6. *Mondays and Tuesdays.* Once I spent the first Tuesday of a certain month in St. Petersburg and the first Tuesday after the first Monday in Riga. In the next month, I spent the first Tuesday in Pskov and the first Tuesday after the first Monday in Vladimir. What were the dates of my visits to each of these cities? (Russia)

7. *Sea story.* A sailboat, a steamer, and a motor boat were called the *Washington*, the *Lincoln*, and the *Jefferson* (not necessarily in this order). Their ports of departure and arrival, in alphabetical order, were Bermuda, Boston, Halifax, London, Newport, and New York. It's known that (a) the motor boat passed by the ship going to Bermuda; (b) the *Lincoln* arrived at Halifax on the same day as the steamer departed from London; (c) the *Washington* departed from New York under full sail but not for Boston, although one of the ships was headed there. What was the name of each ship, which port did it depart from, and what city did it sail to? (United States)

8. *Perfect repetition.* Find a four-digit perfect square whose first two digits are the same and whose last two digits are the same, too. (Poland)

9. *Prime ages.* There is a family with six children. Five of them are 2, 6, 8, 12, and 14 years older than the youngest child, respectively; the age of each child is a prime number. How old is the youngest child in the family? (Australia)

10. *The chase is on.* A car started out from Varna for Sofia at a speed of 75 km/hour; 20 minutes later another car started along the same route at 90 km/hour. At what distance from Varna will the second car catch up with the first one? (Bulgaria)



11. *Olympic equality.* The five olympic rings cut the plane into 15 pieces (not counting the infinite piece on the outside). Arrange the numbers 1 to 15 in these pieces, one number in each, so that the sum of the numbers inside each ring is 39. (Austria)

12. *120° in the shade.* The sides of a triangle have the lengths $x^2 + x + 1$, $2x + 1$, and $x^2 - 1$. Prove that one of its angles is 120°. (Belgium)

ANSWERS, HINTS & SOLUTIONS ON PAGE 60

Hard-core heavenly bodies

How a hypothetical crystal planet sheds light on the inner nature of all planets

by Yuly Bruk and Albert Stasenko

DID YOU EVER WONDER how the various planets differ? "They have different masses and they're different sizes," you'll say, and you'll be right. In fact, you'll be more right than you even know, because the mass and radius of a planet determine its other characteristics to a great extent. All the planets consist of atoms. What *sort* of atoms? An astrophysicist will say, "All kinds of atoms." But in our solar system, and in the universe as a whole, atoms of the various chemical elements are far from equally represented. For instance, the relative amounts (by mass) of hydrogen, helium, and all the remaining elements are expressed by the ratio 0.73 : 0.25 : 0.02.

The planets of our solar system are likewise made of different elements. The biggest of them, Jupiter and Saturn, whose masses are 318 and 95 times that of the Earth, respectively, consist primarily of hydrogen and helium. However, both the hydrogen and helium in these planets are in the liquid state rather than the gaseous or solid states, and the average densities of these planets surpass by far the densities of the planetary atmospheres.

The planets Uranus and Neptune have masses that are 15 and 17 times that of the Earth, and they consist predominantly of ice, solid methane (CH_4), and ammonia (NH_3) in the metallic phase. Notice how the average atomic numbers of the elements in the planets increase as the planetary mass decreases. Is this an accident? It would seem not, because this tendency holds true as we move down the scale to the smaller planets. The planets of the Earth group (Mercury, Venus, and Mars) are not more massive than the Earth, but the characteristic element for all of them (Earth included) is iron (Fe). They also contain a large amount of silicates—for example, silicon dioxide (SiO_2). The tendency is absolutely clear: the greater the planetary mass, the smaller the average atomic number of the elements in the planet. A rather natural question arises: is there some correlation between planetary mass and the masses of the atoms in the planet?

It would, of course, be wrong to assert that the masses of the atomic nuclei depend on the planetary mass. The atoms of each chemical element are absolutely identical not only in different planets but throughout the universe as a whole. However, a correlation between the masses of the atoms constituting the planets and the masses of the planets themselves certainly exists. It is this correlation that we will discuss in this article.[1]

We'll discuss a very simple model. However, as the Nobel laureate Sir Philip Warren Anderson noted, "quite often a simplified model casts more light on the actual mechanisms of natural phenomena than any number of *ab initio*[2] calculations for a particular event—calculations which, even if correct, often contain so many details that they hide the truth more than they elucidate it."

Surprisingly, the planets of our solar system do not deviate very much from the model presented in this article. Still, we must caution our readers against taking the formulas developed below and rigidly applying them to the actual planets. In our order-of-magnitude estimates we'll be using qualitative considerations and dimensional analysis. We won't worry about the numerical coefficients that arise in more precise calculations. This approach is

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[1]Of course, we're not saying that there are no atoms of hydrogen or uranium in the Earth (for instance). There are such elements in (and on) the Earth, but their relative amounts (by mass) are small.

[2]*Ab initio* means "from the beginning" in Latin; in this context, it means "from first principles."

Art by Sergey Barkhin


MERCURY
PLUTO
ПЛУТОН
MARS
МАРС
SATURN
САТУРН
VENUS
ЗЕМЛЯ
EARTH
НЕПТУН
NEPTUNE
ЮПИТЕР
JUPITER
URANUS
URANUS

justified when the coefficients are of the order of 1. And this is exactly the situation that arises so often in physics and astrophysics (but not always, of course). There are more fundamental reasons for this, but we won't discuss them here. We'll just take it for granted that dimensionless coefficients won't spoil our conclusions (at least qualitatively).

On the way to our primary goal—finding the correlation between a planet's mass and its chemical composition—we'll take a little side trip into solid-state physics. We'll also calculate the energy and Young's modulus of an ionic crystal. These calculations will ultimately help us learn a thing or two about the planets.

Young's modulus and ionic crystal

First off, let's consider a model of an ionic crystal similar to a crystal of table salt (NaCl) except that all the atoms have approximately equal masses. This difference between our model and a crystal of NaCl is not very significant for our analysis, but it will simplify our calculations. We can neglect the mass of the electrons in comparison with the mass of the nuclei.

Let the crystal density be ρ and let the atomic numbers of its elements be $A_1 \cong A_2 \cong A$. The masses of nucleons—that is, the protons and neutrons composing a nucleus—differ insignificantly, and the difference will be neglected here. With these assumptions, the mass of each atom can be considered equal to that of the nucleus: $m \cong Am_p$, where m_p is the mass of the nucleon.

If there are n atoms per unit volume, then the mass per unit volume (that is, the density) is

$$\rho = nm. \tag{1}$$

It's convenient to rewrite this simple formula in another way. For the estimates we're planning to make, we can consider our model crystal to be a cube. This means that the atoms sit at the corners of an elementary cube—the unit cell of the crystal lattice. We'll denote the edge of the cube as a (fig. 1). The magnitude n is linked by its very nature with a: $na^3 = 1$. Therefore, equation (1) can be rewritten as

$$\rho = \frac{m}{a^3}. \tag{2}$$

Formula (2) is curious in that its right-hand side contains the "microscopic" values m and a, and the left-hand side consists of just the "macroscopic" crystal density.

The crystal lattice here is made out of alternating positive and negative ions. For simplicity's sake, the charge of each ion will be considered equal to the electron charge with the corresponding sign ($+e$ or $-e$). The usual electrostatic forces act on each ion. Were there only two ions with a distance a between them, the potential energy of their interaction would be $\sim e^2/\varepsilon_0 a$, where ε_0 is the permittivity of free space and the tilde (~) means that it's an order-of-magnitude estimate. The energy of interaction of the ion pair is a very important and useful characteristic in our estimates. Of course, there are many more than two particles in a crystal. Taking $2 \cdot 10^{-8}$ cm as the mean distance between particles, we can easily calculate that there are $\sim 10^{23}$ particles per cm^3.

Physicists usually discuss the density of electrostatic energy of the ionic system that forms a crystal. The word "density" is used here to mean the energy contained in a unit volume. In other words, this value is the sum of all potential energies of interaction among all pairs of ions in a unit volume. It's rather difficult to calculate this sum precisely, and we won't be able to do it here, because we'd have to take into account the interaction of a huge number of ions at different distances from one another. We can, however, draw an analogy with equation (2). It should be noted that the energy density E has the dimensions J/m^3, and that the dimensions of the potential energy of an interacting pair of ions are $[e^2/\varepsilon_0 a]$ = J. The brackets denote the dimensions of the expression inside them.[3] Now divide the "microscopic" expression $e^2/\varepsilon_0 a$ by a^3, which is also "microscopic." This results in an expression with the same dimensions as the energy density E. We can think of this value as an estimate of E. Of course, this reasoning is not a strict proof that the density of electrostatic energy of the crystal's ionic system is $e^2/\varepsilon_0 a^4$. Nevertheless a precise analysis of the ionic crystal results in the following equation:

$$E = \alpha n \frac{e^2}{\varepsilon_0 a} = \alpha \frac{e^2}{\varepsilon_0 a^4},$$

which differs from the above estimate by the numerical factor $\alpha \sim 1$.

The elastic properties of a substance are determined by the interatomic forces. The most important characteristic of these properties is Young's modulus Y. Usually we obtain this value from Hooke's law as the force at which the linear deformation of a body $\Delta l/l$ is equal to 1, or in other words, when the corresponding length l is doubled or

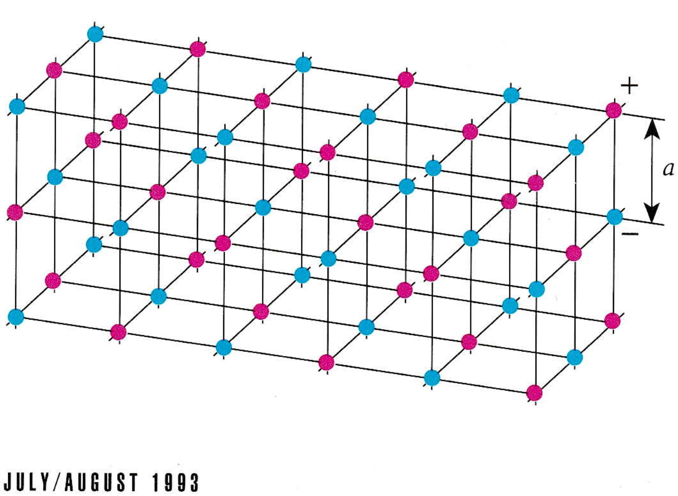

Figure 1

[3]See "The Power of Dimensional Thinking" in the May/June 1992 issue of *Quantum*.—*Ed.*

halved. However, the value Y doesn't at all depend on whether we know Hooke's law or even whether it's correct. We should also note that the dimensionality of the elastic modulus is $N/m^2 = J/m^3$. We can therefore interpret Y as some sort of characteristic energy density.

To make this a little clearer, let's look at two other examples. The first has to do with the common parallel-plate capacitor. If its plates are charged with $\pm q$, then there will be an electrostatic field inside the capacitor, and the plates will attract each other. Let the area of each plate be S and the distance between them h. The attractive force can be calculated, and when divided by S, it results in the "characteristic pressure." It's also possible to calculate the energy accumulated in a capacitor and, dividing it by the volume Sh, determine the energy density. Both estimates result in the value $\sigma^2/2\varepsilon_0$, where $\sigma = q/S$ is the surface charge density on the plates. Thus, the characteristic pressure and the energy density are the same in this case—not only in their dimensionality but numerically as well.

Now it's time to return to the ionic crystal. Its characteristic energy is electrostatic, and its elastic properties are determined by the electrical interactions of the particles forming the crystal. So we can assume $E \sim Y$. Again we take for granted that the proportionality factor here is 1. In this way we've learned how to evaluate Young's modulus for the ionic crystal:

$$Y \sim E \sim \alpha \frac{e^2}{\varepsilon_0 a^4} \cong \left(\frac{\rho}{m}\right)^{4/3} \frac{e^2}{\varepsilon_0} = e^2 m^{-4/3} \rho^{4/3} \varepsilon_0^{-1}. \qquad (4)$$

We used equation (2) to obtain equation (4). From equation (4) it directly follows that E has an upper limit. Indeed, while the ionic lattice exists, the interionic distance cannot in any case be less than the atomic (ionic) size. Were it not so, the electron shells of the adjacent ions would overlap. As a result, the electrons would bunch up, and we'd have a metal instead of an ionic crystal. On the other hand, the value of E for the ionic crystal has a lower limit as well. We'll demonstrate this with another example.

Imagine that a deforming force is applied to a crystal rod. When the force is large enough, the rod breaks. The force at which breakage occurs divided by the rod's cross-sectional area perpendicular to the force is known as the tensile strength p_t, which is always less than Young's modulus. This last statement looks to be true at face value. As we said above, a force equal to Young's modulus causes a twofold change in the length of the sample tested.[4] As we know from experience, it's practically impossible to stretch (or compress) any crystal to twice (or half) its length—it would break long before it reached that point.

Now let p represent the pressure acting on the crystal. Logically, one of the conditions for the existence of a crystal structure is that the following relations hold true:

$$E > p_t > p. \qquad (5)$$

Another obvious condition is that the temperature of the crystal be less than the melting point of the crystal lattice.

Now this question pops up: if Young's modulus is defined as the pressure that changes the length of a rod by a factor of 2, then what about a spherical or cubic crystal deformed from all directions simultaneously? Here it's more convenient to consider the relative change not in the length but in the *volume* of the crystal ($\Delta V/V$). We now reformulate Hooke's law, for small deformations, as

$$\frac{p}{B} = \frac{\Delta V}{V}.$$

[4]We should note that it is generally not possible to use Hooke's law with large deformations, but the qualitative conclusions we make in this article remain valid (even without Hooke's law).

This equation is very similar to the one written for stretching or compressing a rod: $p/Y = \Delta l/l$). But Young's modulus here is replaced with the *bulk modulus B*. The bulk modulus B can also be interpreted as the characteristic energy density, and the values Y, B, and E can be considered equal in their order of magnitude. This is quite sufficient for our purposes in this paper.

An ionic-crystal planet

Now let's move on to our main topic. Consider a hypothetical planet made up of almost identical atoms arranged in a crystal lattice. For the planet to be *entirely* crystalline, the pressure at its center (where it is maximal, of course) must not exceed the value E. The pressure at the center of a planet of mass M and radius R can be calculated by using the following equation:

$$p \sim G\frac{M^2}{R^4}. \qquad (6)$$

This formula can be deduced by dimensional analysis.

We'll remind you how to do this. The pressure p at the center is assumed to depend on the planet's mass M, its radius R, and the gravitational constant G:

$$p \sim G^x M^y R^z,$$

where x, y, z are as yet unknown numbers. Now we write down the dimensions of the physical values in this equation: $[p] = \mathrm{kg \cdot m^{-1} \cdot s^{-2}}$, $[G] = \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$, $[M] = \mathrm{kg}$, $[R] = \mathrm{m}$. Comparing the dimensions on the left and right sides of the equation, we get

$$\mathrm{kg \cdot m^{-1} \cdot s^{-2} = m^{3x} \cdot kg^{-x} \cdot s^{-2x} \cdot kg^{y} \cdot m^{z}}.$$

For this equation to be valid, the numbers x, y, z must satisfy the combined equations

$$1 = -x + y;\ -1 = 3x + z;\ -2 = -2x.$$

From this it follows that $x = 1$, $y =$

2, and $z = -4$, which results in equation (6).

Equation (6) can be understood in yet another way. The gravitational energy of a ball of mass M and radius R must be of the order of GM^2/R. So the density of the gravitational energy is equal to this value divided by the ball's volume ~ R^3. Just as the elastic moduli can be interpreted as the density of electrostatic energy, the gravitational energy can be considered to be of the same order of magnitude as the pressure at the center of a gravitating globe.

