QUANTUM

JANUARY/FEBRUARY 1991

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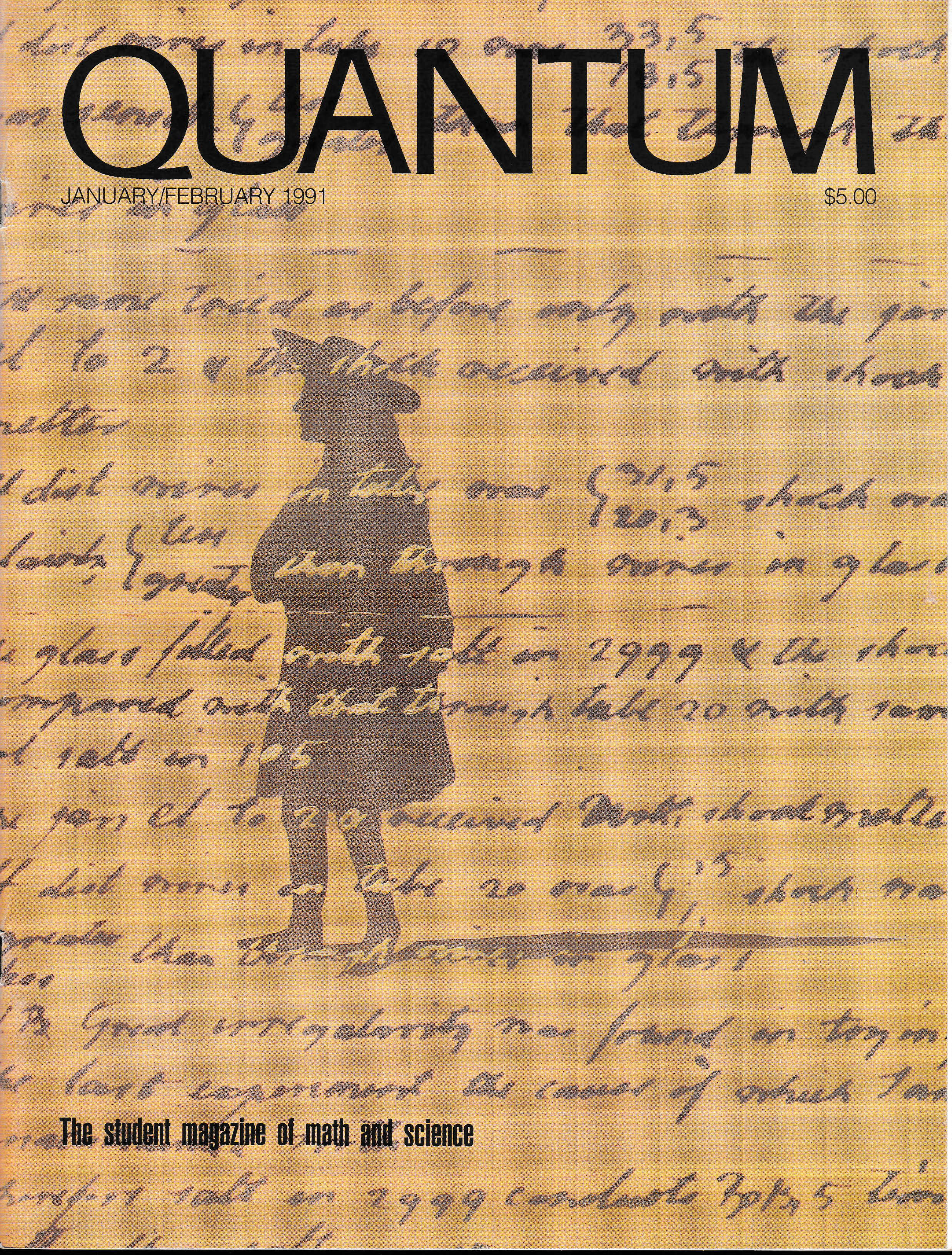

The student magazine of math and science

GALLERY Q

National Gallery of Art, Washington (Chester Dale Collection) © NGA

The Sacrament of the Last Supper (1955) by Salvador Dalí

The name Salvador Dalí (1904–1989) tends to conjure up minutely detailed dreamlike images, visual tricks, and philosophical statements in paint. Maybe you've seen "The Persistence of Memory," with its ominously bleak landscape and strangely limp watches, or other beautiful but unnerving pictures by this Spanish Surrealist.

So it may come as a surprise that many of Dalí's most important paintings during two decades in the prime of his life—1950 to 1970—were concerned with religious themes. "The Sacrament of the Last Supper" is a modern treatment of a traditional theme—you might find it interesting to compare this vision of the scene with Leonardo da Vinci's or Tintoretto's (both of which can be found in recent editions of the *Encyclopædia Britannica* in the article "Visual Arts, Western").

One curious aspect of Dalí's painting is the use made of the "golden section," a proportion that has been considered aesthetically pleasing since antiquity. The Renaissance mathematician Lucas Pacioli defined this ratio as the division of a line so that the shorter part is to the longer as the longer is to the whole (approximately 8 to 13). His treatise was entitled *Divina proportione*, and so the name "divine proportion" came to be applied to this ratio as well.

After you've read more about the golden section in the Kaleidoscope, come back to Gallery Q and see if you can figure out how Dalí used this technical device in his painting.

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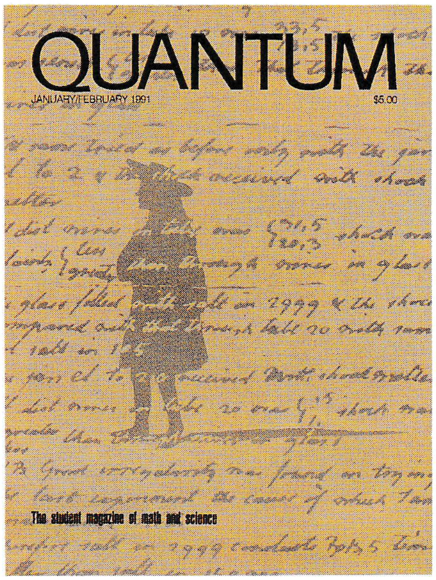

FEATURES

DEPARTMENTS

When you read about the eccentric 18th-century English scientist Henry Cavendish, whose rumpled silhouette graces our cover, you're simply amazed at how much he anticipated. This is a way of describing discoveries that remained unknown and were rediscovered by others: Cavendish's work in electrostatics "anticipated" Coulomb's, his work on the capacity of condensers "anticipated" Faraday's, and so on. While some scientists seem in a rush to publish even dubious results, Cavendish represents the opposite extreme. A brilliant experimenter, he all too often was content to hide his light under a bushel basket. It's said he detested competition and cared not a whit for fame, but it could be argued that he simply wasn't on good terms with the human race! He died a wealthy man, yet curiously enough he didn't leave a penny to science (an "oversight" corrected by his descendants, as you'll see when you read the portrait of Cavendish in Looking Back, page 41).

Here's a shocking bit of anticipation: to establish how electrical potential is related to current (which he found to be directly proportional, as would George Simon Ohm years later), Cavendish used his own body as a meter! He would grab the ends of the electrodes and estimate the strength of the current by feeling how far up his arms the shock went: fingers, wrists, elbows . . . Now *that's* hands-on science!

QUANTUM

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Happy New Year!

I made a resolution: to learn Russian!

IT'S BEEN A LONG TIME SINCE I had to make a concentrated effort to learn something new—actually, not since college. For those of us who teach or work as I do in an administrative job, we don't always appreciate the difficulties and stresses that students face in learning under pressures of time and high expectations. This is especially true of those of you who read *Quantum*, since you're probably under more pressure than most students. Well, my young friends, I am joining you in your learning miseries and pleasures. I'm trying to learn Russian, and I have only until July 27, 1991, to achieve my now public, self-imposed deadline for speaking, reading, and writing Russian at some reasonable level of literacy.

In my three trips to Moscow, I've always felt inadequate and helpless, not knowing even one word of Russian. Our Russian colleagues have always provided interpreters and transportation, but there's a limit on the extent to which we should impose ourselves on their generosity. Also, almost all of my friends there speak English. So why don't we learn Russian?

I've purchased audio tapes, books, and other materials, and I'm spending my substantial travel time on airplanes listening to the tapes and muttering incomprehensible Russian sounds (I hope), while my fellow travelers shake their heads in dismay or express their curiosity. I've asked our consultant Ed Lozansky to purchase books that are used by children in the USSR at the preschool age and in grades 1–4 of their schools. I thought that by starting where young children start, maybe it will be easier to learn the language. For a person who has studied such wonderful and advanced topics as quantum electrodynamics and theory of functions of a complex variable and integral equations, it's a pleasantly humbling experience to be struggling to do what young Russian children manage with great ease.

Actually, it's really fun to be engaged in learning something new at my age, and I intend to achieve my goal. Our Russian friends, of course, will have to judge the extent to which I succeed. I've also sent Russian language tapes to NSTA's president, Bonnie Brunkhorst, to our president-elect, Lynn Glass, and to the chairman of our international committee, John Penick. By this column, I am putting a little gentle pressure on them to make a similar effort, so that all of us have at least tried to prepare ourselves for July.

And why July? That's when we're holding the first Russian–American science education convention, which will take place at Moscow State University. More than 200 Americans have already signed up for the convention, and we expect that number to increase to the planned 500–600. An equal number of Russian science teachers will be there, so it will be a fine opportunity to learn about science education from each other and to gain many new friends.

If you've never considered learning Russian, you should do so. I've only been at this task for two weeks or so, but it's very interesting, although at times difficult. One of the features of the Russian language that I really like and find helpful is the alphabet and the sounds associated with each letter. Many of the Cyrillic letters are from the Greek alphabet, which, as you know, you learn in mathematics—for instance, π, ρ, Γ. Other letters are common to the English alphabet. But most importantly, the phonetic sounds for the letters are almost always the same from word to word, so it's quite easy to sound out words. I've already found that I can spell a Russian word in Cyrillic when I hear the sounds. Being able to spell the word or sound it out will be very helpful as I try to learn to read the language. All in all, studying Russian is fun, but sometimes it's discouraging when I forget things I just learned. Also, some Russian sounds aren't used in English, so you have to get your lazy mouth to try some new gymnastics, which can be quite a challenge.

When I skip ahead in the textbooks or look at the grammar rules, or when I try to read a copy of *Kvant* given to me by our Soviet colleagues, I realize how far I have to go. When you're a student, looking ahead like that can be very intimidating. Yet we can't let that discourage us. We have to just take it day by day, one part at a time, and sooner or later, with hard work and perseverance, we find ourselves there at the end of that complicated material where we thought we'd never be. We almost wonder how we got there. (*Almost.*)

It's in this spirit that I'm going to learn Russian, and it's in this spirit that you should keep studying and learning mathematics and science. You'll be surprised someday at how much you were able to learn. Keep at it. By the way, go ahead and learn a language or two—and why not let Russian be one of them?

We at *Quantum* wish all our readers the best in 1991. May it be peaceful and prosperous for all of us, all over the world.

—Bill G. Aldridge

С новым годом!

Diamond latticework

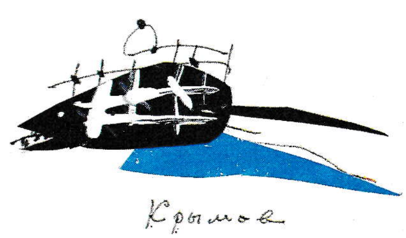
The fearful symmetry of crystalline structures

by R. V. Galiulin

THIS ARTICLE IS DEDICATED to two anniversaries: the centenary of the prominent Russian mathematician B. N. Delone (1890–1980), who made a decisive contribution to mathematical crystallography, and the centenary of this branch of mathematics itself, which was born when the pioneering work of E. S. Fyodorov and A. Schoenflies was published in 1891.

The extraordinary geometric perfection of crystals has amazed the human mind since time immemorial. Our ancestors saw them as either the creations of angels or the products of subterranean evil forces. The first attempt to provide a scientific explanation of crystalline form was given by Johannes Kepler in his work "On Hexagonal Snowflakes" (1611). Kepler suggested that the shape of snowflakes (crystals of ice) is due to the special positioning of the particles composing the crystal. Three centuries later it was finally established that the properties of crystals are due to the special arrangement of atoms in space similar to the patterns we observe in kaleidoscopes. These types of arrangement were classified in 1891 by E. S. Fyodorov (1853–1919), a Russian scientist and founder of modern crystallography. The regular forms of crystalline polyhedrons are easily explained within the framework of his classification.

From the geometrical point of view the positioning of atoms in space is defined by the system of points corresponding to their centers. So the problem can be formulated like this: what are the geometric conditions that distinguish systems of points with "crystalline structure" from all other systems? Since our goal is to find the reasons for regularity in the position-



Art by Dmitry Krymov

ing of real atoms in real crystals, special attention should be paid to physical motivation. The simplest geometric property of a system of points corresponding to atomic centers in any atomic array (and not only in crystals) is its discreteness.

DISCRETENESS CONDITION. *The distance between any two points of the system is greater than a fixed value r.*

The physical meaning of this condition is obvious. The tendency of atoms to spread uniformly in space can be expressed by the following restriction on the corresponding system of points.

COVERING CONDITION. *The distance from any point in space to the nearest point of the system is less than a fixed value R.*

The name of this condition stems from the following fact: if a system of points complies with it, the set of spheres of radius R with centers at these points covers the whole space. (Prove it!) The discreteness condition doesn't allow the points of the system to be spread too densely, while the covering condition outlaws too thin a distribution. Taken together, they ensure the approximately homogeneous distribution of points in space. Systems of points satisfying both conditions simultaneously are called *Delone systems* after B. N. Delone,[1] the Russian geometer who first introduced them.

The simplest example of a Delone system (in a plane) is provided by a set of nodes on an infinite sheet of graph paper. Similar systems play an exceptionally important role in crystallography, and later we'll consider them in more detail. This system can be used to obtain a Delone system of a more general type by giving each node an arbitrary shift not greater than, say, one third of the distance between adjacent nodes (fig. 1).

Exercise 1. Prove that such a system of points satisfies both the discreteness and covering conditions; find the corresponding values of r and R.

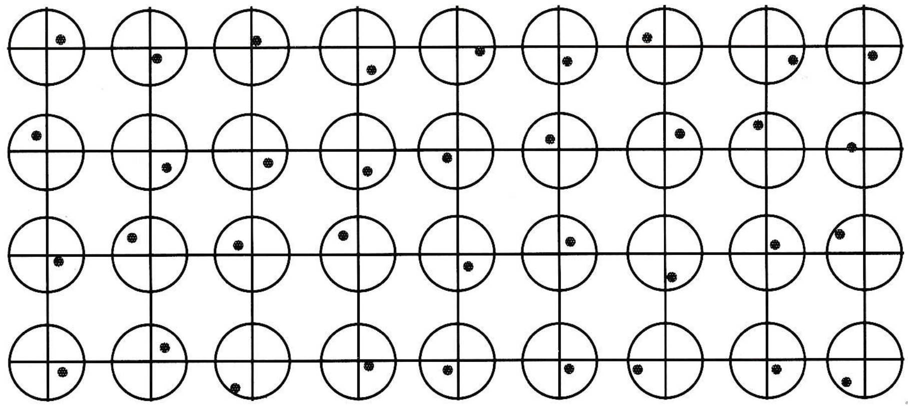


Figure 1

Delone systems provide the most general geometric model of distribution of atoms in any atomic structure. So any theorem about these systems can be interpreted as a property of the structure itself. This makes the theory of Delone systems especially important in various applications. But the general theory of Delone systems (which is still in its early stages) isn't the subject of our story. We'll consider only particular cases: systems describing the position of the centers of atoms in crystal structures. Such systems are distinguished by the primary geometrical property of crystals: their symmetry.

What is symmetry? Intuitively, it's not difficult to distinguish a symmetric pattern from a nonsymmetric one. A symmetric body can always be divided into equal parts, sometimes in many different ways. This property alone, however, isn't enough to guarantee symmetry in the pattern. A heap of bricks isn't symmetric though it consists of identical bricks. Even the brick wall in figure 2a doesn't appear very symmetrical, especially when compared to the bricks in figure 2b. To make the intuitively felt difference between the two walls clearer, consider the bricks surrounding any one of them. In figure 2b any two bricks have identical surroundings, whereas in figure 2a this is true only for the bricks in the same row.

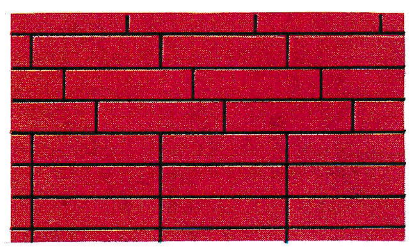
By the "equality" of two figures we mean here that one of them can be superimposed onto the other after an *isometry*—that is, any transformation of the plane that preserves the distance between any two points. An isometry that takes a figure or a pattern into itself is called its symmetry, and a figure or pattern that allows at least one symmetry (other than identity) is said to be symmetric.

For instance, the masonry in figure 2a goes into itself only if translated along the rows by a number of brick lengths, while the masonry in figure 2b allows many other symmetries: vertical translations, half-turns, and also line reflections followed by translations along the reflection axis. (Find all of them!) So both patterns are symmetric, though the second one is "more symmetric" than the first.

Exercise 2. (a) Find all symmetries of a regular n-gon. (b) Prove that a cube has 48 symmetries (including reflections) and find them.

The set of all symmetries of an object together with the operation of their composition is called the *sym-*

a



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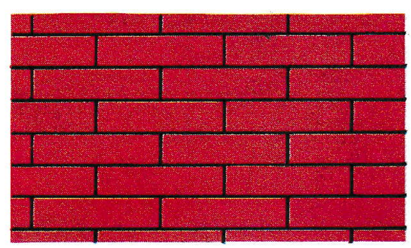

Figure 2
Brick walls: (a) less symmetric, (b) more symmetric.

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[1]Pronounced "deh-law-NAY."—*Ed.*

metry group of this object. This is a very important notion in mathematics, lying at the boundary between geometry and algebra.

So one way of ensuring that a Delone system describes a crystal is to require that it be symmetrical. Another way to describe them is to use the "equal surrounding" notion—that is, join an arbitrary point A of a Delone system to all its other points (fig. 3). You get an infinite set of intervals called the *global star* of point A. In the general case global stars of different points are not equal to (congruent with) each other. If there are at least two points with congruent global stars, however, the system is symmetrical. The converse statement is also true: any symmetrical Delone system has points with the same global stars. So the congruence of global stars for at least two points is a necessary and sufficient condition for a Delone system to have a symmetry.

Exercise 3. Prove that any symmetrical Delone system has an infinite number of pairs of points with congruent global stars.

Exercise 4. Construct a Delone system in the plane that fits onto itself after rotation through (a) 90° (b) 60°.

Exercise 5. Prove that if the rotation through angle α is a symmetry of a plane figure, the rotation through the angle $n\alpha$ (where n is any integer) around the same center is also a symmetry. If the figure is a Delone system, the ratio α/π must be a rational number.

Nevertheless, it is not true that any symmetric Delone system corresponds to a system of atomic centers in a crystalline structure. The symmetry of crystals is a special one. For example, there are no regular dodecahedrons, or icosahedrons, or any polyhedrons with symmetry axes of the fifth order (that is, taken into itself after a rotation through $2\pi/5$ around this axis) among the crystalline polyhedrons. Why are crystals so picky about the shapes they take?

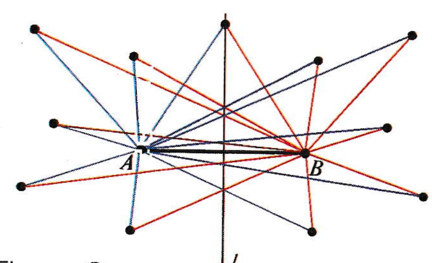


Figure 3
Part of a symmetric Delone system. Its only nonidentical symmetry transformation is the reflection in line l. Points A and B have congruent global stars.

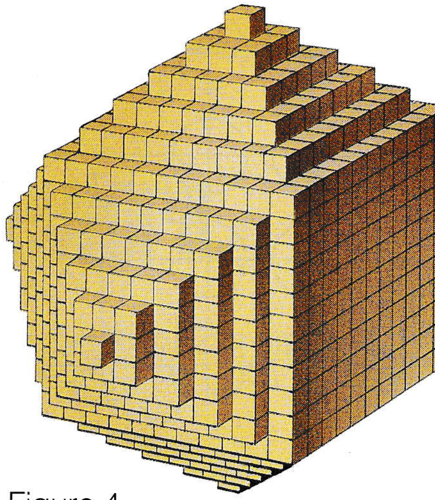


Figure 4
Crystal structure according to R. J. Haüy.

In 1783 R. J. Haüy, a French abbot and mineralogist, suggested that a crystal is made of equal parallel particles touching one another along their entire facets (fig. 4). In 1824 L. A. Seeber, a professor of physics at Freiburg (and a student of great Carl Friedrich Gauss), proposed that Haüy's polyhedrons be replaced with their centers of mass in order to explain the thermal expansion of crystals. Such systems of points have been called lattices.

More precisely, a *lattice* is defined as the set of all points having integer coordinates with respect to an arbitrary (not necessarily rectangular) coordinate system (fig. 5a–5c). The points of the lattice are called *nodes*. Each coordinate system defines a unique lattice. The converse statement is not true: there are an infinite number of ways to choose a coordinate system determining a given lattice (fig. 5b). One can easily check that each lattice satisfies both the discreteness and the covering conditions and is, therefore, a Delone system.

Let's prove they're symmetrical. The following lemma holds.

Lattice Lemma. *A lattice goes into itself under a parallel translation along the vector connecting any two of its nodes as well as under the central symmetry with respect to any node.*

To prove the first statement, notice that for any pair of nodes A and B of the lattice, vector **AB** has integer coordinates (since it's equal to the difference between the respective coordinates of points A and B). A transformation along this vector is equivalent to adding integers (coordinates of the vector) to the coordinates of each node. The resulting coordinates are again integers. So each node matches a node of the same lattice. I'll leave it to you to come up with a proof for the case of central symmetry.

It's the lattice structure of crystals that makes their symmetry so special. Any spatial lattice can (in an infinite number of ways) be divided into an infinite number of congruent and parallel plane sublattices (fig. 5c). It's usually assumed that the planes of all the faces of a crystal contain the plane sublattices of one and the same three-dimensional lattice. Plane sublattices of a three-dimensional lattice related by symmetry transformations are identical in their structure. When a crystal grows, all its faces corresponding to such plane sublattices grow similarly, so that the symmetry of the crystal repeats the symmetry of the lattice.

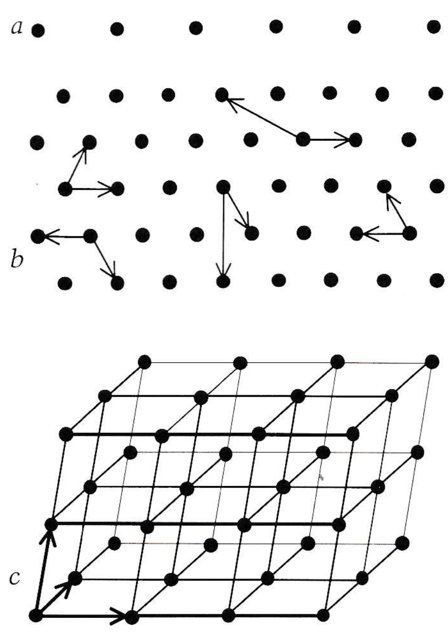


Figure 5
Types of lattices: (a) one-dimensional, (b) two-dimensional (plane), (c) three-dimensional. Arrows show the base vectors of coordinate systems defining the lattices.

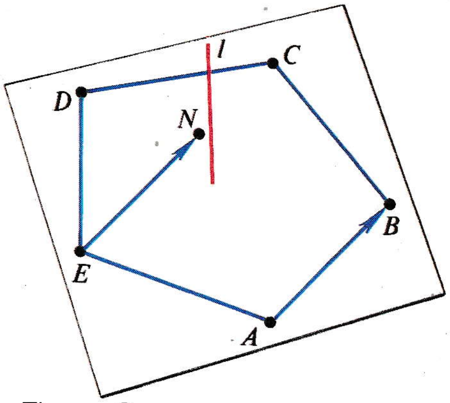


Figure 6

Now we'll prove that no crystal has a symmetry axis of the fifth order. Let's assume that such a crystal exists. Then the lattice corresponding to it also has a fifth-order axis l. Draw a plane perpendicular to l through any node and choose a node A in it that is nearest to l (the existence of such a node follows immediately from the discreteness condition). Since the lattice fits onto itself after a rotation through any angle that is a multiple of $2\pi/5$ around the axis l, all the images of point A under those rotations are also nodes of the lattice. They form a regular pentagon $ABCDE$ (fig. 6). If we now shift the lattice along the vector $\mathbf{AB}$, then (according to the lattice lemma) node E fits onto a node N lying inside the pentagon closer to l than A, thus contradicting the choice of A.

Exercise 6. Construct lattices with symmetry axes of the second, third, fourth, and sixth orders. Prove that no lattice has a symmetry axis of an order higher than six.

It should be pointed out that fifth-order symmetry axes are quite common in the realms of plants and small organisms (viruses). In the vivid words of N. B. Belov (1891–1982), a famous Soviet crystallographer: "For small organisms, a fifth-order axis is a special tool in their struggle for survival, safeguarding them from crystallization and fossilization, the first step toward which would have been the 'capture' of the organism by a crystal lattice."

But not all the facts known about crystals fit the lattice model. For example, there are crystalline polyhedrons like those of the precious stone tourmaline (fig. 7) that have no central symmetry, whereas the lattice lemma

Boris Nikolayevich Delone (1890–1980) was a famous Russian mathematician. He got his family name—Delone—from his French great-great-grandfather, the Marquis de Launay. This Marquis de Launay was the governor of the Bastille on July 14, 1789, when a furious mob marched against the royal fortress on the first day of what was to become the French Revolution. The governor and his small garrison bravely resisted but eventually surrendered to the overwhelming opposing forces after being promised a pardon and immunity. The promise, however, was easier to give than to honor, and the marquis was later massacred by the mob while being escorted through the streets of Paris.

His son, Pierre de Launay, entered on a military career, served as an officer in Napoleon's army, took part in Napoleon's Russian campaign, and, after being taken prisoner, stayed in Russia for the rest of his life.

The classic work of his worthy descendent on the theory of third-order Diophantine equations, which Delone wrote when he was still quite young, marked a breakthrough in number theory, the greatest achievement since the renowned work of Euler and Lagrange on second-order Diophantine equations. He also obtained profound and important results in algebra, geometry, and the application of mathematics in crystallography. "Because of his constant preoccupation with crystals," Delone used to say, "a crystallographer is, in his intuition, already a geometer."

Delone enjoyed spending his time with young students and organized the first mathematical olympiads for high school students in the USSR. He had an exceptionally wide range of interests. One of his childhood friends in Kiev was Igor Sikorsky, the renowned aircraft designer. They built gliders together, and Delone flew them. He was also a famous mountain climber, and a mountain summit in the USSR bears the name "Delone peak."

The last years of his long and exceptionally fruitful life were clouded by the tragic story of his grandson, Vadim Delone, a talented poet. Vadim was an active member of the human rights movement and was one of the seven people who in August 1968 made a desperate attempt to stage a demonstration in Moscow's Red Square to protest the Soviet invasion of Czechoslovakia. All of them were arrested and convicted. After serving his sentence, Vadim Delone emigrated back to France. He died in Paris, but at much a earlier age than his great-great-great-great-grandfather.

The wheel of history had taken an ironic turn.

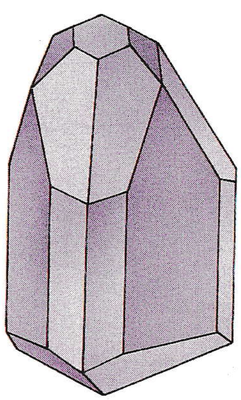


Figure 7
Tourmaline crystal.

implies that all lattices have central symmetry. To cover such phenomena it was necessary to expand the palette of allowable distributions of particles in space. In 1879 L. Sohncke, an eminent German crystallographer, suggested that particles in crystals are organized in *regular systems*.

A Delone system is called "regular" if it looks the same from any point in the system—that is, if global stars of all points in the system are congruent (fig. 8a–8c). If you were asleep and were taken from one point in a regular system to another, you wouldn't notice any change upon waking up. In other words, any point in a regular system can be taken into any other by a symmetry transformation of the whole system. The symmetry groups of three-dimensional regular systems are called *Fyodorov* or *spatial crystallographic groups*. There are 230 Fyodorov groups. (On a plane there are only 17 crystallographic groups.) It is these groups that describe the distributions of atoms in crystal structures that were mentioned at the beginning of this article.