Again it should be stressed that the question is not about the identity of pressure and energy density (which would be an false assertion) but about the equality of these quantities in their orders of magnitude.

The condition for the existence of an ionic crystal at the center of our hypothetical planet is as follows:

$$G\frac{M^2}{R^4} < E \sim e^2 m^{-4/3} \rho^{4/3} \varepsilon_0^{-1}. \qquad (7)$$

Obviously a planet that is entirely crystal can exist only if it's relatively cold—in other words, the temperature at its center must not be close to the melting point. Otherwise a liquid core would appear inside the planet—the crystal would melt. Again taking into account that $\rho \sim M/R^3$ and $m \cong Am_p$, equation (7) can be rewritten as

$$A < \left(\frac{e^2}{\varepsilon_0 G}\right)^{3/4} \frac{1}{m_p \sqrt{M}}. \qquad (8)$$

From this it's evident that our assumptions (the planet is *completely* crystal, the density at its center is of the order of the mean density) lead us to limitations on the masses of atoms that can be used to form *these kinds* of planets.

The assumption that the planet's mean density is of the same order of magnitude as the density at its center seems to be absolutely natural and reasonable in those cases where the substance in the planet's core isn't compressed very much. But if the compression is too great, the *ionic* crystal would not exist. If an ionic-crystal planet has the radius and mass of Earth, then the density of matter at its center and near the surface do not differ markedly—the ratio is about 3 : 1. So the mean density is indeed the same *in its order of magnitude* as the density at the planet's center. This holds true for other planets and stars as well.

The restrictions imposed on the mass of the atoms from which completely crystalline planets could be formed are thus determined by the parameters of the planets themselves. For the simplest model of an ionic-crystal planet, we obtained

$$A_{max} \cong \text{constant} \cdot M^{-1/2}.$$

Let's draw a graph of the function $M(A_{max})$. In the strictest sense, this graph relates only to our hypothetical situation in which planets are composed of ionic crystals and have no liquid cores to speak of. Remember what we said at the outset about the elements and compounds that are characteristic of actual planets.

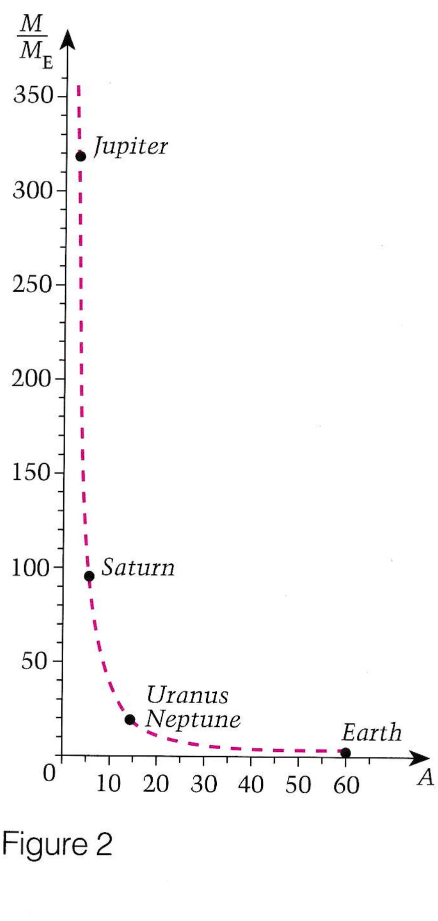

Figure 2

If the planets in our solar system were of the ionic-crystal type, and if we took the average atomic number for the planets of the Earth group to be about 60, for Uranus and Pluto—about 16, and for Jupiter and Neptune—about 2 to 4, then the corresponding data points fall rather nicely on our curve. Along the abscissa we plotted the mean atomic number of the planet's elements (A), and along the ordinate we plotted the mass of the ionic-crystal planet in relative units, the mass of Earth being 1 (see figure 2).

Our calculations do not mean that the *real* planets have no liquid cores. On the contrary, such cores probably do exist. However, crystal structures also exist in these planets. The fact that the real planets are similar to those in our model, at least qualitatively, justifies our assertion that we have indeed captured and understood a law describing the link between a planet's mass and the atomic mass of its predominant substance.

In conclusion, we should add that a similar approach is valid for planets that are not made of ionic crystal but of metal. A substance is *metallic* when free electrons are detached from their atoms. In this case the gravitational compression is said to be "counterbalanced" by the pressure of the electron gas, and it is this balance that makes it possible for such planets to exist.[5]

So you can see once again the "power of dimensional thinking," and how far the human mind casts its net!

[5]The principle of calculation that leads to the link between the masses of the planets and the characteristics of the atoms composing them remains the same, but the calculations are too complex to be considered here. For those who want to do it themselves, we note that the order of magnitude of the electron gas pressure in metals is equal to $\frac{\hbar}{m_e} n_e^{5/3}$, where $\hbar \cong 10^{-34}$ J · s is Planck's constant, $m_e \cong 10^{-30}$ kg is the electron's mass, and n_e is the number of electrons per unit volume.

FOLLOW-UP

Counting random paths

And computing equilibrium temperatures

by Vladimir Dubrovsky

MATHEMATICS IS OFTEN called the "Queen of the Sciences." Perhaps one of the reasons for such high esteem lies in its remarkable power to bring out intrinsic links between phenomena that seem to have nothing to do with each other. A dramatic instance of such an unexpected connection is mentioned in the article "Superheated by Equations" (p. 4), which deals, in particular, with a special kind of serial heat exchange between small portions of two substances. The intermediate equilibrium temperatures that emerge during this process are interpreted in terms of a certain two-dimensional random walk. In this short follow-up article I'll explain why this interpretation is true and, further, I'll adapt it to the *one-dimensional* random walk considered in another article in this issue—"Randomly Seeking Cipollino."[1] Not only that, we'll see that the main results of both articles (I'll refer to them as *SBE* and *RSC*, respectively) can be derived from each other! And in so doing we'll get to know a simple geometric method of counting the paths of a random walk that yields many interesting and important properties.

[1]This random process is the simplest discrete model of Brownian motion (the chaotic motion of particles suspended in some liquid or gas), which is, in a certain sense, governed by the same differential equations as heat conduction. Here the connection between the two processes manifests itself even more vividly and, perhaps, looks more reasonable than in our discrete case.

From probabilities to temperatures

Consider a square grid on the plane (the set of points (i, j) with integer coordinates) and a "particle" that "walks"—or rather, "jumps"—in this grid: from the point (i, j) it can jump to $(i - 1, j)$ or $(i, j - 1)$ with equal probabilities of 1/2, as illustrated in figure 1 (where the axes are directed so as to correspond to the heat-transfer diagrams in *SBE*). Every jump decreases the sum of the coordinates by 1, so, starting from (i, j) with $i \geq 0$, $j \geq 0$, $i + j \geq 1$, the particle will reach one of the axes in no more than $i + j - 1$ jumps. Let's study the probability p_{ij} that it will reach the i-axis first.

As was claimed in *SBE*, the probabilities p_{ij} must be equal to the temperatures t_{ij} that emerge during the heat exchange process described there. Without going into physical details, which would take a lot of space and would be of no purpose here, I'll just note that the equality $p_{ij} = t_{ij}$ can be proven simply by verifying, for all $i \geq 1$, $j \geq 1$, the initial conditions

$$p_{i0} = 1,\ p_{0j} = 0 \qquad (1)$$

and the half-sum law

$$p_{ij} = \frac{1}{2}\left(p_{i-1,j} + p_{i,j-1}\right), \qquad (2)$$

because these relations are true for the temperatures t_{ij} (see *SBE*) and uniquely define the numbers that obey them.

The conditions (1) are obvious—if you start on one of the axes, you reach it sooner than the other one! To prove equation (2), we'll use the "classical"

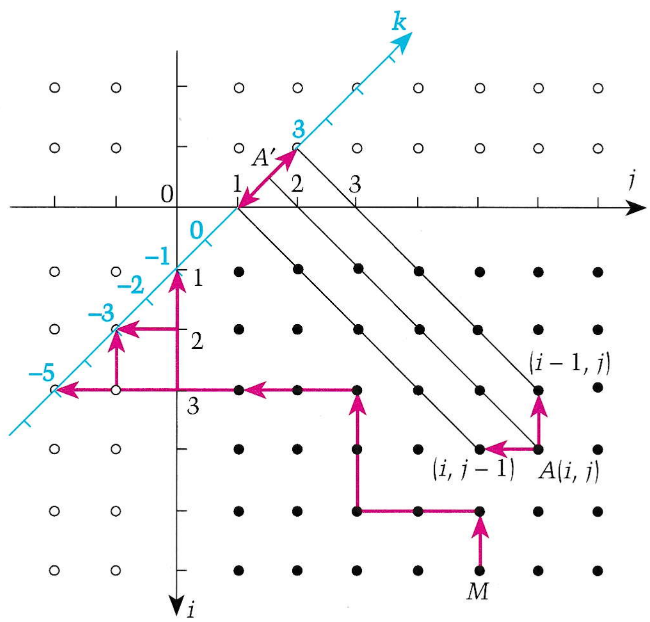

Figure 1

definition of probability: if a random experiment has N equally likely outcomes and exactly M of these outcomes constitute a certain event, then the probability of this event is M/N (see, for instance, "The Symmetry of Chance" in the last issue of *Quantum*). In our case, it's natural to take as "equally likely outcomes" all possible paths of the particle that start at a fixed point (i, j) and have the same length s—that is, consist of s jumps. Let's take $s = i + j - 1$. Then all the paths end up on the line $i + j = 1$ (the blue line in figure 1). Since each path must end on a lattice point, this point must be on or "behind" the axes, and so must cross an axis. Since there are $N = 2^s$ paths of length s, starting at a fixed point (point (i, j)), we have

$$p_{ij} = \frac{N_{ij}}{2^s},$$

where N_{ij} is the number of those that attain the i-axis before the j-axis. (In fact, it's easy to see that these paths are too short to reach the j-axis at all.) Similarly, $p_{i-1,j} = N_{i-1,j}/2^{s-1}$ and $p_{i,j-1} = N_{i,j-1}/2^{s-1}$. Now we notice that all the N_{ij} paths in question fall into two classes: those starting with a jump from (i, j) to $(i - 1, j)$, and those that jump first to $(i, j - 1)$. It follows that $N_{ij} = N_{i-1,j} + N_{i,j-1}$, and so

$$p_{ij} = \frac{1}{2}\frac{N_{i-1,j} + N_{i,j-1}}{2^{s-1}} = \frac{1}{2}\left(p_{i-1,j} + p_{i,j-1}\right).$$

So the first piece of work is done—we've established the link between the temperatures from *SBE* and probabilities. The next step will be to pass from the two-dimensional random walk considered above to the one-dimensional symmetric random walk studied in *RSC*.

From the plane to the line

Let's project the integer grid in the ij-plane onto the blue line $i + j = 1$ supplied with a coordinate scale as shown in figure 1. Let's call this line the "k-axis," although the coordinates along this "axis" are not the same as those along the i- or j-axis. Then an arbitrary node $A(i, j)$ of the grid will be projected onto the point A' with the integer coordinate $k = j - i$ on the blue axis. So the projection of our particle will jump along the integer grid on the k-axis according to the rule described in *RSC:* from any point k it jumps to one of the two neighboring points, $k - 1$ or $k + 1$, with equal probabilities of $1/2$. What does the probability p_{ij} mean in terms of this projection?

It's clear from figure 1 that if a path that starts at a point M with positive coordinates crosses the i-axis before the j-axis, then it meets the k-axis at a point with coordinate $k < 0$, and vice versa. So p_{ij} can be interpreted as the probability that our particle, *starting at point (i, j), arrives in the negative half of the k-axis after $s = i + j - 1$ jumps*, or that its projection, starting at point $k = j - i$, ends up at a point with a negative k-coordinate after s jumps. Now, the probabilities of jumping left or right along the k-axis do not depend on the position of the point we start at. It follows that any path along the k-axis can be shifted along that axis (preserving the direction of each jump) without changing its probability. Let's start our particle at the origin rather than at k. Then the above discussion shows that the probability that the particle will find itself at a point with a coordinate less than $-k$ (that is, less than $i - j$) after s jumps is also equal to p_{ij}. Denoting the k-coordinate of the particle after s jumps by X_s and the probability of an event A under the condition that the walk begins at the origin by $p(A)$, we can write

$$p_{ij} = p(X_s < -k) = p(X_s > k) \qquad (3)$$

(the second equality follows from the symmetry of our random walk).

One immediate consequence of these expressions is the equation $p_{ij} + p_{ji} = 1$, established for the temperatures t_{ij} in *SBE*. Here is a proof.

Remember that $s = i + j - 1$ and $k = j - i$. First we note that the coordinate of a particle starting at the origin always has the same parity as the number of jumps it has made. The expressions for s and k show that these two numbers must have opposite parity. Hence $p(X_s = k) = 0$. Now, $p_{ij} = p(X_s < j - i)$, so we can write

$$\begin{aligned} p_{ij} + p_{ji} &= p(X_s > k) + p(X_s < k) + p(X_s = k) \\ &= p(X_s \neq k) \qquad\qquad + 0 \\ &= 1 - p(X_s = k) \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Another consequence is the explicit formula for p_{ij}, also mentioned in *SBE*. The right side of equation (3) can be written as

$$p(X_s > k) = p(x_s = k + 1) + p(x_s = k + 2) + \ldots + p(X_s = s).$$

Of course, $p(X_s = n) = 0$ for $n > s$. In fact, all the probabilities $p(X_s = k + 2n)$ are also equal to zero, because the numbers s and $k + 2n$ have different parities. According to exercise 5 in *RSC*, $p(X_s = n) = \binom{s}{(n+s)/2}\Big/2^s$ (this probability was denoted by $P_s(n)$ in *RSC*), so

$$p_{ij} = \left[\binom{s}{j} + \binom{s}{j+1} + \ldots + \binom{s}{s}\right]\Big/2^s.$$

However, we won't be using this expression here.

Symmetry at work

My goal in this section is to prove that $p_{n+1,n} \to 1/2$ as $n \to \infty$. This rather technical result—and its proof below—are interesting in three respects. First, it's a pivotal fact in explaining the paradoxical heat transfer discussed in *SBE;* second, it turns out to be closely connected, even equivalent, to the central statement in *RSC* (that the random walk in a line returns to the starting point with probability 1); third, it provides an opportunity to demonstrate a simple but beautiful and useful geometric trick—the so-called "method of reflection."

Applying equation (3), we get

$$p_{n+1,n} = p(X_{2n} < 1) = p(X_{2n} > -1).$$

Since $p(X_{2n} < 1) = p(X_{2n} \le 0)$ and $p(X_{2n} > -1) = p(X_{2n} \ge 0)$, we have

$$\begin{aligned} 2p_{n+1,n} &= p(X_{2n} \le 0) + p(X_{2n} \ge 0) \\ &= p(X_{2n} < 0) + 2p(X_{2n} = 0) + \\ &\quad p(X_{2n} > 0) \\ &= 1 + p(X_{2n} = 0), \end{aligned}$$

or

$$p_{n+1,n} = \frac{1 + u_{2n}}{2}, \qquad (4)$$

where we write u_{2n} for $p(X_{2n} = 0)$.

We'll see below that u_{2n}, the probability of return to the origin after exactly $2n$ steps, is equal to the probability that the particle *does not* return to the origin at any moment of time from 1 through $2n$. But, according to *RSC*, the return takes place (at some moment) with probability 1, so u_{2n} approaches 0 as $n \to \infty$, which means that $p_{n+1,n} \to 1/2$ (see equation (4)).

Let's use the following notation. Let $N_s(m, k)$ be the number of paths of length s that start at point m and end at point k. Let $Z_s(m, k)$ be the number of paths of length s that start at m, end at k, and in addition pass through the origin. In particular, a path counted by $Z_s(0, k)$ touches the origin at least twice: once at the beginning of the path and once after s moves. Certainly $N_s(m, k) = N_s(k, m)$, and we will use this relationship freely in what follows. Similarly, $Z_s(m, k) = Z_s(k, m)$.