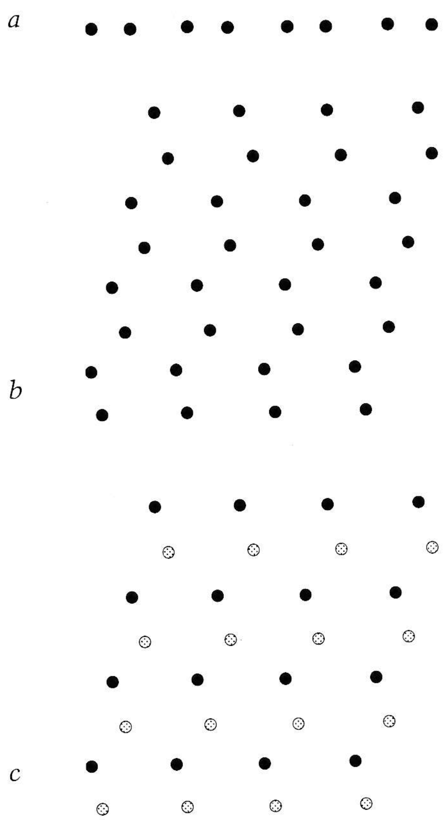


Figure 8
Regular systems of points: (a) one-dimensional, (b) two-dimensional, (c) decomposition of a regular system into lattices.

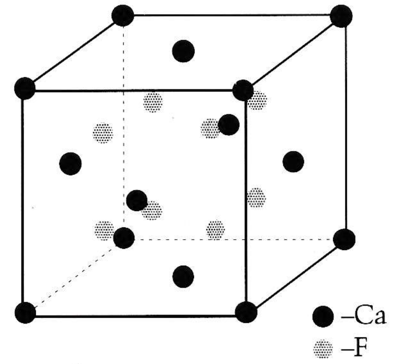


Figure 9
Structure of fluorite, composed of two different regular systems (the gray dots are fluorine atoms, black are calcium).

Exercise 7. The plane can be paved with congruent regular triangles without gaps or overlaps. Prove that their vertices form a regular (plane) Delone system and describe all its symmetries. Do the same for squares and regular hexagons. Are all these systems lattices?

The lattice lemma implies that any lattice is a regular system. The converse is not true, but it can be shown that any regular system is composed of congruent parallel sublattices (fig. 8c). An outline of the proof of this fact (which is not at all simple) was given by E. S. Fyodorov in his classic book *Principles of the Study of Figures*, which he began working on when he was only 16 years old. The proof was completed by A. Schoenflies, but it was so complicated that in the first edition of his work on symmetry of crystal structures (1891) he placed it at the very end of the book in order not to scare readers away.

At the beginning of this century experimental evidence confirmed that atoms in crystals form one or several regular structures with a common Fyodorov group (fig. 9). But these observations didn't explain why atoms in crystals are arranged in an ordered way. They only reflect the fact that such an ordering does exist. This was pointed out by the founder of Soviet crystallography, A. V. Shubnikov (1887–1970): "We have a good understanding of the way in which crystals are built, but the question of why are they built in such a way has never been seriously considered."

Imagine a growing crystal at a stage when the next atom gets included into its structure. What causes this atom to occupy its strictly predetermined place? In order not to break the system's regularity (in the sense of the definition given above) this atom should "know" and "take into account" the positions of all the other atoms, including the most distant ones. It's much more natural to require that for any atom, all the atoms lying at a relatively small distance from it (this distance defined by the effective range of chemical forces) form the same surrounding system. The fact is that even such a loose condition ensures that the system is regular! The following theorem is valid.

LOCAL THEOREM. *If all the points of a Delone system have "equal surroundings" within a sphere of radius kR, where $k = 4$ for a plane system and $k = 10$ for a spatial one, the system is regular.* (Recall that R is the parameter given by the covering condition).

This theorem was proved by B. N. Delone and his colleagues. There are good reasons to suggest that one can take $k = 4$ in the three-dimensional case as well, but no proof of this has yet been given.

The fundamental importance of the local theorem lies in the fact that the "equal surroundings" domain in

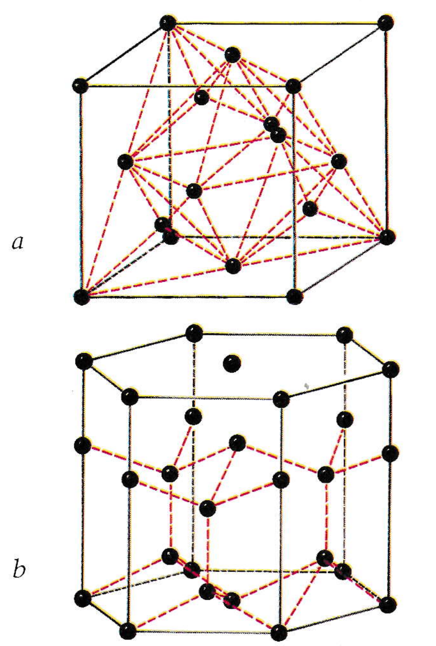


Figure 10
Structure of (a) diamond, (b) lonsdalite.

its statement is approximately the same as the effective range of chemical forces acting between atoms. So the regular structure of crystals can be explained in terms of chemical interaction between the atoms.

Now we can formulate the third natural condition that (together with the discreteness and the covering conditions) distinguishes regular Delone systems.

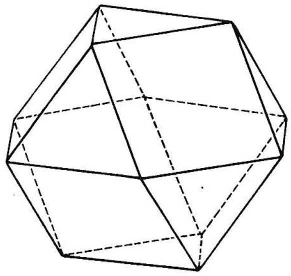
LOCAL EQUALITY CONDITION. *All the points of a system have equal surroundings within a sphere of radius 10R.* (Recall that the number 10 can probably be replaced with 4.)

Let's take a diamond crystal as an example. What happens if the equal surroundings domain gets smaller?

The closest neighbors to each carbon atom in a diamond structure are four other carbon atoms forming a regular tetrahedron (fig. 10a). This is in good agreement with the structure of carbon's electron shell, which is capable of providing four equivalent bonds. The same surrounding structure (the four closest atoms forming a regular tetrahedron) is observed in another modification of carbon—lonsdalite (fig. 10b), the microcrystals of which have thus far been found only in the craters of large meteorites.

So what's the difference between the structures of diamonds and lonsdalite? In diamonds the atoms lying on the second sphere surrounding the initial atom (called the second coordinate sphere) form an Archimedean cubic octahedron—a cube with truncated corners (fig. 11a). In the lonsdalite structure the atoms of the second coordinate sphere form a so-called hexagonal cubic octahedron, which can be obtained from the Archimedean cubic octahedron by rotating its lower half through 180° (fig. 11b). Under the condition that carbon atoms have equal surroundings on both the first and second coordinate spheres, the resulting crystal structure is one of these two pure types of monocrystals.

a



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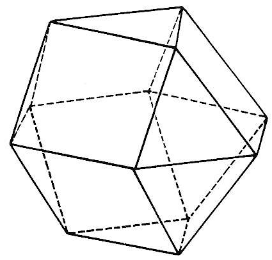


Figure 11
Second coordinate spheres in (a) diamond, (b) lonsdalite.

If carbon atoms are capable of establishing bonds only within the first coordinate sphere (that is, of forming regular tetrahedrons), then mixed structures can arise in which diamond layers are sandwiched between layers of lonsdalite. This happens in so-called twins (fig. 12), in which two diamond crystals are connected to each other by a layer of lonsdalite.

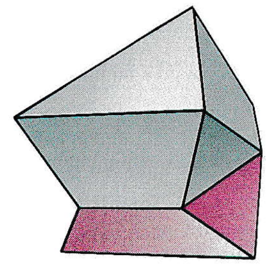
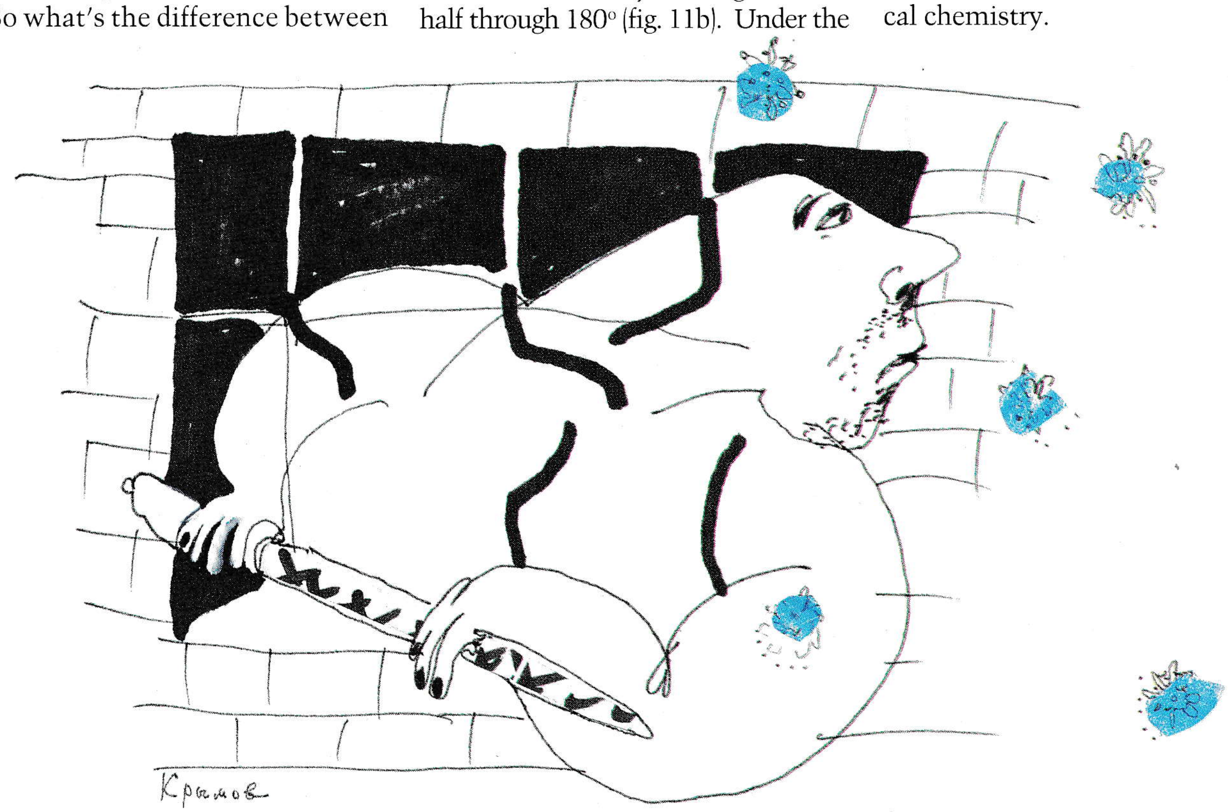


Figure 12

Of course, the problem of the formation of crystal structures is far from being completely solved. Here I've merely tried to show the important role played by mathematics in a problem that might have been thought to reside squarely in the realm of physical chemistry. ◘



Wave watching

"When you throw stones in the water, pay attention to the rings produced; otherwise this habit would be a mere waste of time."
—*K. Prutkov,* Fruits of Meditation

by L. Aslamazov and I. Kikoyin

WHAT IS A WAVE? EVERYone understands the word "wave" and, in most cases, knows that it's related to some kind of motion. Throw a pebble into the water and you'll see waves running along the surface. But if a wave runs into a floating branch, the motion of the branch has nothing in common with the propagation of the wave. Instead of moving with the wave it oscillates, bobbing up and down. So what is it that's actually moving when a wave propagates? Let's look at some examples.

Some historians claim that Elizabeth, Empress of Russia and daughter of Tsar Peter the Great, expressed the royal desire that the solemn moment of her coronation be marked by cannon fire from the Peter and Paul fortress in St. Petersburg. But the law prescribed that Russian tsars be crowned in the Assumption Cathedral of the Moscow Kremlin. Nowadays there's no problem in sending any information from Moscow to Leningrad: you send a radio signal at the moment of coronation from Moscow and the gun is fired in St. Petersburg (now Leningrad). In the 18th century, however, one had to find another way of telling the gunners that the patriarch had just laid the crown on the empress's head.

And a solution was found. Along the entire length of the Moscow–St. Petersburg road (about 650 kilometers), from the cathedral to the fortress, a line of soldiers was drawn up, one in direct sight of the other (about 100 meters). You can readily compute that the whole chain consisted of about 6,500 soldiers. Each soldier was given a small flag, which he had to raise the moment he saw the signal from his neighbor. At the moment of coronation the first soldier raised his flag, then the second, then the third, and so on. A person's reaction time is several tenths of a second, so the message reached St. Petersburg in 10 to 20 minutes.

What actually moved from Moscow to St. Petersburg? Each soldier remained in place. The only move he made was to wave the flag. A scientist would say that by raising or lowering his hand, each soldier for a short time changed his state. It is this change that moved along the line.

A change of state propagating in space is called a wave.

The year 1905 in Russia was marked by strikes that started in St. Petersburg. The newspapers wrote that "a wave of strikes swept through Russia and reached the most distant regions of the empire." What moved in this case was the state when workers stop their work at industrial plants and put forward political and economic demands.

Another example is the way rumors spread. A rumor started by one person can quickly spread over a whole city. The time it takes is much shorter than that needed for this person to visit (or phone) all the city's inhabitants. Rumormongers can remain motionless. What moves is the state of being informed.

But enough of news and rumors—let's look at a physical example. Some billiard balls are lined up on a billiard table (fig. 1a). Another ball hits the string in the direction of the string's axis. After impact the moving ball stops, while the last ball in the string jumps away (fig. 1b). Although the momentum is transferred to the first ball in the chain, the ball that moves is the last one. It is a wave of deformation that propagates along the chain. At the moment of impact the first ball gets compressed, thereby deforming the neighboring ball, which in turn deforms the next one, and so on. Each ball is subjected to equal elasticity forces on both sides acting in opposite directions (fig. 1c) and, therefore, stays at rest. The only exception is the last ball, which is acted upon from one

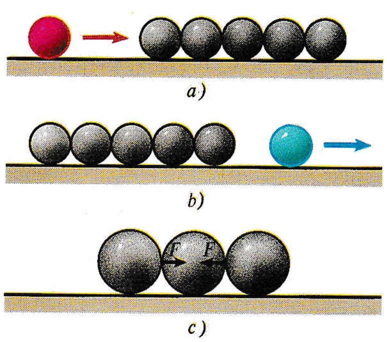


Figure 1

Art by Sergey Ivanov


С. Иванов

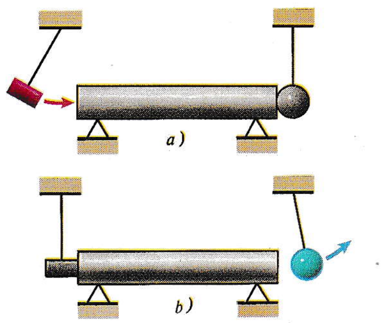


Figure 2

direction only. The resulting nonzero force gives an impulse to the last ball in the chain, setting it in motion.

Deformation waves propagating in elastic media are called acoustic waves. So what we actually got by hitting a string of balls was an acoustic wave. This kind of wave can propagate in any other elastic body. For instance, if you strike a fixed rigid rod (fig. 2a) with a hammer on one end, a deformation (acoustic) wave starts propagating along the rod. After reaching the other end of the rod, the wave sets the ball hanging next to it in motion (fig. 2b). Using a piston instead of a hammer, we can excite an acoustic wave in a liquid or a gas.

Let's examine the propagation of an acoustic wave in an elastic body in more detail. First, what does the velocity of the wave depend on? Let's start with a simple model.

Think of a string of balls of mass m connected by springs with rigidity k (fig. 3). The balls are small compared to the distance between them, and the mass of the springs is negligible compared to the mass of the balls. Actually, it's the same string of billiard balls we just looked at—we've merely separated their inertia (mass) and elasticity (rigidity).

This model is close to the actual situation in solids. In a crystal lattice, atoms are positioned in such a way that the vector sum of forces applied to each atom by the rest of atoms is zero in the equilibrium state. But if an atom is displaced from its equilibrium position, it starts to "feel" attraction and repulsion forces similar to elasticity forces.[1]

Let's give an impulse to one of the balls—for instance, the first one on the left—in the direction of the string by giving it a kick. The wave of elastic deformation runs along the string until it reaches the right end. But the last ball is connected to its neighbor by a spring that makes it impossible for it to go away. The stretched spring forces it to go back, and the ball, because of its inertia, compresses the spring again. The deformation wave now starts moving from right to left as if reflected at the end of the string. Then it reflects from the left end again, and so on. The reflected waves complicate the picture, so let's analyze an "endless" string (that is, a string without ends). This can be made by connecting a large number of balls in a ring (fig. 4). Along such an endless string a wave of elastic deformation goes in circles without any reflections until its energy dissipates and it dies away.

Now push one of the balls from its equilibrium position (for instance, in the clockwise direction) and set it free. Because of the action of the attached springs the ball starts a periodic motion in space, which is called oscillation.

Oscillations play an important role in nature and engineering. Oscillatory motion can be encountered in clock pendulums; the motors in electric devices are driven by alternating current; the succession of day and night, as well as that of the seasons of the year, can also be regarded as oscillatory processes caused by the motion of the Earth. All rotating mechanisms cause vibration in their foundations, which must be taken into account in designing them.

The simplest type of oscillation is simple harmonic motion. In simple harmonic motion the displacement of a body from its position of rest varies in time according to the formula

$$\begin{aligned}\alpha &= \alpha_M \sin(2\pi t/T)\\ &= \alpha_M \sin 2\pi\nu t\\ &= \alpha_M \sin\omega t,\end{aligned}$$

where α is the angular deviation of a ball from its position of rest. Any simple harmonic motion is described by two parameters: maximum displacement (amplitude) α_M and the period T (the time interval between two successive equivalent phases of motion). The frequency ν is the number of complete oscillations performed in a unit of time, and the cyclic frequency $\omega = 2\pi\nu$ is introduced to simplify the mathematical description. The number $\varphi = \omega t$ defining the position of the ball at time t is called the phase angle of the oscillation.

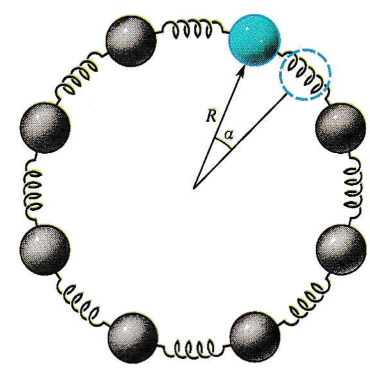


Figure 4

Consider the following example. A ball makes a complete oscillation cycle over an interval of time $T = 4$ s and at the initial moment is at the position of rest. The maximum displacement of the ball is $\alpha_M = 0.1$ rad. Then in simple harmonic motion its displacement from the position of rest is given by the formula

$$\alpha = 0.1\sin(\pi t/2).$$

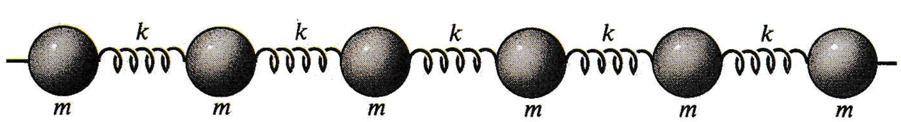


Figure 3

[1] Attraction forces predominate at large interatomic distances, but when atoms approach each other they're subjected to repulsion (quantum mechanics prohibits atoms from penetrating each other). Only at a certain (equilibrium) distance (about the size of an atom) the resultant force acting between atoms is zero.

For $t_1 = 1$ s, the phase angle is equal to $\varphi_1 = \pi/2$; for $t_2 = 2$ s, it's $\varphi_2 = \pi$; for $t_3 = 3$ s, we have $\varphi_3 = 3\pi/2$; and so on.

The frequency of oscillation (and, consequently, the period and cyclic frequency) depends on the properties of the system. For example, the cyclic frequency of an oscillating ball of mass m attached to a spring with rigidity k_0 is equal to

$$\omega_0 = \sqrt{\frac{k_0}{m}}. \qquad (1)$$

(See the appendix to this article.)

Oscillations can propagate in space. For instance, the balls in our string repeat the oscillations of the first, each of them with a certain delay. Each successive ball reaches the state of maximum displacement from the position of rest somewhat later than the preceding one. Similarly, when the first ball gets back to the position of rest, the next one is still displaced and comes to the position of rest only after a certain delay.

This delay can be described in mathematical terms by using the concept of a phase shift. The angular displacement of the nth ball is given by the following expression:

$$\alpha_n = \alpha_M \sin(\omega(t - \Delta t_n))$$

$$= \alpha_M \sin(\omega t - \Delta\varphi_n).$$

The value $\Delta\varphi_n = \omega\Delta t_n$ is called the phase shift (Δt_n is the delay time in the oscillation of the nth ball). In our example every ball in the string undergoes a simple harmonic motion. The amplitude of oscillations α_M and the cyclic frequency ω are the same for all the balls, but their phase shift $\Delta\varphi_n$ differs. The greater the distance to the nth ball, the longer the delay and, consequently, the greater the phase shift.

Figure 5 shows three graphs of oscillations having phase shifts $\Delta\varphi_1 = \pi/8$, $\Delta\varphi_2 = \pi$, and $\Delta\varphi_3 = 15\pi/8$ with respect to the oscillations plotted as a broken line. In the first case, the phase shift is small and the balls oscillate almost synchronously. They are said to be almost in phase. In the second, there is a complete disagreement between the oscillations: the maximum displacement of one ball corresponds to the maximum but opposite displacement of the other. In this case, the balls are said to oscillate in a counterphase mode. In the third case, the phase shift is close to 2π and, as the figure shows, the balls again oscillate almost synchronously—that is, almost in phase. This is to be expected since 2π is the period of the sine function (that is, oscillations with a phase shift equal to a multiple of 2π coincide).

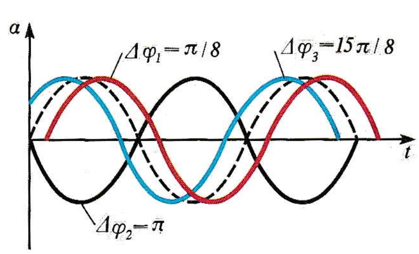


Figure 5

Since the phase shift of a ball increases with distance, there is a distance at which the phase shift equals 2π. Balls separated by that distance, called the wavelength λ, oscillate in unison.

How many wavelengths can fit into our string? Since the ends of the string are joined together (the string is actually a ring!), the number is obviously an integer. This is because the motions of the first and last balls must coincide (since they are actually one and the same ball). If the string's length is L ($L = Na$, where a is the distance between a pair of neighboring balls in the state of rest and N is the number of balls), the longest wave that can propagate along the string has the wavelength $\lambda_1 = L$.

The length of the next, shorter wave is $\lambda_2 = L/2$; the length of the third one $\lambda_3 = L/3$; and so on. What's the shortest wavelength that can go around the ring?

The shorter the wavelength, the greater the phase shift between the adjacent balls. The maximum "disorder" occurs when the phase shift between neighboring balls is π. The two balls then oscillate in counterphase (fig. 6), and the corresponding wavelength is $\lambda_{min} = 2a$.

Let's calculate the frequency of oscillations corresponding to the minimum wavelength (and thus estimate the velocity of the wave propagation along our string). If the oscillation of a ball in the string is described by the function

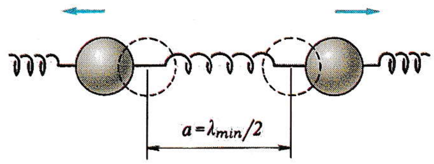


Figure 6

$$\alpha_n = \alpha_M \sin\omega t,$$

then the oscillation of the preceding ball satisfies the expression

$$\alpha_{n-1} = \alpha_M \sin(\omega t + \pi),$$

and that of the next one

$$\alpha_{n+1} = \alpha_M \sin(\omega t - \pi).$$

From the motion of the strings' ends we can readily obtain their deformation as a function of time. Hooke's law ($F = kx$) makes it possible to evaluate the elasticity force acting on the middle ball. The resultant force is

$$F = kx_M(\sin(\omega t - \pi) - \sin\omega t + \sin(\omega t + \pi) - \sin\omega t)$$

$$= -4kx_M \sin\omega t,$$

where $x_M = R\alpha_M$ is the maximum linear displacement of the ball from the position of rest (R is the radius of the ring). The middle ball moves as if attached to a single spring with a rigidity four times that of a real spring. Substituting $k_0 = 4k$ into formula (1), we get the frequency $\omega = 2(k/m)^{1/2}$ for the shortest wavelength ($\lambda_{min} = 2a$) that can travel around the ring. This is the maximum frequency of oscillations for a closed string of balls.

There is also a maximum frequency for atomic oscillations in real solids.

What's the speed of the wave motion? The period corresponding to the oscillation frequency ω is $T = 2\pi/\omega$. A wave propagating with velocity v covers the length $l = vT = 2\pi v/\omega$ over an interval of time T. This length is equal to the wavelength since the oscillations separated by time T are synchronized. Thus,

$$\lambda = vT = 2\pi v/\omega,$$

and so

$$v = \frac{\lambda\omega}{2\pi} = \frac{2}{\pi} a \sqrt{\frac{k}{m}}.$$

And what's the wave velocity and the oscillation frequency for waves with larger wavelengths? We can get the answer the same way (though the problem is a bit more complicated): The oscillation frequency decreases as the wavelength increases, while the wave velocity increases, albeit more slowly than the wavelength. For longer waves ($\lambda >> a$) the speed becomes almost constant, approaching the value of

$$v_0 = a\sqrt{\frac{k}{m}}.$$

So our result gives a good approximation of the wave velocity for other wavelengths as well.

Let's get back to oscillations in solids. What does the speed of acoustic waves depend on? The analogy with the string of balls shows that the velocity depends on the elastic properties of the medium, the mass of the atoms constituting the substance, and the interatomic distances. A decrease in interatomic distance or an increase in the atom mass results in a higher density ρ of the substance. In our model the rigidity k can be considered proportional to Young's modulus E. The exact expression for the speed of sound propagating in a solid body is

$$v = \sqrt{\frac{E}{\rho}}.$$

For instance, this formula gives a value of $v \cong 5{,}000$ m/s. This is almost a miracle! A very simple model explains the propagation of sound in elastic bodies.

Oscillations of other physical values can also propagate in space. Periodic variation in electric strength and magnetic induction is described as the propagation of an electromagnetic wave. Other examples are temperature waves, magnetization waves (induction oscillations of a magnetic field in a medium), and so on. In a manner of speaking, the entire house of modern physics is riddled with different kinds of waves.

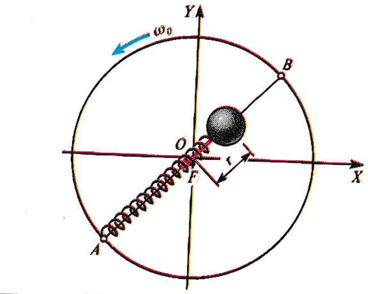


Figure 7

Appendix

Imagine a spring winding around a rod AB that is positioned along the diameter of a circle (fig. 7). One end of the spring is attached to a ball and the other to the rod's end A. The ball can slide along the rod, and its position of rest coincides with point O. The ring is set in rotation in the horizontal plane with a constant angular velocity ω_0. The ball then deviates from the center. Denoting its displacement by r, we have (by Hooke's law) a force $F = k_0 r$ acting on the ball in the direction of the point O. By Newton's second law this force provides centrifugal acceleration $a_c = \omega_0^2 r$:

$$m\omega_0^2 r = k_0 r.$$

So the ball is in a stable position when the rotation velocity is

$$\omega_0 = \sqrt{\frac{k_0}{m}}.$$

Therefore, the projection of the ball's velocity on a fixed axis is in simple harmonic motion, with the cyclic frequency equal to the angular rotation rate. For example, $x = r\sin\omega_0 t$. So for a ball of mass m attached to a spring with rigidity k_0, the frequency of the simple harmonic motion is given by the formula

$$\omega_0 = \sqrt{\frac{k_0}{m}}.$$

Exercises

1. For those of you who haven't experienced a "wave" in the stands at a sporting event, here's a good way to get a feel for what a wave is. Have a number of your friends stand in a ring hand in hand. Let one of them squat down and then stand up again, the person on the right repeating the motion after a certain delay, and so on. What does the speed of this wave depend on?

2. The length of an elevated power line is 3,000 km. The frequency of the voltage is 50 Hz. By what fraction of the oscillation period do the phases at the input and the output of the line differ? What is the corresponding phase shift?

3. Evaluate the collision time t of steel balls with diameter $d = 0.01$ m. The density of steel $\rho = 7.8 \cdot 10^3$ kg/m^3, Young's modulus $E = 2 \cdot 10^{11}$ N/m^2.

4. The eminent Soviet physicist P. L. Kapitsa used the following setup to obtain strong magnetic fields. The rotor of a generator rotating in the magnetic field of the stator was abruptly stopped, resulting in high voltage induction. The rotor was connected to a coil having a small resistance. The powerful impulse of the electric current created a magnetic field inside the coil with flux density of about 30 T (a record at the time). Why was the coil (which contained a sample whose properties under strong magnetic fields had to be investigated) placed far from the generator? Evaluate the minimum distance l between the generator and the coil if the experiment lasted $\Delta t = 0.01$ s and the laboratory had a concrete floor.