It's not hard to see that $N_s(m, k)$ depends (for a fixed value of s) only on the distance between m and k and not on their positions on the line. That is, $N_s(m, k) = N_s(m + a, k + a)$ for any integer a (and similarly with $Z_s(m, k)$). In particular,

$$N_s(m, k) = N_s(0, k - m)$$

and

$$N_s(0, k) = N_s(0, -k).$$

Now the paths that start at m can be divided into those that make their first jump to $m + 1$ and those that jump first to $m - 1$. It follows that

$$\begin{aligned} N_s(m, k) &= N_{s-1}(m+1, k) + N_{s-1}(m-1, k) \\ &= N_{s-1}(0, k-m-1) + N_{s-1}(0, k-m+1). \end{aligned}$$

In particular,

$$N_s(0, k) = N_{s-1}(0, k-1) + N_{s-1}(0, k+1). \qquad (5)$$

Similarly,

$$Z_s(0, k) = Z_{s-1}(1, k) + Z_{s-1}(-1, k). \qquad (6)$$

Fix some positive k. Any path that starts at the point -1 and arrives at k must run through the origin, so $Z_{s-1}(-1, k)$ is equal to $N_{s-1}(-1, k) = N_{s-1}(0, k + 1)$. Let's now count the paths that start at the point 1, arrive at k, and visit the origin in between. We can draw a graph of such a path (fig. 2) by plotting the position of the particle (vertical axis) against time (horizontal axis). We now use the "reflection" trick mentioned above. We take the portion of the graph from its starting point (at 1) to the first moment when the particle hits the origin, and reflect this section about the time axis. This trick turns the graph into the graph of a path starting at the point -1 and arriving at k! Conversely, every path that joins -1 to k can be transformed in the same way into a path from 1 to k that hits the origin at least once. Thus, we obtain a one-to-one correspondence between the paths of these two classes, which means that they are equal in number. We can write this as $Z_{s-1}(1, k) + Z_{s-1}(-1, k)$, and we can use this result to rewrite equation (6) as

$$\begin{aligned} Z_s(0, k) &= 2Z_{s-1}(-1, k) \\ &= 2Z_{s-1}(0, k+1) \end{aligned}$$

(this last equation is obtained by sliding every path we are counting up one unit on the k-axis).

Now we can find the number of paths of length s from the origin to some $k > 0$ that *never* return to the origin by subtracting the paths that hit zero from all the paths and using equation (5). This number is

$$\begin{aligned} N_s(0, k) - Z_s(0, k) &= N_{s-1}(0, k-1) + N_{s-1}(0, k+1) \\ &\quad - 2N_{s-1}(0, k+1) \\ &= N_{s-1}(0, k-1) - N_{s-1}(0, k+1). \qquad (7) \end{aligned}$$

We first use this result to count those paths that (a) start at the origin, (b) contain evenly many steps s, (c) do not return to the origin, and (d) end up in the positive half-axis. It's not hard to show that such a path, because it contains evenly many steps, must end at a point with an *even* coordinate. Hence we can get the count we want by setting $k = 2, 4, 6, \ldots, 2n = s$ in equation (7), then adding. This sum telescopes, and we have

$$\begin{aligned} &N_{2n-1}(0, 1) - N_{2n-1}(0, 3) \\ &+ N_{2n-1}(0, 3) - N_{2n-1}(0, 5) \\ &+ N_{2n-1}(0, 5) - N_{2n-1}(0, 7) \\ &\quad\vdots \qquad\qquad \vdots \\ &+ N_{2n-1}(0, 2N - 1) \\ &\qquad\qquad - N_{2n-1}(0, 2N + 1) \\ &= N_{2n-1}(0, 1) \end{aligned}$$

(the number $N_{2n-1}(0, 2N + 1)$ above equals zero, since a path of length $2n - 1$ cannot get as far as $2n + 1$ starting from 0).

Similarly, the number of paths that (a) start at the origin, (b) contain evenly many steps s, (c) do not return to the origin, and (d) end up in the *negative* half-axis is $N_{2n-1}(0, -1)$, which, by symmetry, is equal to $N_{2n-1}(0, 1)$. The sum of these two is the number of paths of length $2n$ that start at the origin and never come back.

But by equation (5) this is equal to

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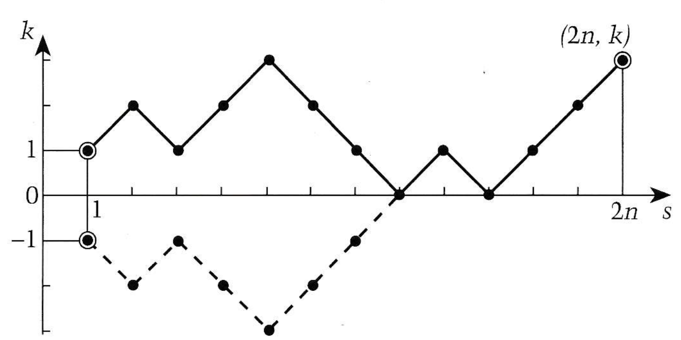


Figure 2

PHYSICS CONTEST

Atwood's marvelous machines

"And your gravity fails
And negativity won't pull you through."
—Bob Dylan

by Arthur Eisenkraft and Larry D. Kirkpatrick

YOU ARE STANDING ON THE fourth floor of a burning building and things aren't looking good. Suddenly you notice that there is a round beam sticking out of the wall with a rope draped over it. One end of the rope reaches to the ground and is tied to a sack of sand that you estimate has a mass of 45 kg. All those contrived physics problems you've solved in your lifetime have prepared you for this moment and you do not panic.

You decide to quickly calculate your chances of surviving the ride to the ground using the rope. You start by drawing a free-body diagram like the one in figure 1 to determine all of the forces acting on the system. Since the beam is highly polished, you decide that you can ignore friction. (And it's just as well, since your physics class never got to the problems where friction was not neglected.) Letting M represent your mass, m the mass of the sack, g the acceleration due to gravity, T the tension in the rope, and a and A the accelerations of the sack and you, respectively, you apply Newton's second law to the problem at hand.

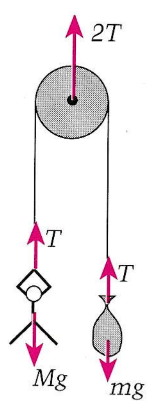


Figure 1

The beam must exert an upward force on the rope of $2T$ to balance the tension in each section of the rope. Let's choose the upward direction to be positive. The net force on you must be equal to your mass times your acceleration:

$$T - Mg = MA. \qquad (1)$$

Likewise the net force on the sack must be equal to its mass times its acceleration:

$$T - mg = ma. \qquad (2)$$

This gives us two equations in three unknowns. But there is a connection between the two accelerations. If you move downward a distance X, the sack must move upward a distance

$$x = -X$$

if the rope doesn't stretch. Since these displacements occur at the same time, the two velocities must have the same magnitudes and opposite directions:

$$v = -V.$$

Likewise the accelerations must be equal in magnitude and opposite in direction:

$$a = -A.$$

This result may seem obvious to you, but relationships that are a bit more complicated than this are often overlooked in solving physics problems like the last one we ask below.

We can substitute for a in equation (2) and reduce the problem to two equations in two unknowns. We can eliminate T by solving each equation for T and equating them to arrive at

$$A = g\frac{M - m}{M + m}.$$

If you have a mass of 90 kg, $A = g/3$, and you will hit the ground with a speed of

$$\begin{aligned} v &= \sqrt{2ax} \\ &= \sqrt{2 \cdot 3.3\ \text{m/s}^2 \cdot 15\ \text{m}} \\ &= 9.9\ \text{m/s}. \end{aligned}$$

Because this is equivalent to jumping from a height of 5 m, you have a good chance of escaping without injury. Of course, if you have less mass or are clever enough to wrap the rope around the beam to maximize the effects of friction, you'll have a safer fall.

This problem is an example of a classic physics problem known as

Art by Tomas Bunk


TOMAS BUNK

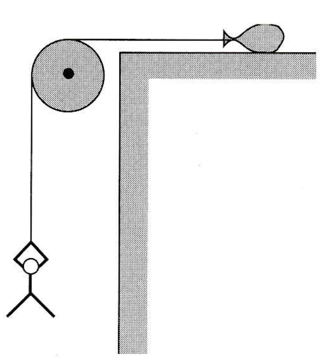


Figure 2

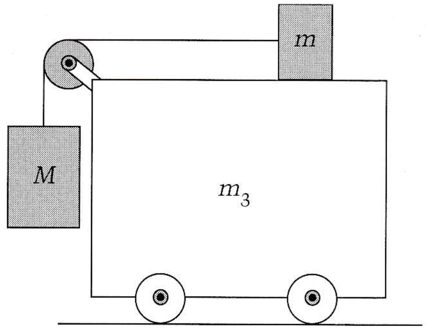


Figure 3

Atwood's machine. In the lab it serves as a means of achieving a constant acceleration of any value less than g. Can you suggest a means of achieving a constant acceleration greater than g? Atwood's machine is often modified to test students' understanding of the applications of Newton's laws. For instance, the beam can be at the top of a wedge with the masses sliding on sloped, frictionless surfaces. And, of course, we could always include the effects of friction.

This month's contest problem consists of two modifications of Atwood's machine. The second one (without the hints) appeared on the first of several examinations used in selecting the members of the US Physics Team that will compete in the International Physics Olympiad this summer in Williamsburg, Virginia.

A. If the sack slides across the floor (assumed to be located at the height of the beam and frictionless) as shown in figure 2, what is your acceleration and the tension in the rope, assuming that your mass is 90 kg? This is sometimes called the "half Atwood's machine" and is often used in introductory physics labs to demonstrate Newton's second law. Be sure to check your answers in the limits of each mass going to zero to see if you get the expected results.

B. Let's assume that we have a scaled-down version of the situation in part A and that the half Atwood's machine is mounted on a car of mass m_3 that is free to move as shown in figure 3. What are the accelerations of all three masses and the tension in the rope just after the masses are released at rest?

If we let the system continue to move, the physics gets very complicated and requires advanced techniques for its solution. However, at time $t = 0$, the solution can be obtained with the techniques used to solve the normal Atwood's machine. Be sure that you draw free-body diagrams for all three masses and then write down Newton's second law for each mass. This will give you three equations in four unknowns (a_1, a_2, a_3, and T). If we assume that the rope does not stretch, you should be able to find a relationship between the three accelerations that will give you the fourth equation you need to solve the problem. As usual, be sure to include checks for the expected answers in the extreme cases.

Please send your solutions to *Quantum*, 3140 North Washington Boulevard, Arlington, VA 22201 within a month after receipt of this issue. The best solutions will be noted in this space and their authors will receive special certificates from *Quantum*.

Row, row, row your boat

The contest problem January/February issue asked readers to row across a river and arrive at a specified point in the least time. Responses arrived from high school students, college students, and physics professors. Geographically, they included solutions from all over the United States as well as from Canada and Great Britain.

The problem assumed that you wish to end up directly across the river and that you are permitted to walk on the far shore if you land upstream or downstream. What path takes the least time? Part A of the problem asked for a qualitative discussion of the range of plausible angles.

The first correct solution was submitted by Ben Davenport of the North Carolina School of Science and Mathematics, a steady reader of *Quantum*. The best angle to take lies between the angle that provides the minimum rowing time and the angle that provides the minimum walking time. For the minimum rowing time, the boat should head directly across the river, or at least at $\theta = 90°$, where θ is measured relative to the upstream bank as shown in figure 4. The minimum possible walking time is zero—that is, the boat arrives exactly at point B across the river. In order for this to occur, the component of the rowing velocity parallel to the river must be equal and opposite to the current. Note that this is also the path of minimum distance. Call this minimum angle θ_w. If the angle is decreased below θ_w, it will take longer to cross the river and the walking time will become nonzero. If the angle is increased above 90°, it will take a longer time to cross the river, and the walking time will also increase. Therefore, we have two boundaries on the angle the boat should take across the river. Just for fun, let's solve for θ_w:

$$v_c = v_r \cos\theta_w,$$

where v_c is the speed of the current and v_r is the speed of the boat relative to the water. Then

$$\theta_w = \arccos\frac{v_c}{v_r} = 48°.$$

So we have limits on reasonable rowing angles:

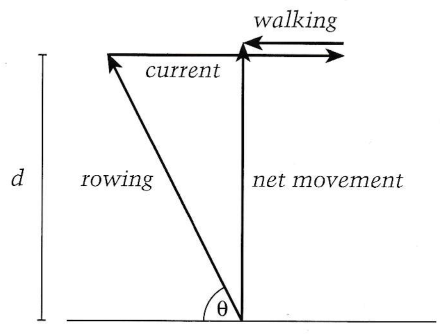


Figure 4

$48° < \theta < 90°$.

Part B required readers to describe this path quantitatively. The rower's velocity component across the river is

$$v_{\text{across}} = v_r \sin\theta.$$

The rower's net velocity parallel to the river is

$$v_{\text{along}} = v_c - v_r \cos\theta.$$

The rower's walking velocity along the shore is given.

The total time is equal to the time spent rowing plus the time spent walking:

$$t = t_r + t_w.$$

The time spent rowing is the distance across the river d divided by the component of the rowing velocity directed perpendicular to the current:

$$t_r = \frac{d}{v_r \sin\theta}.$$

The time spent walking depends on the distance the boat lands from point B and the walking speed v_w. This distance depends on both the net velocity parallel to the river and the time spent rowing:

$$t_w = \frac{t_r\left(v_c - v_r \cos\theta\right)}{v_w},$$

$$t = t_r\left(\frac{v_c - v_r\cos\theta}{v_w} + 1\right).$$

Plugging in for the time spent rowing from above, we get the following expression for the time to cross the river:

$$t = \frac{d\left(v_w + v_c - v_r\cos\theta\right)}{v_w v_r \sin\theta}.$$

One reader, W. Kenneth Beard of Cornwall, England, realized that this problem could also be solved by using Snell's Law, which he states "usually applies to waves experiencing a change of velocity on passing between two media of different density. The waves take the path giving the shortest time, which is precisely what is required here."

$$\boxed{\frac{dt}{d\theta} = \left(\frac{d}{v_w v_r}\right)\left[\frac{\sin\theta\left(v_r \sin\theta\right) - \cos\theta\left(v_w + v_c - v_r\cos\theta\right)}{\sin^2\theta}\right] = 0}$$

Part C required a calculation of the least time using the values provided for the speed of the current, the rowing speed, and the walking speed. Most readers differentiated their equation for the total time and set the derivative equal to zero to find the minimum-time path (see the equation in the box above). Since $\sin^2\alpha + \cos^2\alpha = 0$,

$$v_r - \cos\theta(v_w + v_c) = 0,$$

$$\theta = \cos^{-1}\left(\frac{v_r}{v_w + v_c}\right)$$

$$= \cos^{-1}\left(\frac{3}{7}\right) = 64.6°.$$

Thus, the rower should row 25.4° upstream of straight across for a minimum crossing time. At this angle, the rower arrives 132 m downstream and requires a total time of 12.6 minutes to complete the trip.

Andrew Menard from Saginaw, Michigan, went one step further in noting that "the width of the river does not matter. It will obviously affect the total time, but the optimal angle is not affected." Andrew also sent in a program for the TI-81 and TI-85 graphing calculators that shows the path and calculates the total time required for the journey. This numerical technique could be used by our readers who have not yet encountered calculus.

"COUNTING RANDOM PATHS"
CONTINUED FROM PAGE 41

$N_{2n}(0, 0)$. That is, there are as many paths of length $2n$ that start at the origin and never return as there are paths that start at the origin and return after exactly $2n$ steps. It remains to calculate the corresponding probabilities, and this is easy. In each case, we just divide the number of paths (which we have seen is the same in both cases) by 2^{2n}, which is the number of all possible paths of length $2n$. Since the numbers of paths are equal in each case, the corresponding probabilities must be equal.

Hence u_{2n}—the probability that the particle returns to the origin after exactly $2n$ steps—is equal to the probability that the particle does not return to the origin before $2n$ steps. As noted earlier, the results of *RSC* guarantee that the return takes place with probability 1, so u_{2n} approaches 0 as n gets larger, and equation (4) assures us that $p_{n-1,n}$ approaches 1/2.

The following exercises will help you master the method used above and will tell you more about the random walk.

1. Prove that $p_{n+k,n} \to 1/2$ as $n \to \infty$.

2. Prove that the number of positive paths of length s that lead to a certain point k ($k > 0$) is equal to $(k/s)N_s(k)$—that is, the probability that a path of the symmetric random walk from the origin to point $k > 0$ stays positive at all times is equal to k/s.

3. Prove that a point that starts at the origin and performs the symmetric random walk returns to the origin for the first time at the moment $2n$ with the probability $u_{2n-2} - u_{2n}$.

For further results about the random walk see, for instance, the classic book *An Introduction to Probability Theory and Its Applications* by William Feller.

Atlantic crossings

Meet me in the middle of the wide blue sea

by A. Rozental

Art by Pavel Chernusky

REMEMBER THIS LITTLE DITty from Rudyard Kipling's *Just So Stories*?

Yes, weekly from Southampton,
Great steamers, white and gold,
Go rolling down to Rio
(Roll down—roll down to Rio!),
And I'd like to roll to Rio
Some day before I'm old!

Well, so weekly (say, each Thursday) from Southampton great steamers go rolling down to Rio . . . It takes 14 days for a great white and gold steamer to cover the entire distance of 9,800 km (700 km per day) and arrive at Rio de Janeiro exactly at noon on Thursday. After a four-day stopover, the ship sets off on the return trip, and in a fortnight, at noon on Monday, it arrives at Southampton. Three days later—again on a Thursday—it leaves on its next voyage to Brazil . . . I wanted to roll to Rio, too, so I stepped onto an ocean liner at Southampton on Thursday and my voyage began. In anticipation, I asked myself some questions:

- How many steamers wending their way home will I see on the ocean?
- On what days of the week will I see them?
- How far from Southampton will I meet them?

I'll use this problem to teach you a graphical method for solving a fairly wide range of problems, including the so-called "motion problems."

Plot the movements of the steamers between the two ports on one graph. We'll mark the days of the week on the horizontal t-axis and the distances from Southampton on the vertical S-axis (fig. 1). Knowing that a steamer travels 700 km per day, we plot the points A, B, C, D representing its position at noon on Friday, Saturday, Sunday, and Monday. Of course, these points lie on one straight line—the graph of the steamer's motion. Graphing the movements of all the steamers in the problem (departing from Southampton on Thursdays and from Rio on Mondays), we get what's shown in figure 2. Then every intersection of the graphs corresponds to a meeting of two ships on the open seas.

Consider, for instance, the line KL from Southampton to Rio de Janeiro. It crosses the lines representing the returning ships four times. Therefore, the answer to the first question above is that I'll see four steamers on their way home.

Then, as we see from the graph, the time it takes our steamer to meet the first returning ship equals the time it takes the other ship to reach Southampton after this meeting. So the first meeting occurs halfway between noon on Thursday and noon on Monday—that is, on Saturday at noon. For similar reasons the second meeting takes place at midnight on Tuesday (more exactly, between Tuesday and Wednesday), and so on. The distances from the points of meeting to Southampton equal 1,400 km, 3,850 km, and so on.

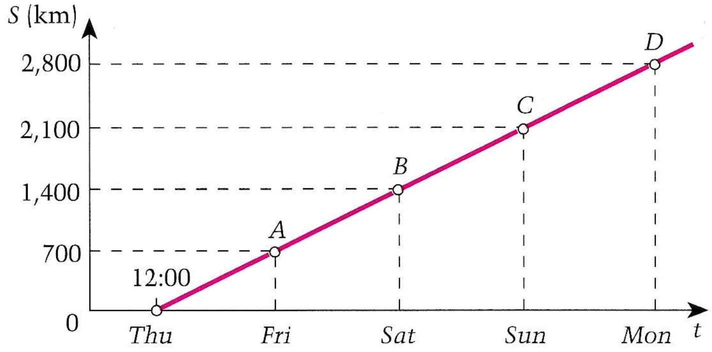


Figure 1

And now answer three questions on your own!

1. Is it true that when two steamers meet in the ocean, two other steamers meet at some other spot? If it's true, then what is the distance between the two points of meeting?

2. How many steamers travel the sea lanes between Southampton and Rio de Janeiro?

3. Answer the very first question—"How many returning ships will I meet?"—for the case when the steamers depart every day, not every week.

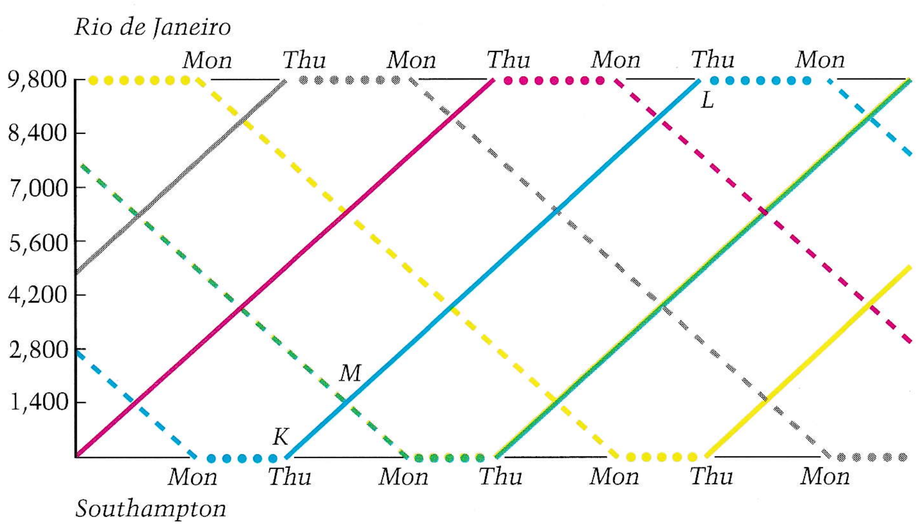


Figure 2

Three problems for landlubbers

Round the clock. The hour and minute hands of a clock meet exactly at noon. Then the minute hand shoots ahead and some time later overtakes the hour hand by a lap and covers it again. When does this happen? At what moments of the day (between 12 A.M. and 12 P.M.) do the two hands form (a) a straight angle, (b) a right angle, (c) an angle of 120°?

Strolling ladies. Two elderly ladies went for a walk at the same time along the same 100-meter path. Mrs. Fields walks at a pace of 1 km per hour, while Granny Smith walks more slowly—at 600 m per hour. When they reach the end of the path, each turns around and walks back at the same speed. Every time they meet, the ladies greet each other with a nod of the head. How many times did they nod during the first 25 minutes of their walk? How long did they walk in the same direction during this time interval?

Sir Isaac Newton's problem. Two mail carriers A and B separated by a distance of 59 miles drive out in the morning to meet each other. Mail carrier A drives 7 miles in 2 hours, while mail carrier B drives 8 miles in 3 hours. If mail carrier B sets off an hour later than mail carrier A, how many miles will mail carrier A cover before she meets mail carrier B?

ANSWERS ON PAGE 61

Ping-pong in the sink

But leave your paddles behind

by Alexey Byalko

THE EXPERIMENTS I'M GOing to talk about in this article are accessible to anyone. They don't require any special contraptions. All you'll need is a Ping-Pong™ ball, a sink (or bathtub), a millimeter ruler, a three-liter jar, and a stopwatch (or a wristwatch with a second hand).

Experiment 1. Close the drain opening in your sink or bathtub and run the water until it reaches the 3- to 5-cm level. Put a Ping-Pong ball in the stream of water at the point where it hits the surface of the standing water and release it. Your intuition says that the ball should be thrown aside. But that doesn't happen! The stream falling from the faucet catches the ball—the ball stays right where you put it.

Take a closer look. The ball doesn't stand still, does it? It oscillates slightly under the influence of the falling water. Cup your hands and push them under the ball so that it can keep floating under the falling stream without touching your palms. Now lift the ball up along the stream. You'll see that the frequency of the oscillations decreases as you lift the ball.

Experiment 2. Open the drain. In an ordinary sink or bathtub the opening is a small cylindrical pit about 20 mm deep and 40 mm in diameter with a latticed bottom.[1] A Ping-Pong ball is a bit smaller: it's 37 mm in diameter, so it can sit in the opening and still leave a small gap of about 1–2 mm (fig. 1).

Put the ball just above the drain opening and let it go. The stream of water will pull it into the opening in spite of the fact that the ball normally resists being submerged. And do you hear a low, buzzing sound? It must be from the ball's oscillations. The first thing that may occur to you is that the ball is bouncing on the lattice. But it's not. Touch the ball on top and you'll be convinced that the ball is oscillating not vertically but horizontally as far as the opening allows.

Now look at the trademark on the ball. You'll notice that the ball is rotating—so slowly, in fact, that you don't need a slow-motion video replay to see it. The ball rotates for a while in one direction, then it switches and begins rotating in the other direction.

How can we explain these experiments? Right off the bat we can say that both of them vividly illustrate Bernoulli's law, one of the first laws of hydrodynamics. It can be qualitatively stated as follows: in flowing liquids the pressure is greater where the velocity of the liquid is less, and vice versa—where the velocity is greater, the pressure is less. In other words, as the velocity of any liquid increases, the pressure inside it decreases. In both experiments the decrease in pressure caused by the stream of water draws the plastic ball into the stream.

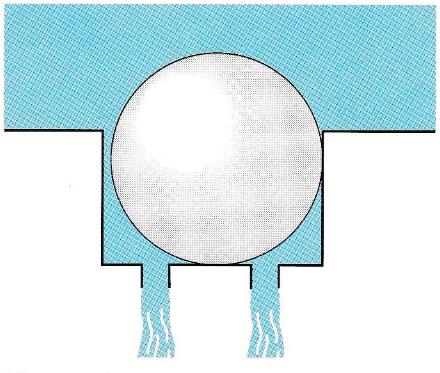


Figure 1

Bernoulli's law will help us understand the ball's oscillations in the stream of water and in the drain opening as well. But first, let's determine the velocity of the water in the stream flowing out of the faucet.

It's not difficult to adjust the flow of water into a sink or bathtub so that the water level remains virtually constant. We can measure the corresponding outflow of water (that is, the mass of liquid drained per unit time) with the three-liter jar and stopwatch. It will be approximately 80–100 g/s. Now let's use the ruler to determine the diameter of the stream near the surface of the water in the sink. It's about 6 mm. This gives us a velocity of about 3 m/s for the stream of water (you can check this). Of course, the velocity of the water can be increased or decreased, but in our calculations let's just use this figure.

To discuss in more detail the effects we observed, we'll need not only a qualitative but a quantitative formulation of Bernoulli's law. Ac-

[1] In the US, the type of drain opening described here is more likely to be found in the utility sink in your basement (laundry room) or in your lab at school.—*Ed.*

cording to this law, in any cross section of the stream of liquid the sum $\rho v^2/2 + p + \rho g h$ remains constant, where ρ is the density, v is the velocity, p is the pressure of the liquid, and h is the height of the section chosen.

The explanation of experiment 1. Let's simplify our task for the time being: instead of the interaction of a cylindrical stream and a spherical ball, let's examine the effect of a flat stream on a cylinder. The essence of the physical phenomenon won't change because of this—it'll just make things easier for us.

First let's look at the case when a cylinder of radius R touches the stream slightly and estimate the force acting on it (fig. 2). According to Bernoulli's law, the pressure inside the curved stream drops by the amount

$$p_0 - p = \frac{\rho}{2}\left(v^2 - v_0^2\right)$$

(we can neglect the term $\rho g R$ since for our case $gR \cong 0.2\ \mathrm{m^2/s^2}$ is much less than $v^2/2 \cong 4.5\ \mathrm{m^2/s^2}$). The trajectory of the particles of liquid enveloping the cylinder is an arc, so they move with centripetal acceleration. The acceleration is due to the difference between the atmospheric pressure p_0 (the same pressure as in a straight stream) and the pressure p in a curved stream:

$$m\frac{v^2}{R} = \left(p_0 - p\right)S,$$

where S is the area of contact between the stream and the cylinder. The velocity v of the stream flowing around the cylinder is but a little more than the initial velocity v_0 of a stream whose thickness d is much less than the radius R of the cylinder. Therefore, the force deflecting the stream is equal to

$$F = \left(p_0 - p\right)S = \frac{\rho v_0^2 d}{R}S.$$

The force acting on the cylinder is of the same magnitude but in the opposite direction.

The result obtained for flow around the cylinder is valid for flow around a ball but, perhaps, with a different numerical coefficient. The area of contact of the stream with the ball is practically always of the order of the square of its radius R. That's why the force pulling the ball into the stream when they touch is of the order of magnitude

$$F \sim \rho v_0^2 R d.$$

Now let's consider the case when the stream falls directly on top of the ball (fig. 3). The water flows around the ball symmetrically, and thus $F = 0$. It's not difficult to determine the thickness of the layer d for this case of centered flow around the ball. Taking into consideration the constancy of the flux, we have

$$d \sim \frac{r_0^2}{R}$$

(check it yourself).

Finally, in the intermediate case when the displacement x of the ball with respect to the axis of symmetry of the stream is not large compared to its radius R (fig. 4), the force F depends on x linearly and is directed so as to bring the ball into equilibrium:

$$F \sim -\rho v_0^2 d \cdot x \sim -\frac{\rho v_0^2 r_0^2}{R}x.$$

(Unfortunately, it's difficult to obtain the exact value of the force F for this case.)

The ball will oscillate under the influence of this force. The frequency of the oscillations can be expressed by analogy with the oscillations of a small mass on a spring:

$$F_{\text{spring}} = -kx \Rightarrow \omega_{\text{spring}} = \sqrt{\frac{k}{m}},$$

$$F_{\text{ball}} \sim -\frac{\rho v_0^2 r_0^2}{R}x \Rightarrow \omega_{\text{ball}} \sim v_0 r_0 \sqrt{\frac{\rho}{RM}}.$$

If the mass of the ball $M \cong 3$ g, we get

$$\omega_{\text{ball}} \sim 40\ \mathrm{s^{-1}},$$

and

$$\nu_{\text{ball}} \sim 6\ \mathrm{Hz}.$$

When we lift the ball in our palms up along the stream, r_0 and v_0 change simultaneously, but the

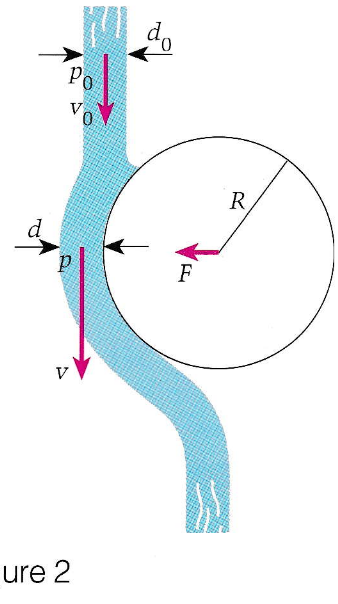


Figure 2

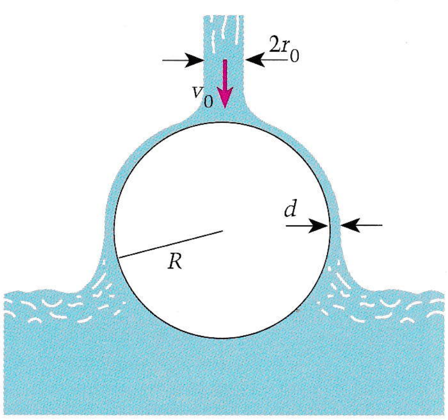


Figure 3

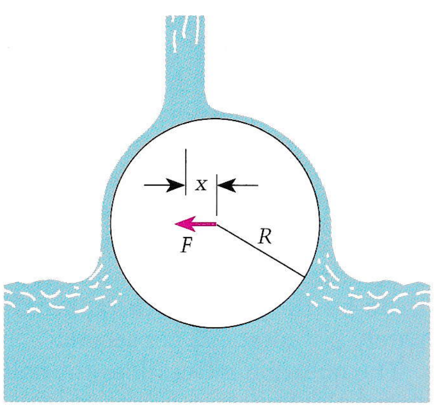


Figure 4

product $v_0 r_0^2$ remains constant. As r_0 increases, the product $v_0 r_0$ decreases. Thus, as the ball is lifted toward the source of the stream, the frequency of the oscillations must decrease, which is what we saw in our experiment.