5. A model of carbon dioxide gas (CO_2) consists of three balls connected by two springs. In the position of rest both springs are lying along the same line. The model can perform different types of motions (shown in figure 8). Calculate the ratio of their frequencies. ◘

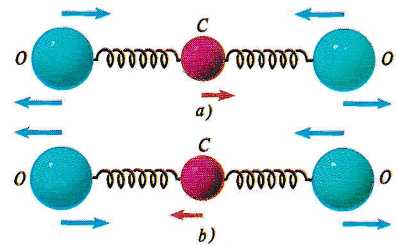


Figure 8

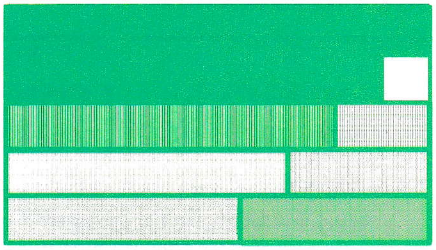
SOLUTIONS ON PAGE 54

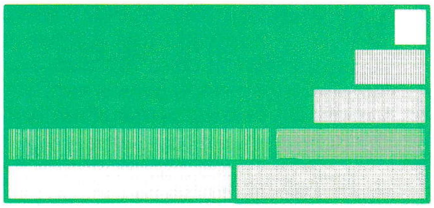
Rearranging sums

For what values of n and k is it possible to rearrange the sum 1 + 2 + 3 + ... + n into k equal summands?

by George Berzsenyi

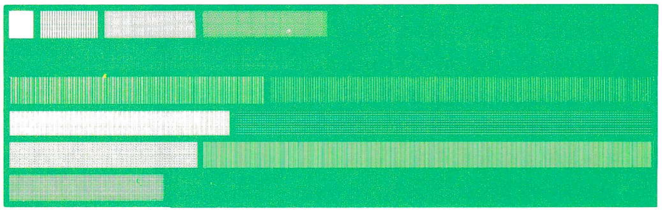
IN THE FIRST ROUND OF THIS year's USA Mathematical Talent Search (see page 58 of the Sept./Oct. 1990 issue of *Quantum* for details), I posed a very special case of the above problem with $k = 5$. Most of the over 250 contestants responded to the challenge and proved that if $n = 5m$ or $n = 5m - 1$, where m is a positive integer greater than or equal to 2, then it is indeed possible to partition the set $\{1, 2, \ldots, n\}$ into five subsets whose elements have the same sum. Two of the figures illustrating this article provide a geometric interpretation of the case $m = 2$; for $m > 2$, one proceeds inductively.

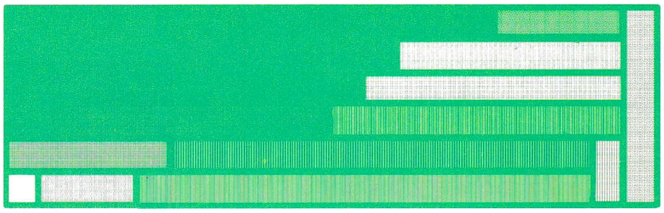




Our first challenge is to treat the more general case of $k = p$, where p is a prime number. That is, **determine the possible values of n for which one can partition the set $\{1, 2, 3, \ldots, n\}$ into p subsets whose elements have the same sum.** There are at least two different inductive procedures that can be applied; it may also be of interest to study their geometric interpretations.

Our second challenge is to treat the case of $k = pq$, where p and q are primes. The third figure illustrates the case of $p = 2$, $q = 3$, $n = 15$, while the fourth figure should be viewed as an invitation to a yet more general question: **what must be the dimensions of a rectangle that can be tiled by rectangles of width 1 and length $1, 2, 3, \ldots, n$?**





You may also wish to consult a recently published research article entitled "Disjoint Subsets of Integers Having a Constant Sum" (*Discrete Mathematics* 82 (1990), 7–11), in which a related result is established along with yet another generalization. I am indebted to my colleagues Gary Sherman (who called this article to my attention), Roger Lautzenheiser, and Bart Goddard for insightful discussions of such problems.

Please send your solutions to these problems to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. The best results will be acknowledged, and their authors will receive free subscriptions to *Quantum* for one year and/or book prizes.

The successful solvers of these problems are also encouraged to present their findings at conferences (such as the Eighth Annual Rose-Hulman Conference on Undergraduate Mathematics, to be held March 15–16, 1991), in publications (such as the *Journal of Undergraduate Mathematics*, *Pi Mu Epsilon Journal*, and *Kenyon Quarterly*), and at science fairs and talent searches (such as the Westinghouse Science Talent Search). I also encourage you to communicate with one another.

What the seesaw taught

"The balance distinguisheth not between gold and lead."
—George Herbert, "Jacula Prudtum"

by Arthur Eisenkraft and Larry D. Kirkpatrick

REMEMBER THE FIRST TIME YOU played on a seesaw? You balanced on one end, your friend balanced on the other. With small kicks off the ground the seesaw tilted one way and then the other. At times, you may have even threatened not to let your friend down as you gloried in the power of your position. Another, quieter way of playing with the seesaw is to try to balance perfectly. You and your friend position yourselves so that the seesaw balances. If you lean back, the seesaw tilts. Lean forward and you can get it to return. How long can you keep it balanced?

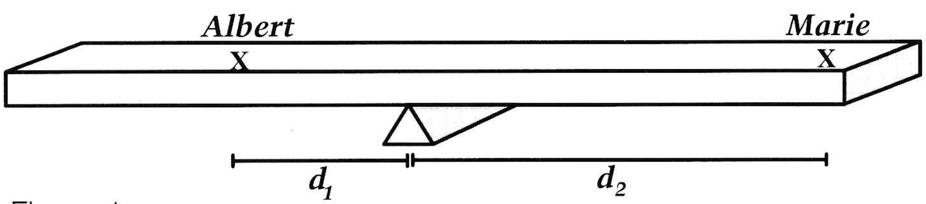


Figure 1

The seesaw is a very good place to begin a study of forces and torques. The forces make the seesaw move up or down. The torques make it rotate about the pivot in the middle. Since the entire seesaw never leaves the ground, we can be sure that the "up" forces equal the "down" forces. If Albert and Marie are on the two sides of the seesaw (fig. 1), their weight plus the weight of the seesaw ("down" forces) must equal the force of the support on the seesaw ("up" force).

We can write this as

$$F_{\text{down}} = F_{\text{up}},$$
$$W_{\text{Albert}} + W_{\text{Marie}} + W_{\text{seesaw}} = F_{\text{support}}.$$

Torque is like a "turning force." The greater the force, the greater the torque. Similarly, the longer the moment arm, the greater the torque. On our seesaw, the moment arm can be chosen as the distance from the pivot to the weight. When Albert and Marie are perfectly balanced, there's no rotation and so the clockwise torques must equal the counterclockwise torques. We can write this as

$$T_{\text{counterclockwise}} = T_{\text{clockwise}},$$
$$d_1 \cdot W_{\text{Albert}} = d_2 \cdot W_{\text{Marie}}.$$

If Marie has 3/4 the weight of Albert, her moment arm (distance from the pivot) must be 4/3 times the moment arm of Albert.

If you know your weight, you can guess someone else's by balancing on a seesaw and measuring distances. Give it a try and let us know how successful you were.

By applying these physical principles you can learn a great balancing act! Take any long rod. A meter stick, a baseball bat, or a curtain rod will all do well. Cradle the stick on the edges of your two hands. Move your hands slowly together. Your two hands will meet at a point—the same point at which the stick can be balanced. We call this position the center of mass. Will it work if your hands start at different locations? Sure it will. Will it work if we add an extra mass to one side of the stick? Sure it will. As the hands slide closer together, one hand always seems to move more easily than the other. **The contest problem for this issue is to describe why this works.**

A second contest problem is offered for those of you who have some extra physics under your belt or some extra time to work on your physics problem-solving skills. In this problem, a uniform stick is resting on two fixed cylinders that rotate with equal velocities in opposite directions. (See figure 2.) The stick's center of mass is somewhat displaced with respect to the

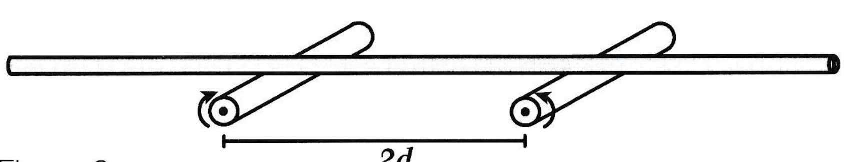


Figure 2

CONTINUED ON PAGE 26

A talk with professor I. M. Gelfand

A student and teacher who followed his own interests and instincts

Recorded by V. S. Retakh and A. B. Sosinsky

ISRAEL MOISEYEVICH GELFAND is one of the greatest living mathematicians. He's the author of around 500 works—books and articles not only on mathematics *per se* but also on mathematical physics, cell biology and neurobiology, and applications in medicine, seismology, and other areas. Gelfand is a member of the Soviet Academy of Sciences, the US National Academy of Sciences, the American Academy of Arts and Sciences, the London Royal Society in England, the French Academy of Science, the Royal Swedish Academy, and many other foreign academies. He has received honorary doctorates from Oxford, Paris, Harvard, and many other universities. He has also received such distinguished prizes as the Kyoto Prize, the Wolf Prize, and the Wigner Medal.

For some 45 years now, first-year students and famous scholars have gathered on Monday evenings at Moscow University for Gelfand's renowned mathematics seminar. Several generations of outstanding mathematicians have been nurtured by this seminar.

Gelfand founded the Mathematics Correspondence School, which has students throughout the Soviet Union, and is the chairman of its governing committee. The main goal of this school is to reach out and help those students who are practically deprived of mathematical literature and contact with scholars. These are generally students who live outside of Moscow, Leningrad, and other big cities where there is access to good books and good mathematicians. Created 25 years ago, this correspondence school was the first such school in the Soviet Union and served as an example for other correspondence schools that followed.

Interviewers from our sister magazine *Kvant* planned this conversation with professor Gelfand in the usual way—that is, by proposing questions that would be of interest to both Gelfand and *Kvant*'s student readers. Gelfand glanced at the list of questions and said they were very interesting but he didn't consider himself competent enough to answer them.

"You see," he said, "I don't think I have the right to impose my opinions on your readers. It would be better if I just tell what I was doing mathematically at their age—13 to 17 years old. I'm not sure I can recall now all the problems I was working on at that time, but the problems I'll talk about I remember very well."

And now—I. M. Gelfand's story.

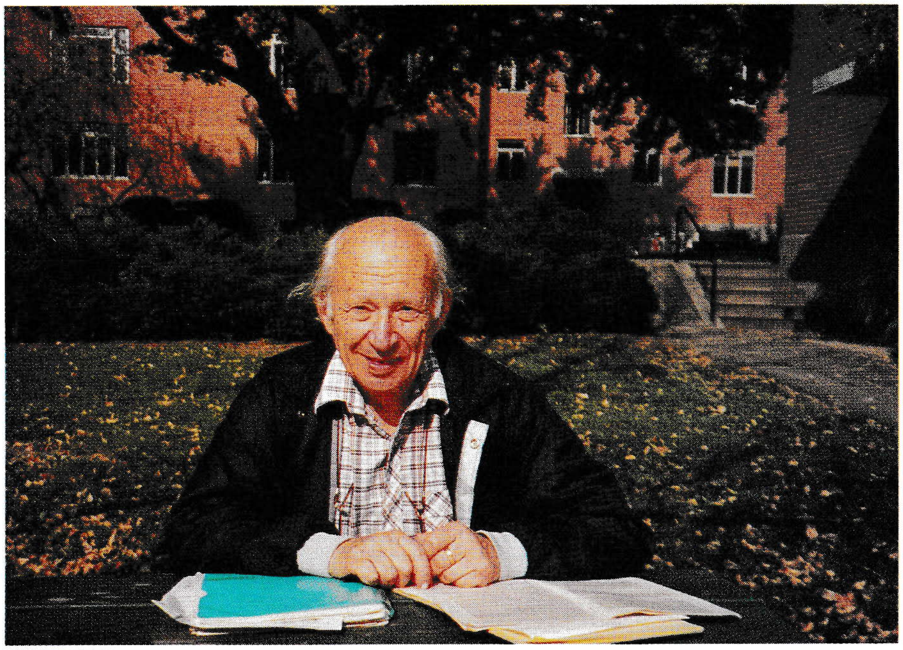


Photo by T. Alekseyevskaya

Professor I. M. Gelfand at home in Boston, October 1989.

ONE OF GRAHAM GREENE'S NOVELS is called *The Loser Takes All.* My mathematical experience was such a wonderful and happy one, for many years it seemed to be the realization of Greene's title. Why was I so fortunate? Briefly stated: first, I didn't study at a university (or any institution of higher learning, for that matter); second, because of certain difficulties in my family life I found myself in Moscow without parents, and jobless, at sixteen and a half years of age.

I'll try to illustrate the meaning of the expression "the loser takes all" with the help of another English writer, Somerset Maugham. The hero of the story, a church sexton, suffers a misfortune: during certification of church personnel it comes to light that he's illiterate, and so he's fired. He starts selling cigarettes, then buys a tobacco stand, then several others, and ends up making a brilliant career in commerce. He becomes the richest man in the city. He becomes the city's mayor. Someone comes to interview him—just as you're doing now—and he explains to the journalist that he's illiterate. The stupefied journalist exclaims, "What heights you could have attained if you had been literate!" Without a pause the mayor replies, "I'd have been a sexton."

So in February 1930, at sixteen and a half, I came to Moscow to live with my distant relatives, and I was often unemployed. I tried many temporary jobs, but mostly I went to the Lenin Library and "pulled together" all the knowledge I didn't get in school and in the technical training I didn't finish. At the library I met university students and started going to seminars. At 18 I was already teaching, and at 19 I found myself in graduate school. The rest of my mathematical career proceeded quite normally, taking the usual track for mathematicians.

But it's not this part of my life that I want to talk about. I want to tell your readers about the earlier period. I'd like to do this for two reasons. First, it's my deeply held conviction that mathematical ability in most future professional mathematicians appears precisely at that time—at 13 to 16 years of age. (Of course, there are exceptions—some who develop earlier, some later, at 20 to 30 and even 40—among very strong mathematicians.) Second, this early period formed my style of doing mathematics. The subject of my studies varied, of course, but the artistic form of mathematics that took root at this time became the basis of my taste in choosing problems that continue to attract me right up to the present time. Without an understanding of this motivation, I think it's impossible to make head or tail of the seeming illogic of my ways of working and the choice of themes in my work. In the light of this motivating force, however, they actually come together sequentially and logically.

The first thing I remember happened when I was around 12. I understood then that there are problems in geometry that can't be solved algebraically. I drew up a table of ratios of the length of the chord to the length of the arc in increments of 5 degrees. Only much later did I learn that there are such things as trigonometric (not algebraic!) functions and that, in essence, I was drawing up trigonometric tables.

At about this time I was working through a book of problems in elementary algebra. I had no accompanying textbook, I didn't know the theory, but sometimes I had to

Receiving an honorary doctorate at Oxford University in 1973.

solve some pretty tough problems, using formulas that I didn't know at the time. When I couldn't figure out how to solve a certain problem, I'd look at the answer, and I learned how to reconstruct methods of solving problems from the way they're set up and from the answers given. In particular, I understood then, and remembered for the rest of my life, that you can master a subject by solving problems and that there's nothing wrong with looking at the answer since we always have a hypothesis about the answer while we're working on any problem. Doing research in mathematics is similar to solving problems in which something about the answer is known. This is the difference between working in mathematics and training for university entrance exams (which is necessary as well, of course).

At the age of 12 or 13 I turned my attention to geometry problems in which there was often a right triangle with sides 3, 4, 5 and even with sides 5, 12, 13. I wanted to find all right triangles with integer sides, and I derived a general formula for their sides. That is, I found all Pythagorean triples.[1] (Of course, I didn't know the term at the time.) Unfortunately, I don't remember how I did it.

I worked at mathematics when I was sick and when I was on vacation. Even now I can't help noticing how much strong students manage to do when they stay home because of illness. And so I would keep my own sons home a few extra days after they got better.

In the geometry textbook we used, some theorems were given as problems. I got my hands on a notebook (not an easy thing in those days) and wrote out the statement of a theorem on each page. Over the course of the summer I covered almost all the pages with proofs. That's how I learned to write out my mathematical work.

I'll skip over a stretch here. I'll mention only the book by Davydov on algebra in which you can find clever ways of solving problems about maxima and minima by means of elementary techniques (that is, without using differential calculus). For example: given $a + b$, find the maximum of ab; for a given perimeter, find the rectangle with the maximum area; find the maximum of the product of nonnegative numbers $a_1a_2...a_n$, given their sum $a_1 + a_2 + ... + a_n$; little squares are cut out of a square with a given side and a box is made out of the remainder—what size must the little squares be for the volume of the box to be maximal?

Combinatorics and Newton's binomial formula made a great impression on me, and I thought about them for a long time.

I lived in a small town with only one school. My mathematics teacher was a kind but stern-looking man by the name of Titarenko. He had a huge Cossack moustache. I haven't met a better teacher, although I knew more than he did and he knew it. He liked me a lot and encouraged me in every way. Offering encouragement is a teacher's most important job, isn't it?

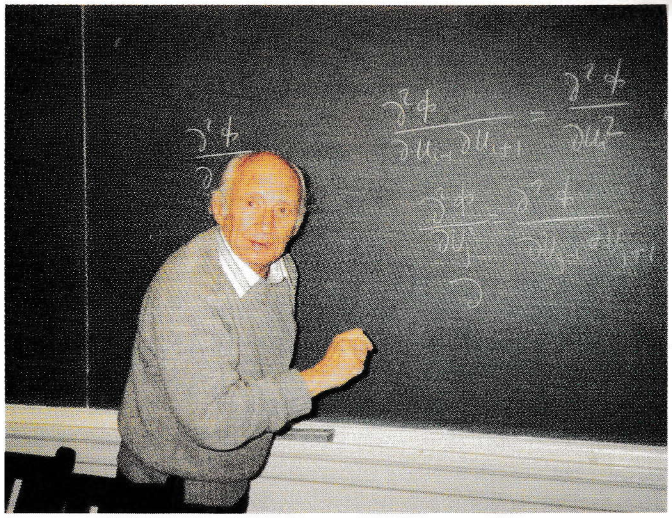


Photo by T. Alekseyevskaya

Lecturing at the Massachusettes Institute of Technology (MIT) in 1989.

There was a definite lack of mathematical books. I saw ads for books on higher mathematics and figured higher mathematics must be pretty interesting. My parents couldn't order these books—they didn't have the money. But once again I was lucky. At the age of 15 I was taken to Odessa to have my appendix taken out. I told my parents I wouldn't go to the hospital until they bought me a book on higher mathematics. My parents agreed and bought me the textbook on higher mathematics written by Belyayev in Ukrainian for use in technical institutes. But they only had enough money for the first part, which was about differential calculus and analytical geometry in the plane.

I was lucky that I didn't start with a full-fledged university course. This was a very elementary book. You can judge the level of Belyayev's book by its introduction—in particular, it says there are three kinds of functions: analytical, as defined by formulas; empirical, as defined by tables; and correlational. I didn't find out about correlational functions until many years later, from a student who was studying probability theory.

On the third day after the operation I picked up the book and read it, alternating it with novels by Émile Zola, for nine days. (In those days you'd stay in the hospital for twelve days after an appendectomy.) That was enough time for me to finish Belyayev's book.

I took away two remarkable ideas from this book. First, any geometric problem in the plane and in space can be written as formulas. (I had suspected this earlier.) I also learned about the existence of some remarkable figures—the ellipse, for example.

The second idea turned my world view upside down. This idea is the fact that there's a formula for calculating the sine: $\sin x = x - x^3/3! + x^5/5! - \ldots$. Before this I thought there are two types of mathematics, algebraic and geometric, and that geometric mathematics is basically "transcendental" relative to algebraic mathematics—that is, in geometry there are some notions that can't be expressed by

[1] See "Genealogical Threes" in the Nov./Dec. 1990 issue of *Quantum*.—Ed.

formulas. Consider, for example, the formula for circumference—it contains the "geometric" number π; or, say, the sine—it's defined in a completely geometric way.

When I discovered that the sine can be expressed algebraically as a series, a barrier came tumbling down, and mathematics became one. To this day I see the various branches of mathematics, together with mathematical physics, as a unified whole.

Of course, I became convinced that problems of the extreme are solved automatically (that is, by means of an exact algorithm). Although they lose their charm, you have in your hands a powerful tool (calculus) for solving them.

Studying differential calculus I learned that there is also integral calculus, which has to do with areas and volumes. But what it consisted of, I had no idea—I didn't have the second volume of Belyayev's textbook!

Now's a good time to mention another problem I recall. The next autumn we studied the volumes of solids of revolution at school. A classmate of mine, D. P. Milman, who later became a famous mathematician, brought the following problem to my attention: find the volume of a body formed by the rotation of a circle about its tangent. To solve it I divided the circle into strips. Then I calculated the differences of the volumes of the corresponding cylinders obtained by rotation. Finally, I found the sum of these differences. This brought me face to face with the need to find the sum

$$\cos\varphi + \cos 2\varphi + \cos 3\varphi + \ldots + \cos n\varphi. \qquad (1)$$

The rest, as usual, was a mixture of inventiveness and stupidity. I passed over an elementary solution based on standard trigonometry, using instead the formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

(This formula is called Euler's formula, but I didn't know that.) I got this formula from the power series for $\sin x$, $\cos x$, and e^x, which had made a deep impression on me. It remained for me to find the sum of the geometric progression $e^{i\varphi} + e^{2i\varphi} + \ldots$ and, from that, to derive the sum (1), which I did.

This problem led to my habit of thinking about a problem even after I'd solved it. And I came up with something else: I moved the circle away from the line and understood that rotation produces a body that looks like the rubber cushion my friend's hemorrhoidal grandfather used to sit on. Knowing the radius r of a circle and the distance d from its center to the line, I used the method described above to determine the volume of the solid of revolution, $2\pi^2 r^2 d$. I was stunned by the simplicity of this formula. I rewrote it in the form $\pi r^2 \cdot 2\pi d$ and understood that if we cut the rubber cushion and pull it into a cylinder whose side equals the length of the trajectory formed by the center of the circle, then the volume of the cylinder would be the same. A similar fact is true for the area of a surface, and I understood that it was not by chance. What will happen if we rotate some other figure instead of a circle—for example, a triangle?

In this case the volume of the solid of revolution coincides with the volume of a prism whose base is a triangle and whose height equals the length of the trajectory formed by the common intersection of the medians of the triangle. From a physics book I knew that this point is the triangle's center of mass. Seeing what happens when a section is rotated, I understood that the center of a circle is its center of mass as well.

I found a general definition of the center of mass in some textbook on the strength of materials—I have no idea where I got a hold of it. Not only did I immediately start rotating various figures, I'd move them along various curves and calculate the volumes of the bodies obtained and their surface areas. The rigor of the thinking was important here. I was very proud that I could find the center of mass of a half circumference (half circle) and of a half disk (half of the interior of a circle) given the volume of a ball and the area of its surface.

And I was lucky yet again. An extraordinarily well-educated man (in my opinion at the time) came to our town. He had graduated from the Odessa Pedagogical Institute in physics and math. Among the books he brought with him were Kagan's *Theory of Determinants* and Hvolson's *Course in Physics*. Kagan's book was useful and detailed. It even contained a chapter on determinants of infinite order.

I should also mention the biology textbook by Filippenko, the well-known biologist from the school of the famous geneticist N. K. Koltsov. This was a fine book, and it naturally influenced my work in biology some 15 or 20 years later.

But to get back to mathematics. I was still interested in problems of areas and volumes. I began with a calculation of the area under the segment between two points of a parabola. This problem reduces to a calculation of the sum $1^2 + 2^2 + \ldots + n^2$, which I did easily.

Then I wanted to find the area under the curve $y = x^p$, where $p = 2, 3, 4, \ldots$; that is, to find the sum $S_0 = 1^p + 2^p + \ldots + n^p$ for every positive integer p.

By analogy with the formula

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

I decided that S_0 is a polynomial in n of degree $p + 1$. I didn't notice that to find the area under the curve it's sufficient to know only the first coefficient of the polynomial S_0, so I started searching for the entire polynomial. This turned out to be quite interesting. First of all, I generalized the problem: instead of x^p I considered $f(x)$ and started looking for the sum

$$S_0 = f(1) + f(2) + \ldots + f(n).$$

Let $F(x)$ be a function such that $F'(x) = f(x)$. From Taylor's formula we get

$$F(2)-F(1)=f(1)+\frac{f'(1)}{2!}+\frac{f''(1)}{3!}+\dots,$$

$$F(3)-F(2)=f(2)+\frac{f'(2)}{2!}+\frac{f''(2)}{3!}+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$F(n+1)-F(n)=f(n)+\frac{f'(n)}{2!}+\frac{f''(n)}{3!}+\dots.$$

I added these equalities and got

$$F(n+1)-F(1)=S_0+\frac{S_1}{2!}+\frac{S_2}{3!}+\dots,$$

where S_0 is the sum that interested me and

$$S_1=f'(1)+f'(2)+\dots+f'(n),$$
$$S_2=f''(1)+f''(2)+\dots+f''(n),\dots.$$

Then I wrote the following system:

$$F(n+1)-F(1)=S_0+\frac{S_1}{2!}+\frac{S_2}{3!}+\frac{S_3}{4!}+\dots,$$

$$f(n+1)-f(1)=S_1+\frac{S_2}{2!}+\frac{S_3}{3!}+\dots,$$

$$f'(n+1)-f'(1)=S_2+\frac{S_3}{2!}+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

This is an infinite system with an infinite number of unknown variables $S_0, S_1, S_2, \dots$. As I mentioned earlier, Kagan's book touched on determinants of infinite order, so I was able to use Cramer's rule to find S_0:

$$S_0=\frac{\begin{vmatrix} F(n+1)-F(1) & 1/2! & 1/3! & 1/4!\dots \\ f(n+1)-f(1) & 1 & 1/2! & 1/3!\dots \\ f'(n+1)-f'(1) & 0 & 1 & 1/2!\dots \\ \dots & \dots & \dots & \dots \end{vmatrix}}{1}$$

I expanded the determinant in the numerator of this "fraction" in the elements of the first column and the corresponding minors and got

$$S_0=B_0(F(n+1)-F(1))+B_1(f(n+1)-f(1)) + B_2(f'(n+1)-f'(1))+\dots, \qquad (2)$$

where $B_0=1, B_1, B_2, \dots$ are numerical determinants of infinite order. The expression I got is called the *Euler–Maclaurin formula*, but of course I didn't know that. To calculate this expression I needed to know the coefficients $B_0, B_1, B_2, \dots$.

To do this, I used arguments that would now be called "functorials." Taking advantage of the fact that the coefficients $B_0, B_1, \dots$ don't depend on f, I picked a function f such that the left part of the system formed a geometric progression (which I knew how to sum). The function $f(x) = e^{\alpha x}$ suits this purpose. Inserting it into formula (2) (I'll leave the intermediate steps for you to work out!), I got

$$B_0+\alpha B_1+\alpha B_2+\dots=\frac{\alpha}{(e^\alpha-1)}.$$

That is, I got the power series for the numbers I was after. (These numbers $B_0, B_1, B_2, \dots$ are called *Bernoulli numbers*, and the polynomial S_0 for $f(x)=x^p$ is called *Bernoulli's polynomial*.)

I remember two other problems from this period. The first arose out of the problem in our book of algebra problems: express $x_1^2+x_2^2$ and $x_1^3+x_2^3$ via the coefficients of a quadratic equation with the roots x_1 and x_2. A natural generalization of this problem leads to another: express the sum $x_1^2+\dots+x_n^2$ and the sum $x_1^3+\dots+x_n^3$ via the coefficients of the equation $x^n+a_1x^{n-1}+\dots+a_n=0$, where $x_1, \dots, x_n$ are roots of this equation. At this point Bezout's theorem helped me, which I knew from Davydov's book. I went further and posed a more general problem for myself: express the sum of kth degrees of the roots of an algebraic equation of nth degree via the coefficients of this equation. I managed to solve this problem (the solution is known as Newton's formula).

The second problem I solved at that time arose when I discovered that the number cos ix is real because

$$\cos ix = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots.$$

I pondered this unexpected fact and came up with the following general theorem: *every even real-valued function takes real values on the imaginary axis.*

To prove this I had to refine the notion of a "function." I thought about what to call a function and arrived at this definition: a function is the sum of a convergent power series. After this, the proof of the theorem is almost self-evident.

This problem was probably the last one I thought about before I came to Moscow. I solved it in the summer of 1929. The next six months were very difficult for my family and me. Mathematics was far from my mind.

The next period of my studies in Moscow was no longer "pure experimentation." In Moscow I was exposed to many completely different influences, and my development no longer drifted on its own course. At this time, as I mentioned earlier, I studied independently in the Lenin Library and lived on occasional earnings from odd jobs. For a while I actually worked behind a check-out desk at the library. I met mathematics students from the university. One of them told me that expressions of the form $f(n+1)-f(n)$, which greatly interested me, were part of a whole science called the theory of finite differences. He told me I had to read Nörlund's book *Differenzenkalkül* on this topic. It was in German, but I mastered it with the help of a dictionary.