When the frequency of the oscillations decreases, we notice that the ball still turns in different directions. In thin, slow streams the oscillation and rotation of the ball combine to create a complex and beautiful movement around the stream. You can check this experimentally on your own.

The explanation of experiment 2. Let the initial position of the ball be as shown in figure 5 (viewed from above). Estimate the velocity of the flow in the gap between the ball and the drain opening. The area of the gap doesn't depend on the position of the ball and equals $\pi(R^2_{\text{drain}} - R^2) \cong$ 1 cm^2. Therefore, the average velocity of the flow around the ball is $v \cong$ 1 m/s. This velocity, however, is far from constant in the cross section of the gap—it's essentially greater in the wider part than it is in the narrower part of the gap (near the point of contact A), where the flow is slowed by friction from the wall.

According to Bernoulli's law the pressure in a moving fluid is greater where its velocity is lower. Therefore, a force arises that is directed at the wider portion of the gap, and this causes the ball to oscillate.

But the real motion of the ball is not a simple movement from wall to wall via the center. It presses against the wall of the drain, and the point of contact A moves around the rim of the drain, while the center of the ball O traces a small circle (the dotted line in figure 5). It's easy to see that the angular velocity of the ball's rotation is $\Omega = \omega(R_{\text{drain}} - R)/R_{\text{drain}}$—that is, it's approximately one tenth of ω. And it isn't hard to measure the angular velocity with a stopwatch by calculating the number of rotations made by the trademark (approximately 2–3 rotations per second).

The question arises: does the ball rotate clockwise or counterclockwise? Before answering, we should

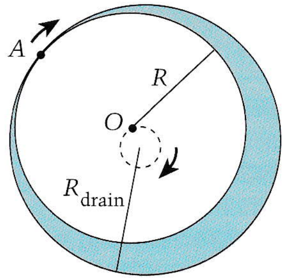


Figure 5

note that the central position of the ball in the drain opening is stable. To convince yourself of this, turn off the faucet and press down on the ball with your finger. Then carefully release your finger—the ball remains stationary. This means that the specific motion you observe depends on circumstances—all the events leading up to it—and three cases are possible: if the ball was stationary, it will remain stationary; if the initial push moved the point of contact A clockwise, the ball will rotate counterclockwise; and vice versa.

Bulletin board

Desktop motion simulation

"Engineers now have a tool where they can build and test physical systems on the computer with a high level of accuracy. Animators can make physically accurate animations in minutes—something they've never been able to do before." So says Gregory Baszucki, Vice President of Marketing for Knowledge Revolution, developers of Working Model™, a professional motion simulation application for the Macintosh. Working Model builds on Knowledge Revolution's Interactive Physics II software and includes a *Smart Editor*™, CAD file import and export, and direct file saving to popular animation programs such as MacroMind Three-D and Wavefront.

Working Model lets users create physical objects on a Macintosh screen by drawing. Objects are given physical properties such as mass, friction, and elasticity. When a simulation is run, Knowledge Revolution's dynamic simulation engine mathematically calculates the accurate motion of each object and displays their movement in smooth animation. The program has a large tool kit of devices that allows users to easily build their physical systems and a complete set of measurement tools for data analysis.

The program also uses Apple Computer's AppleEvents® to create real-time data links with other applications; Working Model can send and receive data to other applications while a simulation is running. For example, a control algorithm for a self-stabilizing unicycle is developed in Microsoft Excel®. The actual physical model of the unicycle is developed in Working Model. AppleEvents allows the control algorithm residing in Excel to communicate with the unicycle model residing in Working Model. When the simulation is running, the information is transferred between both programs. Any changes in the control algorithm can be tested on the unicycle model in a matter of seconds.

Working Model is available from Knowledge Revolution and Macintosh resellers nationwide. For more information, write to Knowledge Revolution, 15 Brush Place, San Francisco, CA 94103, or call 800 766-6615.

Molecular biology series

What if cows could make human milk, and pigs, human blood? What if we could cure alcoholism or Alzheimer's disease by simply changing one gene? In factories and operating rooms, on the farm and at the supermarket, we must all face a revolution in our understanding of life. Fresh advances in the science of molecular biology challenge our ethics, our economy, our very notions of what we are.

To help us understand these changes, WGBH-TV presents *The Secret of Life*, an eight-part public television series premiering this fall (check local listings for specific airdates and times). Hosted by geneticist and television personality David Suzuki, this series explores the "new biology" for both scientists and nonscientists. Using a studio set designed to illustrate our genetic archive, Suzuki explains exactly what we know—and don't know—about DNA. Most importantly, he defines the moral, financial, and political implications of this new technology. Microphotographs created for the series illuminate intimate biological details, and computer animation takes viewers into the core of our genetic code: the double helix of DNA and beyond.

To extend the educational impact of the series, WGBH is offering free print materials for science and social studies teachers. If you would like more information about the series or would like to order the teacher's guide or classroom poster, please contact Marisa Wolsky, Outreach Coordinator, at 617 492-2777, extension 4390, or write her at WGBH, Educational Print & Outreach, 125 Western Avenue, Boston, MA 02134.

International student fairs

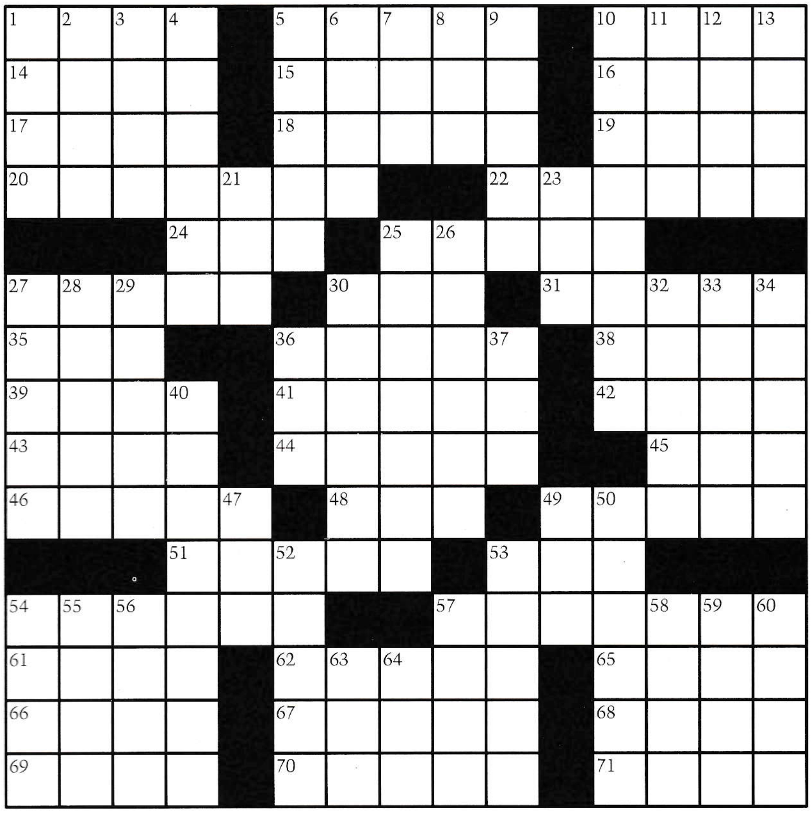
The first International Student Fairs & Conferences—events that will provide high school students an opportunity to view, evaluate, and meet school and industry representatives from all over the globe—have been set for March and October 1994.

Produced by Communication Alliance of the Americas, Inc. (a Chicago-based company), the International Student Fairs & Conferences have been organized to answer the mutual needs of a new generation of student consumers, corporations, and providers of postsecondary education whose requirements, competitive boundaries, and methodologies were transformed during the 1980s. New realities in business and higher education have created the need for an event which encompasses all postsecondary education

CONTINUED ON PAGE 63

Criss × cross science

by David R. Martin



Across

1 Speck of dust
5 Nostrils
10 Chemical compound
14 Asleep
15 Garnish fruit
16 Easy gait
17 Tokamaks, e.g.
18 Started
19 Type of parity
20 Type of strength
22 Trig. function
24 Songlike poem
25 Like Argon
27 Units of area
30 Disapprove
31 Units of mass
35 Honest ___
36 Male voice
38 Exclude
39 Rational
41 Get up
42 10^{-9}: pref.
43 Article
44 Demolished
45 Luminous plasma sphere
46 ___ & Clark expedition
48 A Kennedy
49 Out-of-date
51 Distance divided by time
53 Plant pouch
54 Ankle bone
57 Computer chip material
61 Decorative case
62 Arched roof
65 City slicker
66 As well
67 Follow
68 Eye part
69 Sharp
70 Plant parts
71 Penny

Down

1 Bible bk.
2 Orchestra member
3 Sea eagle
4 Am. inventor
5 Like Neon
6 Toward shelter
7 Equip
8 A Peron
9 Smell or sight, e.g.
10 Fundamental particle
11 Brilliant star
12 Uncovered
13 Fasting period
21 Impulsive psyches
23 Unit of energy
25 Charged (electrically)
26 Hung to death
27 Seasoning
28 Decrease
29 Refresh
30 Scold
32 Accumulate
33 Calculator button
34 Rock
36 Distilled coal
37 Ruby
40 Electron ejection
47 Mineral spring
49 Chum
50 Having pH below 7
52 Sprites
53 Locations
54 Hard wood
55 Small evergreen
56 Trick
57 Ghetto
58 Remedy
59 Norse god
60 Cozy home
63 Colonist
64 Employ

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE MAY/JUNE PUZZLE

S	L	A	B		M	E	T	E	R		S	T	A	T
H	A	I	R		E	R	O	D	E		C	O	L	E
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A	N	S	W	E	R		A	M	I	L	L	I	O	N
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P	A	R	T	I	C	L	E		T	A	R	S	A	L
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M	I	S	T	L	E		H	E	R	C	U	L	E	S
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P	I	C	O		O	P	T	I	C		E	P	E	E
A	L	A	D		G	I	A	N	T		R	O	S	A
L	O	N	E		Y	A	R	D	S		S	S	T	S

Volume 3 (1992—93)

ANSWERS, HINTS & SOLUTIONS

Math

M86

It's convenient to reverse the statement and try to find the shortest way of transforming an arbitrary n into zero by means of the two inverse operations: "dividing by 2" (D) and "subtracting 1" (S). Let $r(n)$ be the number of operations D and S in such a shortest sequence. Since only operation S can be applied to an odd n,

$$r(2k+1) = 1 + r(2k).$$

Now let's show by induction over k that

$$r(2k) = 1 + r(k)$$

—that is, in the shortest sequence of operations an even number must be always divided by two.

For $k = 1$ this equality is clear (here D and S coincide). Suppose it's been proven for all even numbers less than $2k$. If we apply S to the number $2k$ (that is, subtract 1), then the total number of operations needed to obtain zero will be no less than

$$1 + r(2k-1) = 2 + r(2k-2) = 3 + r(k-1)$$

(by our assumption). But if we apply D first and then choose the shortest path, the number of operations will be

$$1 + r(k) \le 2 + r(k-1),$$

so the second method is better.

To find $r(n)$ for a given n, represent n as the sum of powers of 2:

$$n = 2^{k_1} + 2^{k_1+k_2} + \ldots + 2^{k_1+k_2+\ldots+k_m},$$

where k_1 is a nonnegative integer and $k_2, \ldots, k_m$ are positive integers. (The exponents $k_1, k_1 + k_2, \ldots$ are the numbers of the places in the binary notation of n in which there are ones.)

Now we find that

$$\begin{aligned} r(n) &= k_1 + r\left(n/2^{k_1}\right) \\ &= k_1 + 1 + r\left(2^{k_2} + 2^{k_2+k_3} + \ldots + 2^{k_2+k_3+\ldots+k_m}\right) \\ &= \left(k_1 + k_2 + \ldots + k_m\right) + m \\ &= \left[\log_2 n\right] + m \end{aligned}$$

($[x]$ is the greatest integer not exceeding x), because $k_1 + \ldots + k_m$ is the exponent of the highest power of 2 not exceeding n. The number m is equal to the number of ones in the binary notation of n.

In particular, for $n = 100$ we get the following answer to part (a): $r(100) = 9$. Shown below is the transformation of the number 100 into zero in decimal and binary notations. Since $100 = 2^6 + 2^5 + 2^2$, $r(100) = 6 + 3 = 9$:

100		1,100,100
↓	$D_2 = DD$	↓
25		11,001
↓	S	↓
24		11,000
↓	D_3	↓
3		11
↓	S	↓
2		10
↓	D	↓
1		1
↓	S	↓
0		0

M87

Let's prove a slightly more general statement: *Let $M_1, M_2, \ldots, M_6$ be the midpoints of the sides of a convex hexagon (fig. 1) and P a point inside*

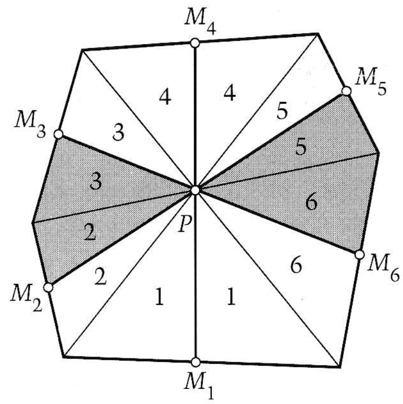

Figure 1

it. If the broken lines M_1PM_4 and M_2PM_5 both divide the area of the hexagon in half, the same holds for M_3PM_6.

To derive the statement of the problem from this one, take for P the point of intersection of two midlines M_1M_4 and M_2M_5. Then the broken line M_3PM_6 bisects the hexagon's area (as does the third midline M_3M_6), so P lies on M_3M_6.

To prove the italicized assertion, join P to the midpoints of the sides and to the vertices of the hexagon. We get six pairs of triangles such that the triangles of one pair (numbered the same in our figure) have equal areas (because a median of a triangle bisects its area). Denoting by a_i the area of either of the ith triangles, we can write the condition that M_1PM_4 bisects the hexagon's area as $a_1 + 2a_2 + 2a_3 + a_4 = a_4 + 2a_5 + 2a_6 + a_1$, or equivalently as

$$a_2 + a_3 = a_5 + a_6.$$

Similarly, the condition that M_2PM_5 bisects the hexagon's area is equivalent to

$$a_6 + a_1 = a_3 + a_4.$$

Summing these two equalities and simplifying, we get

$$a_1 + a_2 = a_4 + a_5,$$

which means that M_3PM_6 also bisects the hexagon's area.

It's not hard to show (do it!) that inside any convex hexagon there is a unique point P such that the broken lines M_iPM_{i+3}, $i = 1, 2, 3$, bisect the hexagon's area. (V. Proizvolov, V. Dubrovsky)

M88

The largest integer not exceeding a positive a is the number of positive integers less than or equal to a. So $[\sqrt[k]{n}] - 1$ is the number of integers y, $y \ge 2$, such that $y \le \sqrt[k]{n}$, or $y^k \le n$, and the sum is the number N of the *pairs* of integers (x, y) such that $x > 1$, $y > 1$, and $y^x \le n$ (there are no such pairs such that $x > n$, because $2^n \ge n$)—that is, the number of integer points in the area shaded in figure 2. We get the sum above when we count these points "down." When we count them "across" (along the lines $y = m$), we get the numbers of integers $x \ge 2$ such that $m^x \le n$, or $x \le \log_m n$—that is, the numbers $[\log_m n] - 1$, $m \ge 2$. Therefore,

$$([\log_2 n] - 1) + ([\log_3 n] - 1) + \dots + ([\log_n n] - 1) = N$$

(for $m > n$ the inequality $m^x \le n$ does not hold for any $x \ge 2$), and both sides of the equation in question turn out to be equal to $N + n - 1$.

M89

Let a_x be the number written near point x, n_x the number of different lines joining x to all the other plotted

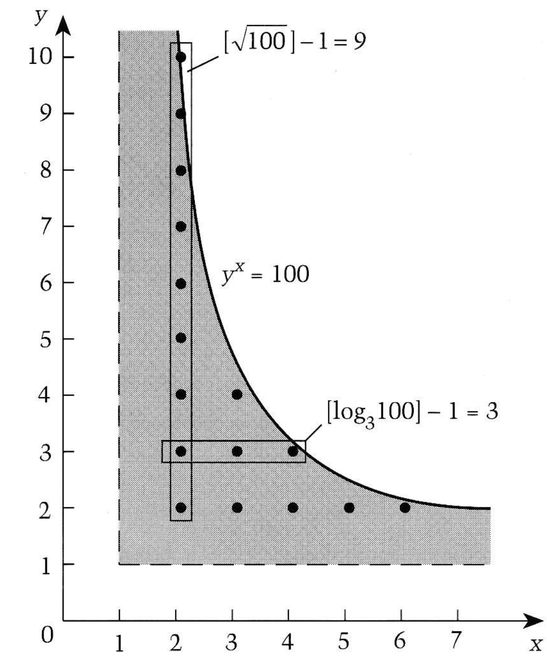

Figure 2

points. If S is the sum of all the numbers a_x, then summing the numbers along each of the n_x lines through x and adding together all the sums we'll get, on the one hand, zero (because every single sum is zero) and, on the other hand, $(n_x - 1)a_x + S$ (because every number except a_x is counted once and a_x is counted n_x times). So for any x,

$$(n_x - 1)a_x + S = 0,$$

which means that a_x and S have opposite signs ($n_x > 1$, since our points do not all lie on one line) or both are zero. We can write this as $a_xS \le 0$. Summing these inequalities for all x, we get $S^2 \le 0$, so $S = 0$ and $a_x = S/(1 - n_x) = 0$ for any x.

M90

Number the cards $0, 1, \dots, 2n$ from the top of the stack to the bottom and lay them out one by one at the vertices of a regular $(2n + 1)$-gon $A_0A_1 \dots A_{2n}$ (see figure 3, in which $n = 4$). Then the operation A will be represented simply as the rotation of the whole circle of cards about the polygon's center through the angle $360°k/(2n + 1)$, where k is the number of cards placed underneath the stack ($k = 1, 2, \dots, 2n$), and under the operation B the cards are moved from the vertices $A_0, A_1, \dots, A_{n-1}$ to $A_1, A_3, \dots, A_{2n-1}$, respectively; and from the vertices $A_n, A_{n+1}, \dots, A_{2n}$ to $A_2, A_4, \dots, A_{2n}$ (fig. 4). So after this operation two cards that were adjacent become one vertex away from each other. And if they were k sides of the polygon away from each other (we can assume $k \le n$), then operation B moves them $2k$ sides apart. It follows that after carrying out a number of operations A and B in any order, the distances between the cards 0 and 1, between 2 and 3, …, and between $2n$ and 0 will be equal to one another. Therefore, the arrangement of the cards is uniquely determined by the positions of the cards 0 and 1: if card 0 is at vertex A_i and card 1 is at vertex A_j, then card 2 is at vertex A_k such that $A_jA_k = A_iA_j$, card 3 is at vertex A_l such that $A_kA_l = A_iA_j$, and

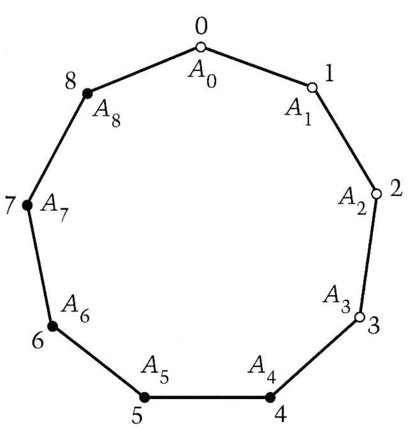

Figure 3

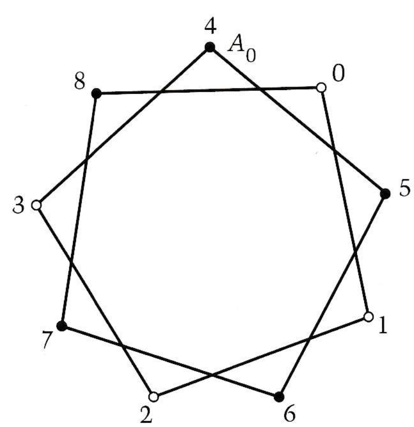

Figure 4

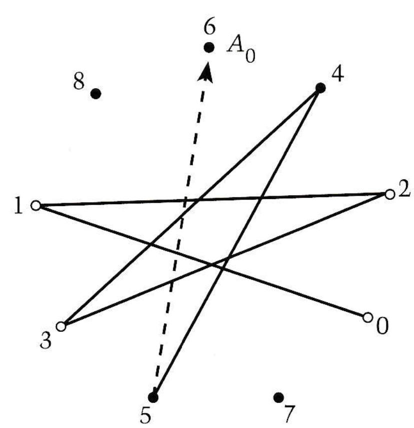

Figure 5

so on (fig. 5). The position of card 0 can be chosen in $2n + 1$ ways, which leaves $2n$ choices for card 1. So the total number of arrangements is not greater than $2n(2n + 1)$.

For a fixed position of card 0, some positions of card 1 can be unattainable: the number of sides of the polygon between these cards after one application of operation B becomes equal to the remainder of the doubled initial number of sides when divided by $2n + 1$, so the values this number can take are all possible remainders of the powers of 2 when divided by $2n + 1$. If there are N such remainders, the exact number of arrangements is $(2n + 1)N$. (D. Fomin, V. Dubrovsky)

Physics

P86

When the bead passes the bottom of the parabola, its velocity is directed along the X-axis (fig. 6). During a tiny time interval t after this moment, the displacement along the abscissa can be approximated as

$$x = V_0 t.$$

The corresponding displacement along the ordinate is

$$y = Ax^2 = AV_0^2 t^2.$$

For a small enough time interval, we can consider the acceleration a along the ordinate to be uniform and write the displacement as

$$y = a\frac{t^2}{2}.$$

Therefore, the acceleration of the bead along the ordinate is equal to

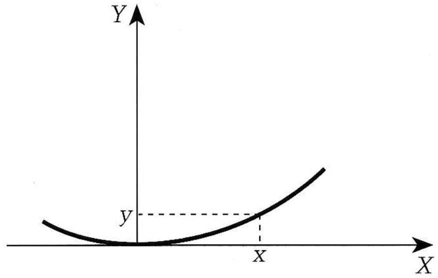

Figure 6

$$a = 2AV_0^2.$$

The force acting on the bead along the Y-axis is equal to

$$F = Ma = 2MAV_0^2.$$

By Newton's third law, an equal force acts on the wire in the opposite direction.

P87

Both the ammeter and voltmeter are clearly nonideal in our case. This means that the electrical resistance of the ammeter is not zero and the resistance of the voltmeter is not infinite. Let's denote the resistance of the ammeter as r, that of the voltmeter as R, and that of the unknown resistance as Z. In the first network (fig. 7a) the current flowing through ammeter is

$$I_1 = \frac{E}{r + \frac{RZ}{R+Z}} = \frac{E(R+Z)}{rR + rZ + RZ}.$$

In the second case (fig. 7b), when the resistor is connected in parallel with the ammeter, the current through the battery is

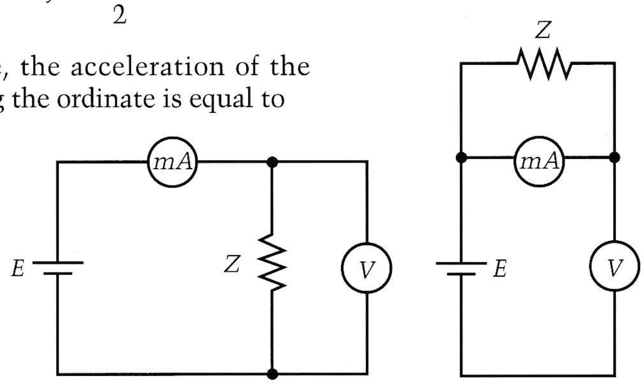

Figure 7 a b

$$I_B = \frac{E}{R + \frac{rZ}{r+Z}},$$

The current flowing through the ammeter in this case is

$$I_2 = I_B\frac{Z}{r+Z} = \frac{EZ}{Rr + rZ + RZ}.$$

Comparing the expressions for I_1 and I_2, we see that they differ by just one factor: in the first formula it's $(R + Z)$, and in the second it's Z. Since the currents differ by a factor of 3, the resistance of the voltmeter R is twice that of the resistor Z. Now it's easy to calculate Z: the sum of the electrical currents flowing through the voltmeter and resistor in parallel in the first network is 3 mA, so the current through the resistor is 2 mA. The voltage drop across the resistor is 2.9 V, and so its resistance is $Z = 1.45$ kΩ.

P88

The low thermal conductivity of cloth—and of the pot holder made from it—is due to the air distributed among the fibers of the fabric. The thermal conductivity of water is higher than that of air, so you might burn your hand pretty severely if you replace the air in the pot holder with water by getting it wet.

P89

Connecting the wires A and B between S_1 and S_2 puts the capacitors $C_1 = \varepsilon_0 S_1/d_1$ and $C_2 = \varepsilon_0 S_2/d_2$ in series—that is,

$$C_{AB} = \varepsilon_0 \frac{S_1/d_1 \cdot S_2/d_2}{S_1/d_1 + S_2/d_2} = \varepsilon_0 \frac{S_1 S_2}{S_1 d_2 + S_2 d_1}.$$

When the wire is connected between S_1 and the metal sheet, we short out this capacitor and the remaining capacitance is just C_2. Likewise, when the wire connects S_2 and the metal sheet, the capacitance is C_1.

P90

The solution will be clear if we draw the path of the light rays through the lens. First consider the most extreme rays—that is, those refracted at the very edges of the lens (fig. 8). After refraction these beams define the region where it's possible to observe the optical image of the light source behind the lens. The continuation of these extreme rays defines the boundaries of the region where the light source can't be seen because the lens blocks it. It's clear that both the optical image and the light source itself can be observed from the point d_1, since the rays bypass the lens in traveling to this point. From the point d_2 it's possible to see the optical image only as result of refraction in the lens. Finally, neither the source nor its optical image can be seen from the point d_3.

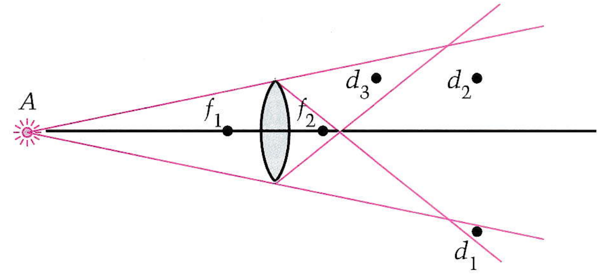

Figure 8

Brainteasers

B86

No, it's impossible, because when the five odd sums are added up we must get an odd number; but on the other hand, each of the inscribed numbers enters this total sum twice, so it must be even.

B87

The answer is 561 and 165. Clearly, the numbers in question must have three digits each. Let one of them be $abc = 100a + 10b + c$ and the other cba. Since $a \cdot c$ ends in 5, one of the numbers a and c—say, a—must be 5; and since $92{,}565 \div 500 < 200$, $c = 1$. To find b we notice that the 6 in the product is the last digit of $5b + b = 6b$, so $b = 1$ or $b = 6$. All that remains is to test the two possibilities. (V. Dubrovsky)

B88

The mass of the log is 300 kg. The distance between its center of gravity and the first scale is one half the distance between the center of gravity and the second scale.

B89

The line EF in figure 9 makes it obvious that the area of the triangle ABE is exactly half that of the bottom rectangle $ABCD$. So the covered part of the rectangle has a greater area. By the way, this will remain true even if we replace the top rectangle with an arbitrary convex polygon that covers the vertices A and B and has a common point with the side CD.

B90

The answer is nine. After Prince Ivan makes h_n ($n = 1, 2$) strokes that the chop off n of the dragon's heads and t_k ($k = 1, 2$) strokes that chop off k tails, the numbers of heads and tails will be equal to $3 - 2h_2 + t_2$ and $3 + t_1 - 2t_2$, respectively. So we must find the solution to the equations

$$2h_2 - t_2 = 3,$$
$$2t_2 - t_1 = 3$$

with the least sum $h_2 + t_1 + t_2$ (of course, Ivan must choose $h_1 = 0$). It follows from the first equation that t_2 is odd; from the second equation we see that $t_2 \geq 3/2$, so $t_2 \geq 3$. Then $h_2=(3 + t_2)/2 \geq 3$ and $t_1 = 2t_2 - 3 \geq 3$, so the total number of strokes is not less than 9. The numbers $h_2 = t_1 = t_2 = 3$ satisfy our equations, but we must make sure that it's really possible to inflict 3 strokes of each sort (so that, for instance, Ivan won't have to chop off two tails when there's only one tail left). One of the possible sequences is to chop off one tail (so that two new tails grow), then two tails (one new head grows), then two heads, and then repeat this series of strokes twice—after each of the three series the dragon loses one tail and one head. (V. Dubrovsky)

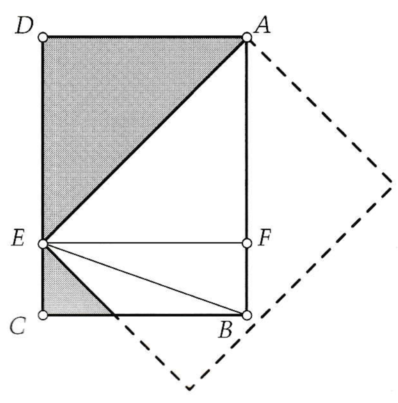

Figure 9

Neutrino

1. The energy per reaction in MeV is given by

$$(26.3 \text{ MeV})(1.6 \cdot 10^{-13} \text{ J/MeV}) = 4.2 \cdot 10^{-12} \text{ J}.$$

Therefore, the number of reactions per square meter of the Earth's surface per second is

$$\frac{1{,}377 \text{ W/m}^2}{4.21 \cdot 10^{-12} \text{ J}} = 3.3 \cdot 10^{14} \text{ m}^{-2}\text{s}^{-1}.$$

Since there are two neutrinos per reaction, the flux F of neutrinos at the Earth is twice this, or $6.6 \cdot 10^{14}$ m^{-2} s^{-1}.