I started going to university seminars, and there I found myself under intense psychological stress. I discovered that my style of doing mathematics wasn't good for anything. New breezes were blowing in mathematics—new demands for rigorous proofs, great interest in the theory of functions of a real variable. (Today this level of rigor and this particular theory are considered old-fashioned and obsolete, but at the time . . .)

Then I realized it's very important that a function doesn't have to be continuous, that a continuous function doesn't have to be differentiable, that a differentiable function doesn't have to be twice differentiable, and so on; that even if a function has derivatives of all orders, the Taylor series for this function isn't necessarily convergent, and that even if it is, its sum doesn't necessarily coincide with the value of the function! If this coincidence takes place, the function is called analytic, and this class of functions (so the devotees of the real-variable function theory maintained) is so narrow that it lies outside the bounds of mainstream mathematics. And these were the only functions I'd been looking at!

Under the pressure of this point of view, I read the "modern, rigorous" textbook on analysis by Vallee Poussin. It's similar to the texts currently used at Moscow University by students of mathematics and mechanics, but better. So I sympathize with those first-year students who are allowed to experience the beauties of mathematical analysis only after a year's probation, a sort of trial by the fire of its "rigorous foundation."

But even here I was lucky. I began reading I. I. Privalov's remarkable book on the theory of functions of a complex variable. While reading this book I understood why, for the function $f(x) = 1/(1 + x^2)$, the Taylor series is divergent at $x = 1$ even though its graph is continuous. (As a matter of fact, the corresponding complex function has a peculiarity for $x = i$). After the first 100 pages I felt a fresh wind. I discovered that if a complex function has a first derivative, it has derivatives of all orders, and then the Taylor series converges at the value of this function in some domain. Everything fell into place, and harmony was restored.

I raced through Gourvits and Courant's book on the theory of functions of a complex variable. I was mostly

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A note on international prizes in mathematics

There is no Nobel Prize in mathematics. One story has it that the woman Alfred Nobel loved left him for the famous Swedish mathematician M. G. Mittag-Leffler. So when Nobel decided what prizes should be awarded by his foundation, he established one in physics, one in economics, one in literature, as well as other areas, but somehow either forgot or wasn't especially enthusiastic about establishing a Nobel Prize in mathematics.

Whatever the real reason, this injustice was corrected by Riccardo Wolf, another wealthy industrialist. Being of Jewish origin, he emigrated from Germany after Hitler came to power and settled in Latin America, making a brilliant career in the steel industry. Even though he was a capitalist, he was a friend of Fidel Castro, who even sent him as the Cuban ambassador to Israel. When Cuba broke off diplomatic relations with Israel after the Yom Kippur war in 1973, he decided to stay in Israel. Wolf, who was already in his eighties at the time, founded the Wolf Foundation. The foundation, which is based in Israel, each year awards several Wolf Prizes for achievements in different areas of science and the arts, and among them is the Wolf Prize in mathematics! Wolf Prizes were awarded for the first time in 1978, and the first Wolf Prize in mathematics went to I. M. Gelfand of the Soviet Union and C. L. Siegel of Germany. In 1978 the Soviet Union rarely allowed its citizens to travel to Israel and back, so Gelfand wasn't allowed to go to Jerusalem to receive the prize. It was only ten years later, in May 1988, when *perestroika* was gathering steam, that Gelfand was able to come to Jerusalem and attend the yearly presentation ceremony in the Knesset (the Israeli Parliament).

The Kyoto Prize, awarded by the Inamori Foundation (established by the well-known Japanese industrialist Kazuo Inamori in 1984), is different. There are no annual prizes in specific predetermined fields like physics, chemistry, economics, or mathematics. Instead, there are three broad categories: advanced technology, basic sciences, and creative art and moral sciences. Each year a specific field is selected from each of the three categories, and a laureate is then chosen from that field. For instance, the 1986 Kyoto Prize in basic sciences was awarded in biology, in 1987—in Earth sciences and astrophysics, and so on. You see that to get a Kyoto Prize for a mathematician is much more difficult since it's not awarded every year. I. M. Gelfand received the Kyoto Prize in 1989 when the field chosen in basic sciences was mathematics.

Another award that should be mentioned here is the Fields Medal. At the 1924 International Congress in Toronto, a resolution was adopted that two gold medals should be awarded at each international mathematical congress, held every four years. Professor J. D. Fields, a Canadian mathematician who was secretary of the 1924 congress, later donated funds establishing the medals, which were named in his honor. Fields wished that the awards be open to the entire world and recognize both existing work and the promise of future development, so the medals are restricted to mathematicians not over the age of forty. In 1966 the number of medals that could be awarded at each international congress was increased to four in light of the great expansion of mathematical research in the world.

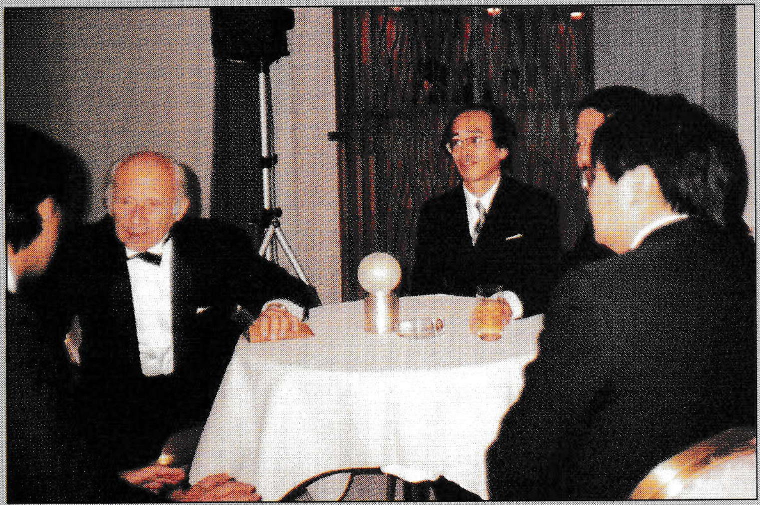

Photo by T. Alekseyevskaya

After the Kyoto Prize award ceremony, Professor Gelfand talks with Japanese mathematicians (November 1989).

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impressed by the chapters on elliptic functions written by Gourvits. And once again fashion made a fool of me—this branch of mathematics was considered obsolete. The theory of elliptic functions was looked down on as "barely extended trigonometry." Many years would pass before this area once again became a focal point of mathematicians' attention.

I gained a lot from the university seminars. Meeting with mathematicians of every stripe, I was able to compare my romantic, antiquated (that is, unfashionable) views of mathematics with what was actually happening then. I studied with many remarkable mathematicians and continue to try to learn this way.

A little later I read—studied in great depth, really—a remarkable book by Courant and Hilbert called *Methods of Mathematical Physics*. I understood then the need to read basic works. Here it's important not to regret the time spent thinking about the very foundations of a theory. The work of Herman Weyl (1925) on the representations of classical groups belongs to that category. But, unfortunately, we didn't have access to even older fundamental works by Cayley, Schur, and other authors of the "pre-Hilbert period."

I learned a lot from L. G. Shnirelman, M. A. Lavrentiev, L. A. Lusternick, I. G. Petrovsky, A. I. Plesner, and even more from Andrey Nikolayevich Kolmogorov.[2] In particular, I learned from him that a true mathematician nowadays must be a philosopher of nature.

But my story has turned into the standard scientific biography. This genre is usually very misleading. A true scientific biography is simply a collection of the scientist's works. One's own impressions about one's works are no more significant than the impressions of any other reader. And so it's time I ended my tale. ◘

[2]For more on A. N. Kolmogorov, see the Innovators department in the Jan. 1990 issue of *Quantum*.—*Ed.*

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point midway between the cylinders. If the distance between the cylinders is $2d$, the stick has a weight w and length l, and the coefficient of friction between each cylinder and the stick is μ, describe how the stick moves.

Please send your solutions to *Quantum*, 1742 Connecticut Avenue NW, Washington, DC 20009. The best solutions will be acknowledged in *Quantum* and their creators will receive free subscriptions for one year.

Click, click, click

We were disappointed that we received no correct solutions to this contest problem. We are confident that our readers could have solved part A. Don't get discouraged. If you can answer part A but not part B or C, send us a note anyway. Your solutions will help us judge what you might like to see.

In the Contest Problem involving Newton's toy you were asked to find the mass of a middle ball so that the velocity of the small ball will be greatest in a three-ball collision. Applying the laws of conservation of energy and momentum to the first collision, we have $m_1v_1 = m_1v_1' + m_2v_2'$, $(1/2)m_1v_1^2 = (1/2)m_1v_1'^2 + (1/2)m_2v_2'^2$.

Solving for v_1' in the first equation and substituting in the second equation, we arrive at $0 = -2m_1v_1v_2' + m_2v_2'^2 + m_1v_2'^2$. Solving for v_2', we find that $v_2' = 0$ and $v_2' = 2m_1v_1/(m_1 + m_2)$. We ignore the solution $v_2' = 0$ since this corresponds to the case of no collision. Since the second collision is similar to the first, we can write down the relevant equation immediately: $v_3'' = 2m_2v_2'/(m_2 + m_3)$. Combining the last two equations, we get

$$v_3'' = \frac{4m_1m_2v_1}{(m_1+m_2)(m_2+m_3)}. \qquad (1)$$

To find when the value of m_2 for which v_3'' will be a maximum, we can take the derivative of v_3'' with respect to m_2 and set it equal to zero. The solution is that the mass m_2 should be the geometric mean of the other masses. Specifically,

$$m_2 = \sqrt{m_1m_3}. \qquad (2)$$

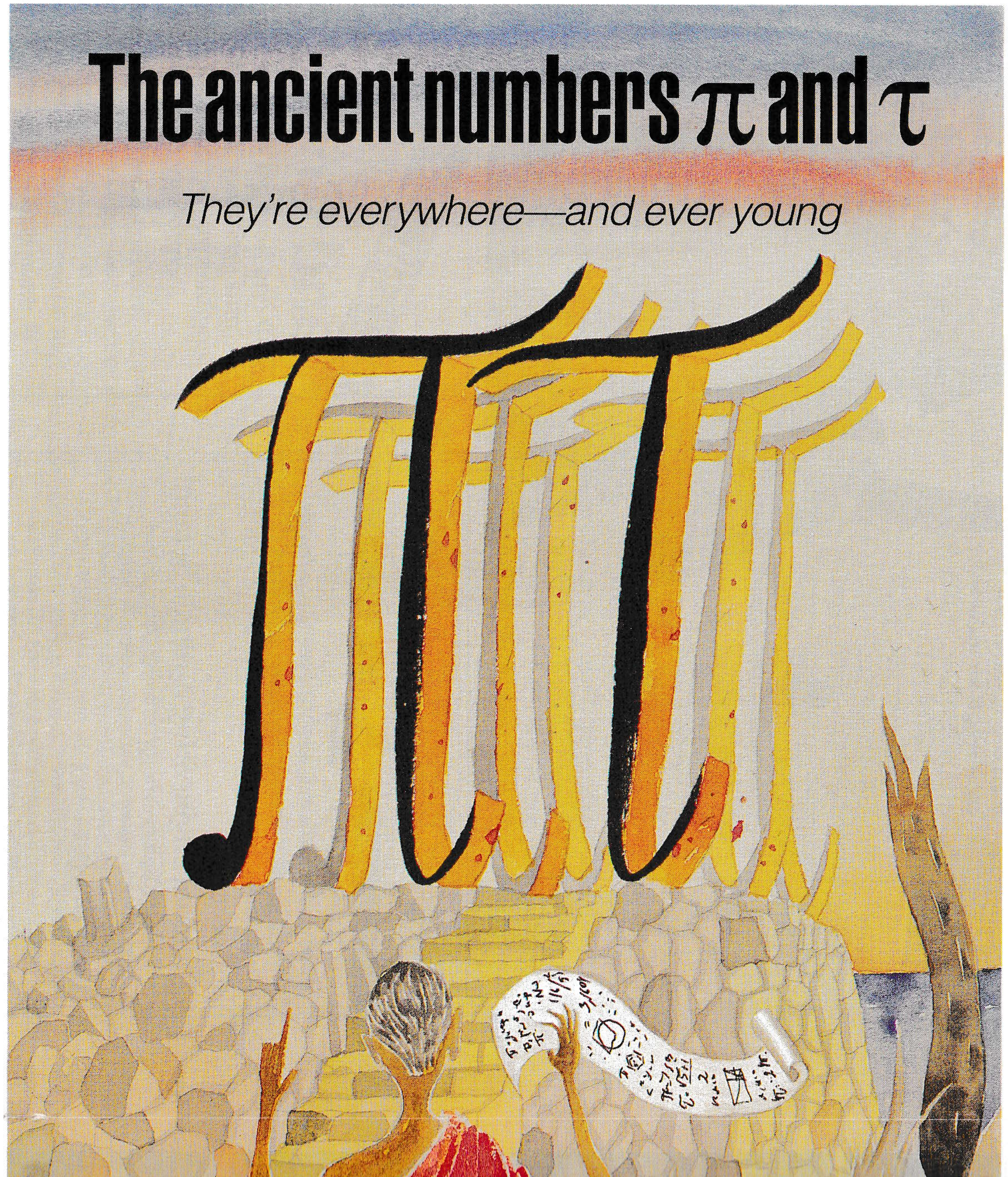
For those of you who aren't knowledgeable about calculus, we suggest that you take arbitrary values for m_1 and m_3 (that is, $m_1 = 1$ and $m_3 = 100$) and plot a graph of v_3 versus m_2 for different values of m_2. You'll find that the graph reaches a peak where $m_2 = 10$, as predicted by equation (2).

Part B of the problem is an extension of this solution to a collision of five balls. In this case, the masses of the balls follow the relation $m_1/m_2 = m_2/m_3 = m_3/m_4 = m_4/m_5$. Part C of the problem asks about the middle mass given a coefficient of restitution e. You may be surprised to find out that the ratio of masses is the same, independent of e, and is therefore the same solution as in part A.

Burt Lowry, our colleague from Whitman High School in Bethesda, Maryland, was quick to point out that other collision possibilities exist mathematically in the Newton toy that obey energy and momentum conservation. These never occur because the masses are independent. One ball always hits a second ball. The incoming ball never "sees" a ball of twice the mass, but rather sees a single-mass ball. This probably explains the importance of always leaving a small space between the balls when you build one of these toys. ◘

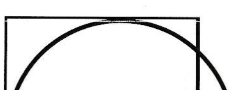
The ancient numbers π and τ

They're everywhere—and ever young



THERE ARE NUMbers in mathematics with the striking ability to spring out all the time in the most unexpected places. Perhaps the most remarkable of them is π = 3.14159265..., which is the ratio of the length of a circle's circumference to its diameter. (The notation goes back to περιφερεια, "circumference" in Greek.) Isn't it a surprise to find π in the expression for the proportion of coprimes among all pairs of integers (equal to $6/\pi^2$)? Or for the sum $1^{-2} + 2^{-2} + 3^{-2} + \ldots$ (equal to $\pi^2/6$)? Or for the probability that a randomly tossed needle will intersect a line in a grid of parallel lines (see *Quantum*, Sept./Oct. 1990, p. 6)?