The volume of the author's body is

$$V = \frac{\text{mass}}{\text{density}} \cong \frac{75 \text{ kg}}{900 \text{ kg/m}^3} = 0.08 \text{ m}^3.$$

The number of neutrinos in the author's body at any instant is therefore

$$N_V = \frac{FV}{c} \cong 1.8 \cdot 10^5.$$

2. One light-year equals $(3 \cdot 10^8 \text{ m/s})(3.15 \cdot 10^7 \text{ s}) \cong 10^{16}$ m. The number of neutrinos detected is equal to

$$N_d = N_0 \frac{A_d}{4\pi d^2},$$

where N_0 is the number of neutrinos emitted by the supernova and A_d is the cross-sectional area of the reconfigured tube:

$$A_d = \frac{V}{MFP} = \frac{4{,}000 \text{ m}^3}{10^{16} \text{ m}} = 4 \cdot 10^{-13} \text{ m}^2.$$

Therefore,

$$N_0 = N_d \frac{4\pi d^2}{A_d} = 20 \frac{4\pi(170{,}000 \cdot 10^{16} \text{ m})^2}{4 \cdot 10^{-13} \text{ m}^2} = 1.8 \cdot 10^{57}.$$

Cipollino

5. The probability that the chip makes r moves to the right and l moves to the left is equal to $\binom{r+l}{r} \Big/ 2^{r+l}$, because this happens if and only if, in a series of $r + l$ tosses of the coin, heads turn up exactly r times, whereas the number of such series is $\binom{r+l}{r}$ (see, for instance, "Combinatorics–polynomials–probability" in the March/April issue, where the notation $C(n, m)$ was used instead of $\binom{n}{m}$ here.) In the exercise, $t = r + l$, $k = r - l$, so $r = (t + k)/2$.

6. Apply the Binomial Theorem from the aforementioned article.

7. The answer to the first question is $P(k) = 1 - k/n$ ($P(k)$ is a linear function taking the values 1 and 0 at points 0 and N, respectively). The second probability is equal to $P(N - k) = k/N$.

8. The segment AB can be removed from all the graphs in figure 6 in the article, because the caterpillar sticks at its endpoints. This turns the question for figures 6a and 6b into the question of exercise 7 for $N = 3$ and $N = 4$, respectively.

If $P(k)$ is the probability that the caterpillar sticks at point A when it starts from point k, then for figure 6c, we have the following equations:

$$P(1) = \frac{1}{3}P(2) + \frac{1}{3}P(3) + \frac{1}{3} \cdot 1,$$

$$P(2) = \frac{1}{2}P(1) + \frac{1}{2}P(3),$$

$$P(3) = \frac{1}{3}P(1) + \frac{1}{3}P(3) + \frac{1}{3} \cdot 0,$$

which yield $P(1) = 5/8$, $P(2) = 1/2$, $P(3) = 3/8$. The probabilities of sticking at B are obtained by switching points A and B, 1 and 3; they turn out to be equal to $1 - P(k)$, so the caterpillar sticks somewhere with a probability 1.

Fundamentally the same solution applies to figure 6d (of course, the equations must be changed). Here $P(1) = 3/4$, $P(2) = 1/2$, and $P(3) = 1/4$.

Royal problem

1. White wins as follows:

(1)	f8	h7 (a)
(2)	f6	g8
(3)	h6	f7
(4)	h7	f8 (b)
(5)	g6	e7
(6)	g7	e6
(7)	h7	f6
(8)	g8	e7
(9)	g6	f8
(10)	h7	e8

Notes: (a) If (1) ... g6, then (2) h8 f7, and continue from (4).

(b) If (4) ... e6, continue as before. If (4) ... f6, continue from (8).

2. White wins as follows:

(1)	e5	g4
(2)	f6	h5
(3)	f4	g6
(4)	e5	h6 (a)
(5)	f5	g7
(6)	e6	h8 (b)
(7)	h6	g8
(8)	g6	h8 (c)
(9)	g5	h7
(10)	f6	g8
(11)	h6	f7
(12)	g5	f8 (d)
(13)	g6	e7
(14)	f5	e8

Notes: (a) If (4) ... f7, continue from (12). If (4) ... h7, continue from 10.

(b) If (6) ... f8, continue from (13). If (6) ... h7, continue from (10).

(c) If (8) ... f8, then (9) h7 e8.

(d) If (12) ... e6, then (13) f4 e7, and continue from (14).

3. Red wins. An article in the next issue will present an argument supporting this conclusion by solving the Royal Problem for all $m \times n$ chessboards, $m \geq n \geq 3$.

Kaleidoscope

1. Each grace had 12 pieces of fruit.

2. The three numbers on every line considered in the problem form an arithmetic sequence with the central number of the square as its second term. So the sum of each triple is three times the central number.

3. The treasury initially held 225 coins.

4. There are 24 shortest paths.

5. There are three solutions to this alphametic. The greatest EULER is given by 12325551 – 29127 = 12354678.

6. I visited St. Petersburg on February 1, Riga on February 8, Pskov on March 1, and Vladimir on March 8.

7. The sailboat was named the *Washington* and sailed from New York to Bermuda; the steamer was named the *Jefferson* and sailed from London to Boston; the motor boat was named the *Lincoln* and sailed from Newport to Halifax.

8. $7{,}744 = 88^2$.

9. The youngest child is 5 years old. The remainders of the ages of all the children when divided by 5 take all possible values, so one of the remainders must be 0. This means that one of the children is 5 years old (all the ages are prime numbers!), and the answer can be only one of the three numbers 2, 3, or 5. But $2 + 6 = 8$ and $3 + 6 = 9$ are not prime numbers.

10. The second car will overtake

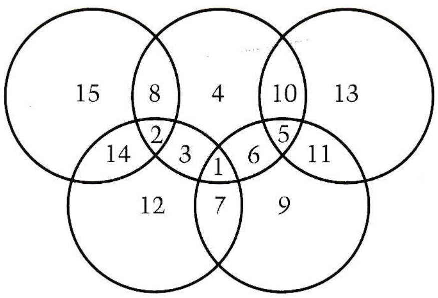

Figure 10

the first at the 150-km mark.

11. See figure 10.

12. Note that $(x^2 + x + 1)^2 = (x^2 - 1)^2 + (2x + 1)^2 + (x^2 - 1)(2x + 1)$ and apply the Law of Cosines.

Atlantic crossings

1. There is always a second pair of ships meeting 4,900 km away from the first pair.

2. Five steamers (as many as the number of zigzag graphs in figure 2 in the article).

3. In the case of daily departures, each ship would meet 29 ships during its one-way voyage (including the two ships it meets at the ports).

Round the clock. The graphs in figure 11 show that the hands meet every 12/11 of an hour. So the first moment they coincide is 12/11 o'clock = 1:05.4545... A.M.; and the first moments they form angles of 180°, 90°, and 120° are 6/11 o'clock, 3/11 o'clock, and 4/11 o'clock, respectively. So the answers are (a) $(6 + 12k)/11$ o'clock, where $k = 0, 1, \ldots, 10$ (see figure 11); (b) $(3 + 6k)/11$ o'clock, $k = 0, 1, \ldots, 21$; (c) $(4 + 12k)/11$ and $(8 + 12k)/11$ o'clock, $k = 0, 1, \ldots, 10$ (the two sequences correspond to the two directions in which an angle of 120° can be measured).

Strolling ladies. The graphs are shown in figure 12. We see that the ladies will meet three times and walk in the same direction for 6 + 2 + 2 + 1 = 11 minutes.

Sir Isaac Newton's problem. The answer is 35 miles. Using graphs you can reduce this problem to some simple geometry (considerations of similar triangles), but here the algebraic solution is just as easy.

A new face

(See the Kaleidoscope in the May/June issue)

1. No. To get a nonregular tetrahedron with congruent faces, take an arbitrary acute triangle and fold it along its midlines (fig.13) so that its vertices meet at one point. The reader may enjoy proving that the projection H of the top of this tetrahedron is the orthocenter of the original triangle.

2. The answer to both questions is yes. The height of the required pyramid must fall on the point of intersection of the extensions of two (first question—fig. 14) or three (second question—fig. 15) sides of its base.

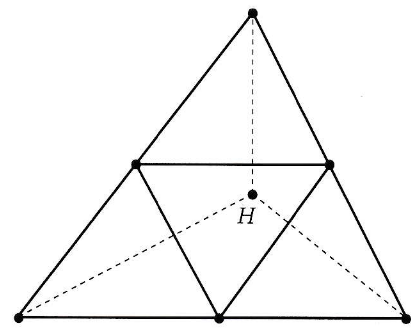

Figure 13

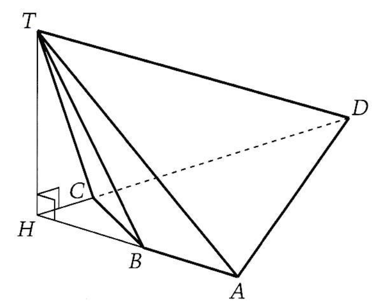

Figure 14

Quadrilateral ABCD is the base of pyramid ABCDT; H is the foot of its altitude.

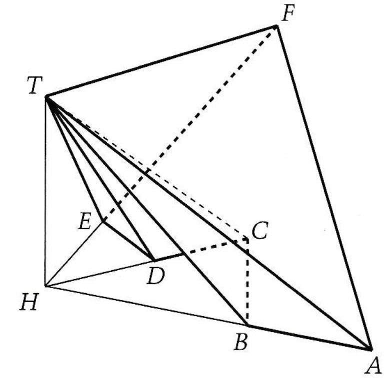

Figure 15

Nonconvex hexagon ABCDEF is the base of pyramid ABCDEFT; H is the foot of its altitude.

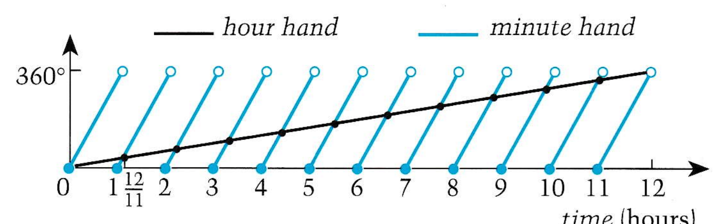

Figure 11

Plotted along the vertical axis is the angular distance to each hand from its initial position (counted clockwise).

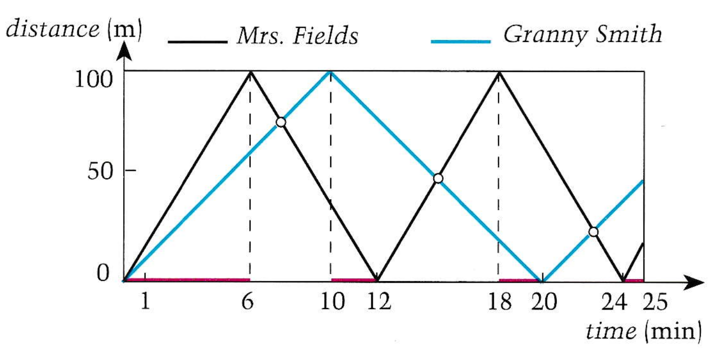

Figure 12

The time intervals during which the ladies walked in the same direction are colored red on the time axis.

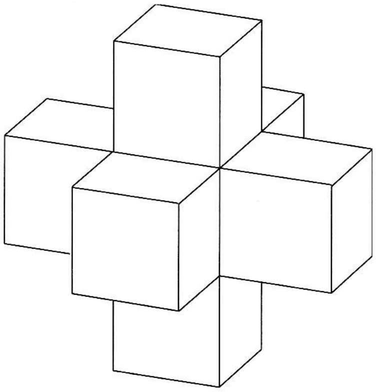

Figure 16

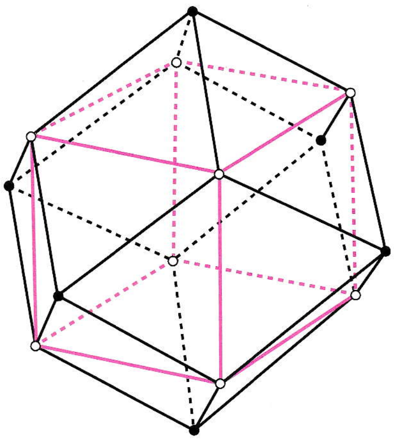

Figure 17

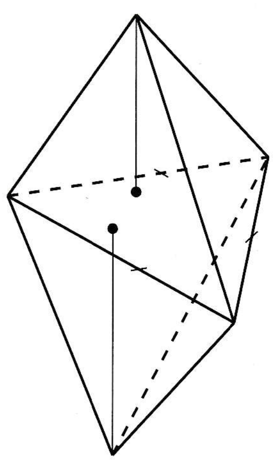

Figure 18

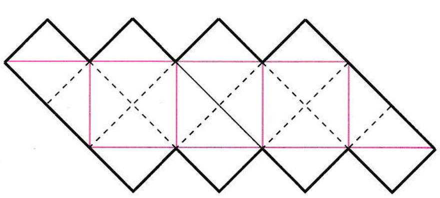

Figure 19

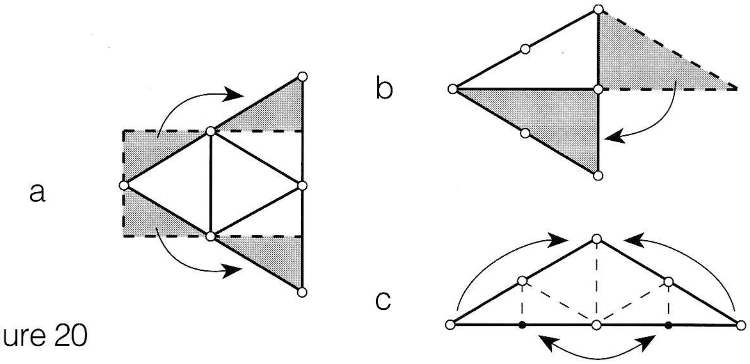

Figure 20

3. No—see the "three-dimensional cross" in figure 16.

4. By stacking two differently tilted oblique prisms with congruent bases, base to base, we'll get a polyhedron satisfying the "Oxford definition" of a prism. It's not a prism, but it will never be convex. A "convex example" is shown in figure 17. It can be constructed from the cross in the previous problem. The cross consists of seven equal cubes: we take only the central one and add to it six quadrilateral pyramids built on its faces with the vertices at the centers of the six other cubes.

All twelve faces of the resulting polyhedron are congruent rhombi. We can choose any pair of opposite faces as the "ends" in the Oxford definition: they are all "similar, equal, and parallel." The remaining faces are the "sides," and of course they are all "parallelograms."

5. Consider two irregular pyramids whose bases are congruent equilateral triangles (but whose other faces are not congruent). By rotating the base of one of the pyramids, we can fit the pyramids base to base (fig. 18) in three different ways, thus creating three different six-faced polyhedrons with the same set of faces.

6. Yes. If you bend the given figure at a right angle along all the red lines in figure 19, you'll get a cube (its bases will consist of four triangles each).

7. Figures 20a and 20b show how the given developments can be transformed into an equilateral triangle with the same rule of pasting on the border (each side must be folded in half), which yields a regular tetrahedron in both cases (compare with problem 1).

The transformations consist of cutting the given developments and pasting the pieces back together differently but, of course, following the prescribed rule for pasting. The solid lines inside the rectangle in figure 20a and the dotted lines in figure 20c are the creases along which the given polygons should be bent to form the tetrahedron.

8. See figure 21 for an example.

9. Figure 6 in the statement of the problem can't be a projection of a polyhedron, because the extended lines AB, DC, and EF must meet at one point—the point of intersection of the three planes $ABCD$, $CDEF$, and $EFBA$.

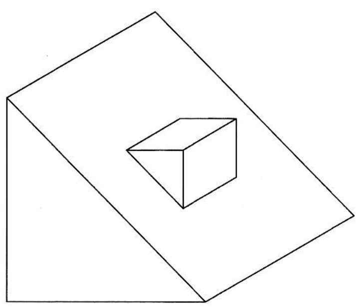

Figure 21

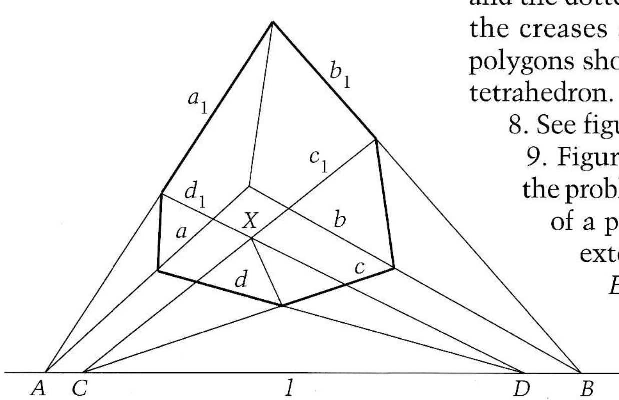

Figure 22

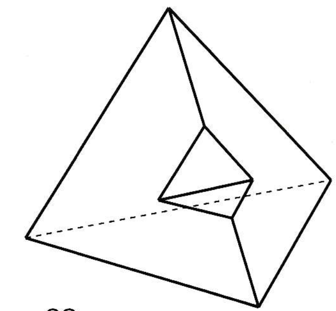

Figure 23

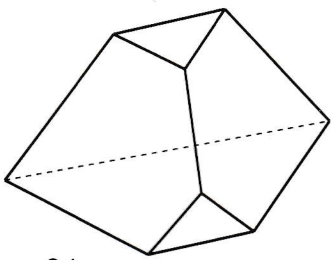

Figure 24

The correct location of point X in the second question can be found by using the construction shown in figure 22. The points A, B, C, and D, where the extended edges of the polyhedron—a and a_1, b and b_1, c and c_1, d and d_1—intersect, must lie on one straight line l, the common line of the planes $abcd$ and $a_1b_1c_1d_1$. Points A and B, and so the entire line $l = AB$, can be constructed irrespective of the position of X. Then we can find C and D at the intersections of l with c and d, draw c_1 and d_1, and locate X at their intersection.

10. See figure 23. The nonconvexity of this polyhedron accounts for its property violating the conditions of Steinitz's theorem: it has two faces with two common edges.

11. (a) See figure 24; (b) see figure 23 again.

Corrections

Vol. 3, No. 3:

p. 29, col. 1, l. 42; col. 2, l. 23; col. 3, l. 7: *for* h *read* $\hbar$.

p. 35, col. 1, l. 7: *for* L *read* ... [an ellipsis].

Vol. 3, No. 5:

p. 15, B81: *for* angles ABD and BCD *read* angles ABD and BDC.

p. 16, col. 2, l. 4: *for* k^{k^k} *read* $k^{k^k} + 1$.

p. 32: Due to an erroneous back translation (English to Russian to English), the name of the polyhedron in "Always a New Face to Show" (col. 1, ll. 6–7) was corrupted. It should read "shaddock with six beaks." These beaks are not easy to discover: they are open, and one of them, facing upwards, is formed by the two triangular faces of the shaddock in figure 1 that have a common edge AB (one of them is ABC); the five other beaks are like this one but face five other directions. One of the big diagonals of the shaddock is erroneously referred to as AB in the text (col. 1, l. 31); in fact it joins A to the vertex right below B in that figure.

p. 57, col. 3, third display equation: *for* $\Delta v/\Delta t$ *read* $\Delta x/\Delta t$.

p. 59, B84: *Quantum* reader Jonathan Wildstrom pointed out that the answer as printed is incomplete: the years 8960 and 8970 are also solutions. (The year 8950 doesn't quite make the cut—since centuries begin with year 1, 8950 falls within the first half of the 90th century.) In our answer we had assumed that the year in question had already passed.

p. 61, col. 2, ll. 24–25: *for* Writing $f(x^2) = f(x^2 + f(y))$ and substituting *read* Writing $f(x^2) = f(x^2 + f(0))$ and substituting $-x$ for x.

p. 61, col. 2, l. 27: *for* $f(-x) \neq f(x)$ *read* $f(-x) = f(x)$.

p. 61, col. 2, ll. 42–43: *for* equation (1) *read* the original equation in the statement.

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The International Student Fairs & Conferences are scheduled at McCormick Place in Chicago, March 24–27, 1994, and at the Boston World Trade Center, October 21–23, 1994. For more information on attending, exhibiting, or submitting a "Call for Presentation," write to Kaureen Duffy, Communication Alliance of the Americas, Inc., 11435 South Bell, Chicago, IL 60643, or call 312 445-2221.

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TOY STORE

Flexland revisited

Still new forms of flexlife!

by Alexander Panov and Anatoly Kalinin

MORE THAN A YEAR AGO we took *Quantum* readers on a trip through Flexland, an exotic country of tricky flexible and transformable creatures, many of which can be modeled with paper and glue.[1] Recently we received a letter from that funny Flexlander, Mr. Flexman himself. You may recall that he was our guide on our last trip. He tells us he's a regular reader and admirer of *Quantum* and considers this magazine the best publication outside Flexland. He thinks any magazine, as a physical object, is a primitive form of a flexagon, so the very existence of the magazine business on Earth suggests long-standing connections between Flexland and this part of world. (Mr. Flexman even thinks that the word "magazine" should be replaced by "flexazine.")

Mr. Flexman was pleased to read our reports about the Flexland tour and was especially impressed by the description of the "ring of tetrahedrons" (July/August 1992). Spurred on by this favorable impression, he published a short paper in his own magazine *Flexum* in which he conjectured that there should exist other as yet unknown ring-shaped forms of flexlife. He gave a hypothetical description of each and delineated their areas of distribution. Soon he became the head of a special expedition that scored a brilliant success: new life forms were discovered! Mr. Flexman compares his discovery to that of Adams and Leverrier (who'd predicted the existence of a planet later called Neptune), and to the great achievement of Mendeleyev (who predicted the existence of previously unknown chemical elements). Actually, he places his own discovery even higher, because he had not only made a fundamental prediction, he confirmed it on his own.

One of the newly discovered flexrings—a ring of triangles and some of its transformations—can be

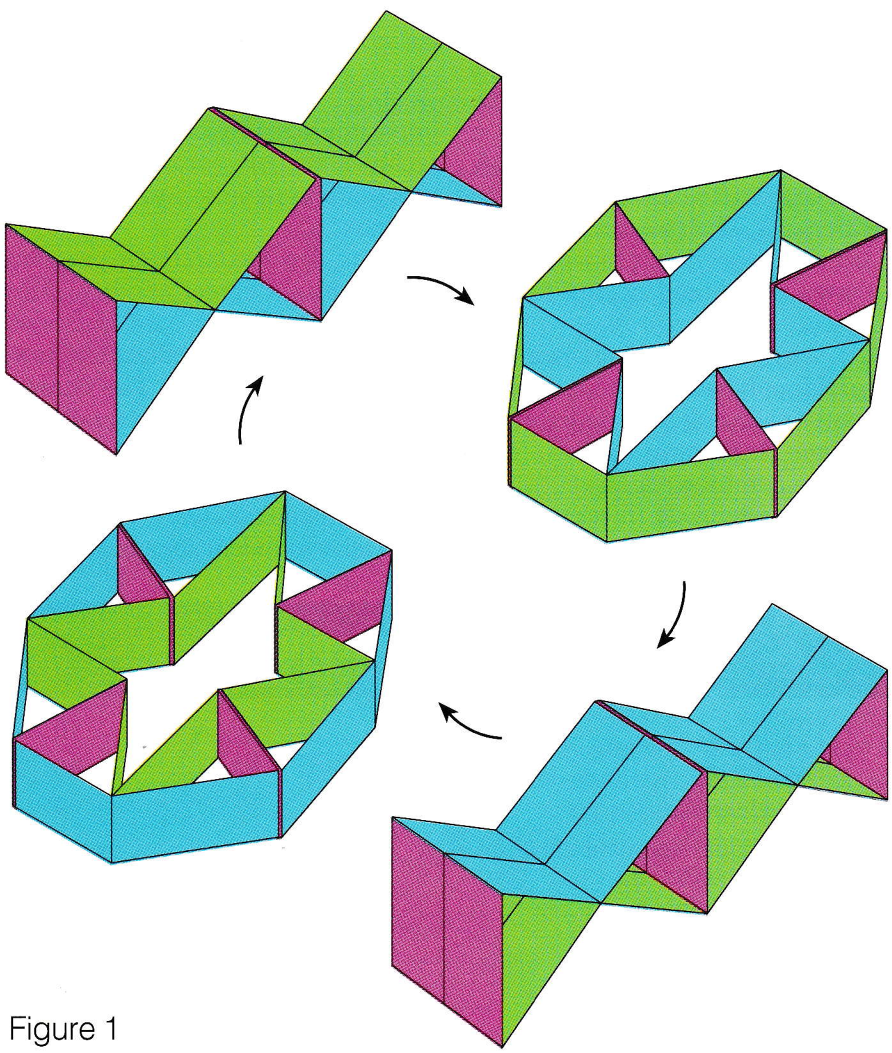

Figure 1

[1]See the Toy Store installments in issues 4, 5, and 6 of volume 2.

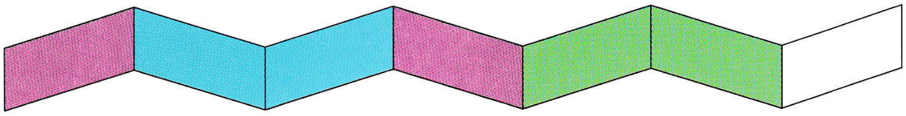

Figure 2

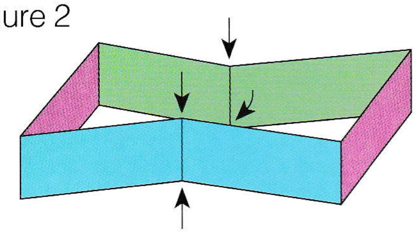

Figure 3

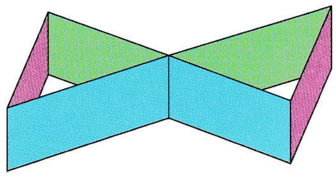

Figure 4

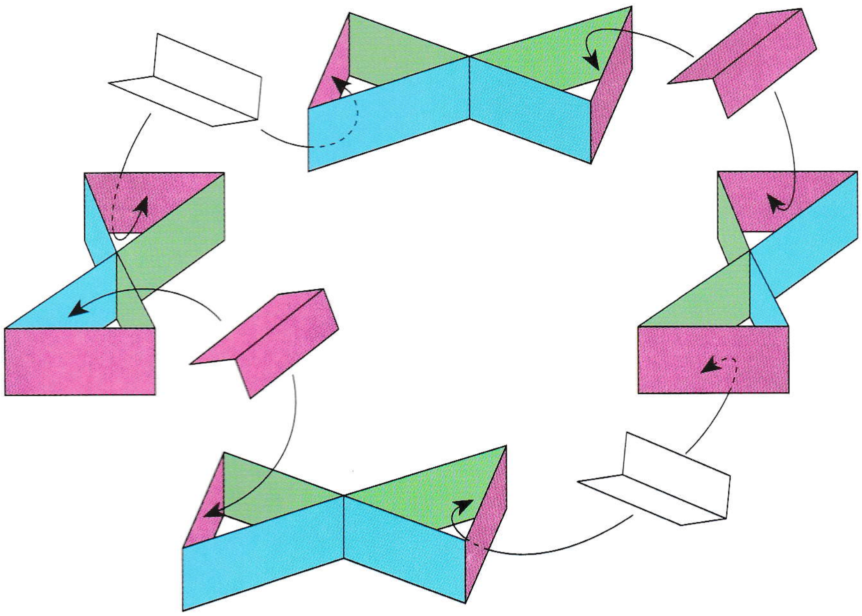

Figure 5

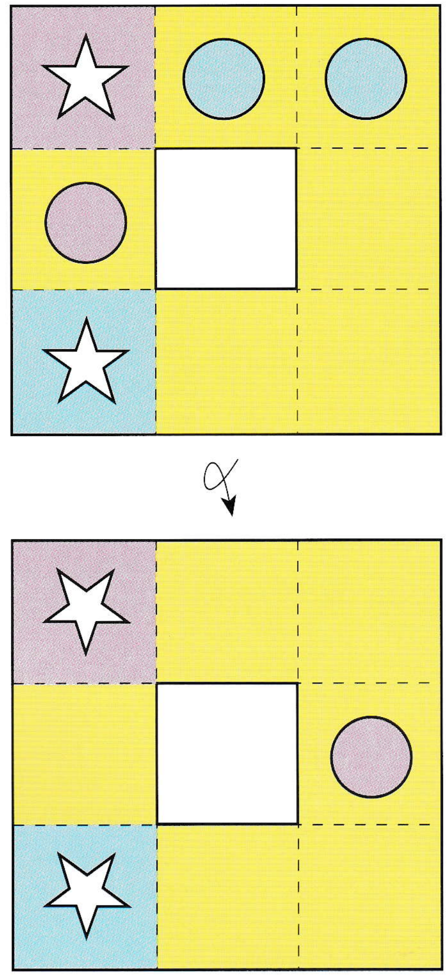

Figure 6

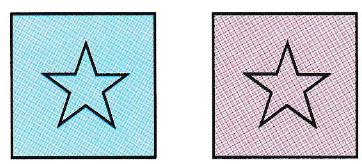

Figure 7

seen in figure 1. To make it out of paper, cut out four strips measuring 3 cm × 42 cm and fold them into seven 3 cm × 6 cm rectangles, as shown in figure 2. Glue the rectangles together at the ends of the strip (fig. 3) and make little cuts along the creases as the arrows indicate. Pass a string through the cuts, draw it tight, and tie it up (fig. 4). You get a flexible piece consisting of two triangles (or, more exactly, triangular prisms having a common edge). Then put the four pieces together using four connecting strips (measuring 5 cm × 5 cm) and glue (or cellophane tape), as shown in figure 5. The flexring is ready for you to play with.

Mr. Flexman classifies this representative of flex fauna with the ring of tetrahedrons mentioned above and conjectures that they have a common ancestor. But the "flexquare" shown in figure 6 must be attributed to another family of flexrings. It's very simple in structure. To make its paper model, take a square measuring 3 × 3 units, cut out a unit square in the center and two star-shaped holes at the adjacent corners, and color the model on both sides according to figure 6. The star-shaped holes should fit inside the colored circles. Note that the positions of the holes determine the way the flexquare is flipped in going from the top of figure 6 to the bottom.

Mr. Flexman's expedition discovered a remarkable property of the flexquare: it can fold itself into a unit square that is either blue or red on both sides, depending on the weather (fig. 7). Mr. Flexman says he asked several puzzle experts to figure out how the flexquare folds itself. At first glance, without touching the model, they usually presume that this must be quite easy. But when they spend some time trying to fold the thing, they usually blurt out that there must be some mistake, that the problem is unsolvable. Only after being reassured that the solution exists do they take up the challenge in earnest, and it generally doesn't take them much time to master it.

Mr. Flexman asked us to offer the flexquare puzzle to *Quantum* readers, which we hereby do with great pleasure.

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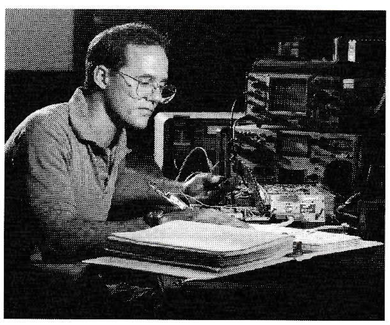

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