The history of π was greatly influenced by the ancient problem of "squaring the circle"—that is, using a compass and ruler to construct a square equal in area to a given circle. The efforts to solve this problem—or, to an even greater extent, refute the "solutions"—instigated an unending pursuit of greater accuracy in determining the number π.



and all the rest (dotted segments), which are also congruent with each other by the symmetry of the construction. This is true whatever the number denoted by τ in the figure. But if this τ is the "real" one—that is, the golden ratio—all the edges turn out to be congruent, and so the solid turns out to be a regular icosahedron, one of the five regular polyhedrons.

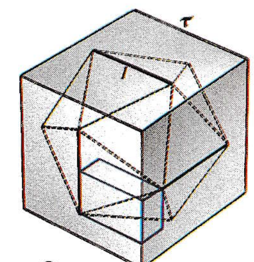


Figure 3

Indeed, one of the dotted edges is a diagonal of the cuboid marked blue in figure 3. Its dimensions are $\tau/2$, $1/2$, and $(\tau - 1)/2 = 1/2\tau$, so its diagonal is equal to

$$\frac{1}{2}\sqrt{\tau^2 + 1^2 + (\tau - 1)^2} = 1$$

(check it, using the equation for τ). Now it's easy to see that every two opposite edges of the icosahedron are the sides of a golden rectangle, while ~~the ends~~ of every five edges issuing from one vertex are the vertices of a regu-

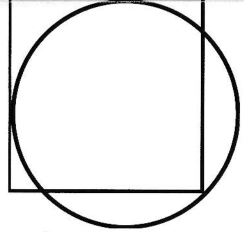


Figure 1

Art by Leonid Tishkov

Problem 1. Figure 1 shows an approximate squaring of the circle. What is the corresponding approximation of π?

For almost 2,000 years the main tool used to arrive at a value for π was a method invented by Archimedes. He replaced a circle with inscribed and circumscribed regular polygons. Starting with a hexagon, he successively doubled the number of sides and went all the way through to a 96-gon to obtain the approximation $\pi \cong 22/7 = 3.1428...$

A short list of subsequent records and record holders includes the remarkable fraction 355/113 = 3.1415929... found by the Chinese astronomer Tsu Chung-ji (fifth century); 16 true decimal digits of π found by al-Kashi from Samarkand, who used regular $3 \cdot 2^{28}$-gons (1424); 32 true digits found by the Dutch calculator Ludolph van Ceulen using 2^{62}-gons (1596); 100 digits first obtained by the English astronomer J. Machin, who used an infinite series representing π (in 1706); 200 digits found by Z. Dase of Hamburg (1844). The absolute record in the precomputer age was established by the Englishman W. Shanks: 707 digits (1873). But, alas, he made a mistake in the 528th decimal place!

In the meantime, mathematicians successfully investigated the intrinsic nature of the number π. It was proved to be irrational (J. H. Lambert, 1767) and then transcendental—that is, not a root of a polynomial with integer coefficients (F. Lindemann, 1882). This result shattered the last hopes of the "squarers"—it implied that the problem was unsolvable.

By that time, calculation of π had become a kind of sport. The computer revolution gave a fresh and powerful impetus to this competition. The best result known to us was achieved several years ago by Jonathan and Peter Borwein of Dalhousie University in Canada. They computed 29,360,128 decimal digits of π!

THE "GOLDEN SECTION" is the partition of a line segment in such a ratio that the whole segment is to its bigger part as the bigger part is to the smaller one. The ratio of the golden section—the "golden ratio"—is denoted by τ ("tau"), from the Greek τομη, "section." By this definition, τ is the positive root of the equation $(\tau + 1) : \tau = \tau : 1$, or $\tau^2 = \tau + 1$; that is, $\tau = (1 + 5^{1/2})/2 = 1.618...$.

Because it's a root of a quadratic equation, τ, unlike π, can be constructed with compass and ruler.

Problem 2. Think of a construction for τ given a unit segment.

Sometimes the golden ratio is denoted by φ in honor of the famous Greek sculptor of antiquity Phidias, whose name is associated with the imposing ensemble of the Acropolis in Athens. Legend has it that he often used this proportion in his works. This is only one of numerous occurrences, real or imagined, of the golden section in art, architecture, and even nature. (For another, see the inside front cover; more details can be found, for example, in M. Ghyka's *The Geometry of Art and Life*, Dover Publications, 1977.)

In plane geometry two figures are most closely associated with the golden ratio. One is the *golden rectangle*—a rectangle whose sides are in the ratio τ.

Problem 3. Cut the golden rectangle into an infinite set of squares whose sides make a geometric sequence. (You'll find more curious properties of the golden rectangle in the solution to this problem.)

Drawing diagonals in a regular pentagon, we obtain the second, even more "golden" figure. Maybe because of the abundance of golden sections in it, the pentagram (five-pointed star) has been considered a magic symbol since ancient times.

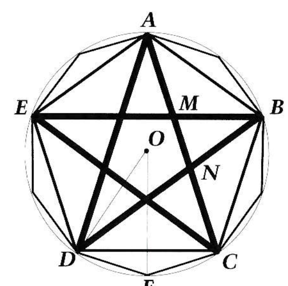


Figure 2

Problem 4. Prove that in figure 2

$$AC{:}AB = AC{:}AN = AN{:}NC = AM{:}MN = OD{:}DF = \tau = 2\cos\pi/5.$$

In figure 2 DF is a side of a regular decagon and $OD = \tau \cdot DF$ is its circumradius. It yields a tolerable approximation of π: $2\pi \cdot OD \cong 10 \cdot DF$, which gives $\pi \cong 5/\tau = 3.09...$.

Regular pentagons and golden rectangles come together in the construction of figure 3. The edges of the 20-faced solid shown in this figure fall into two classes: the unit segments lying on the cube's faces the ends of every five edges issuing from one vertex are the vertices of a regular pentagon.

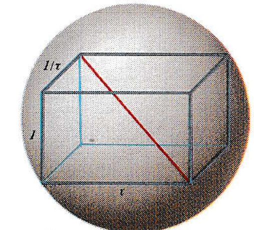


Figure 4

A cuboid whose edges are in the ratio t:1:1/t, like the one we used above, is called a *golden cuboid* (fig. 4).

Problem 5. Prove that the surface area of a golden cuboid is to the area of its circumscribed sphere as $\tau : \pi$.

Two beautiful algebraic expressions for τ can be easily derived from the equation $\tau^2 = \tau + 1$:

$$\tau = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}} .$$

Their rigorous proof, however, requires some advanced techniques as well as many other impressive and important algebraic properties of τ, which we'll examine in future issues of *Quantum*. ◘

SOLUTIONS ON PAGE 54

Just for the fun of it!

Problems offered for your enjoyment by S. Genkin, A. Domashenko, D. Fomin, and V. Proizvolov

B21

Bobby added together three consecutive integers, then the three next numbers, and multiplied one sum by the other. Could the product be equal to 111,111,111?

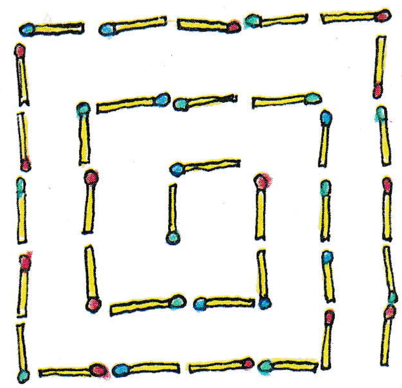
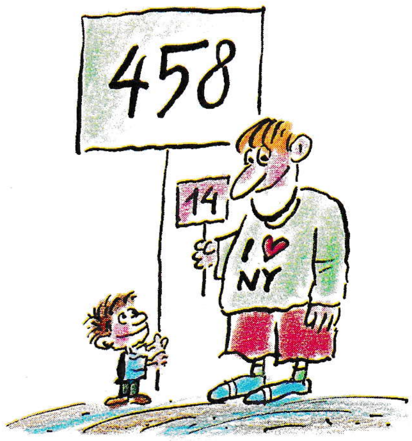


Figure 1

B22

In figure 1 a spiral made of 35 matchsticks is wound clockwise. Shift four matches to rewind it counterclockwise.



B23

You can do either of two things to a number written on the blackboard: you can double it, or you can erase the last digit. How can you get 14 starting from 458 by using these two operations?

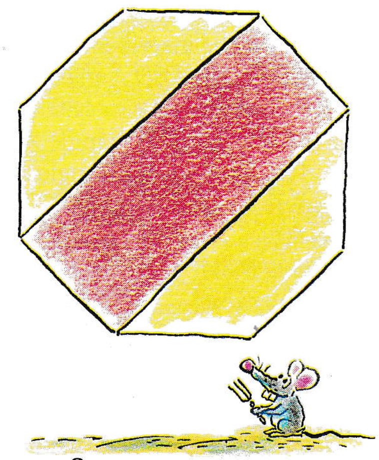


Figure 2

B24

Two parallel diagonals are drawn in a regular octagon (fig. 2). Prove that the area of the rectangle thus obtained is half the area of the entire octagon.

Art by Edward Nazarov

ANSWERS ON PAGE 53

B25

The smoke we see consists of small particles of unburned fuel. Each of the particles is much heavier than air. So why do they fly upward?

Do you have a brainteaser for Quantum*? Send it to Managing Editor,* Quantum*, 1742 Connecticut Ave. NW, Washington, DC 20009.*

Circumcircles to the rescue!

A technique for certain traditional problems

by D. F. Izaak

IT'S WELL KNOWN THAT THE angular value of an arc is equal to the value of the central angle it subtends and is twice the value of the inscribed angle it subtends. In solving problems in plane geometry, it's often useful to draw the circle circumscribed about a triangle or a quadrangle. The properties I mentioned above can then be formulated in the following way.

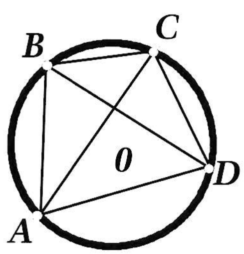


Figure 1

I. If a quadrangle $ABCD$ is cyclic—that is, if it can be inscribed in a circle (fig. 1)—then angle ABD = angle ACD, angle ABC + angle ADC = 180° (also, angle DBC = angle DAC, angle ADB = angle ACB, angle BAC = angle BDC, angle BCD + angle BAD = 180°).

II. If points B and C are on the same side of a straight line AD and angle ABD = angle ACD or angle ABC + angle ADC = 180°, then quadrangle $ABCD$ is cyclic (see figure 1).

III. If triangles ABC and AOC lie on the same side of the line AC and O is the circumcenter of triangle ABC, then angle AOC = 2 angle ABC (fig. 2).

IV. If triangles ABC and AOC lie on the same side of the line AC, OA = OC, and angle AOC = 2 angle ABC, then O is the circumcenter of triangle ABC (see figure 2).

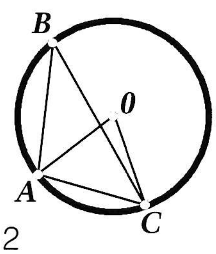


Figure 2

Here are some examples of how these properties can make it a lot easier to solve some rather complicated problems in which the degree values of both the given angles and the angles in question are integers.

Example 1. A triangle ABC is given in which angle A = 70°, angle B = 50°. Point M lies inside the triangle and angle MAC = angle MCA = 40°. Find angle BMC.

Solution. Since angle AMC = 100° = 2 angle B and MA = MC, then, according to property IV, M is the circumcenter of triangle ABC (fig. 3). So, by property III, angle BMC = 2 angle A = 140°.

Example 2. Triangle ABC is constructed such that AB = AC, angle BAC = 40°. Point M lies outside triangle ABC but inside angle ABC such that angle AMB = 30°, angle BMC = 20° (fig. 4). Find angle ABM.

Art by Sergey Ivanov

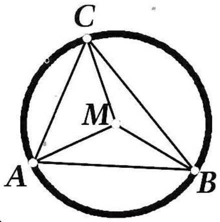


Figure 3

Solution. Property IV implies that point A is the center of the circumcircle of triangle BCM. So $AB = AM$ and angle ABM = angle $AMB = 30°$.

This problem can be generalized to refer to any triangle ABC in which $AB = AC$, angle BAC = 2 angle BMC, angle $AMB = \alpha$. In such a triangle, angle $ABM = \alpha$.

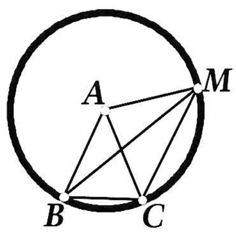


Figure 4

Example 3. Triangle ABC is constructed such that $AB = BC$, angle $ABC = 80°$. Point M lies inside the triangle and angle $MAC = 10°$, angle $MCA = 30°$. Find angle BMC (fig. 5).

Solution. Angles BAC and BCA are both equal to 50°. Let O be the circumcenter of triangle AMC. Then, according to property III, angle AOM = 60°, and so triangle AOM is equilateral. BO is the perpendicular bisector of segment AC (since $AB = BC$ and $AO = OC$). Right triangles ABK and AOK are congruent since angle $OAK = 60° - 10° = 50°$ = angle BAK. Consequently, $AB = AO$; and taking into account that $AO = AM$, we have $AB = AM$. In an isosceles triangle ABM, angles ABM and AMB are equal to $(180° - 40°)/2 = 70°$. Finally, angle $BMC = 360° - 140° - 70° = 150°$ (see figure 5).

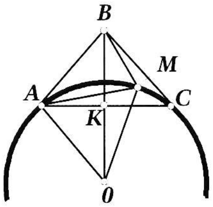


Figure 5

I'll leave you with five more problems for you to tackle on your own.

Problems

1. In a triangle ABC angle $A = 50°$, angle $B = 60°$. Points D and E are chosen on sides AB and BC, respectively, such that angle DCA = angle $EAC = 30°$. Find angle CDE.
2. In a triangle ABC angle $A = 30°$, angle $B = 80°$. Point M lies inside the triangle and angle $MAC = 10°$, angle $MCA = 30°$. Find angle BMC.
3. In a triangle ABC angle $A = 20°$, angle $C = 30°$. Point K lies inside the triangle and angle KAC = angle KCA = 10°. Find angle BKC.
4. In a triangle ABC $AB = BC$, angle $C = 40°$. Outside triangle ABC but inside angle BAC a point M is taken such that angle $AMC = 50°$, angle ABM = angle ACM. Find angle BCM.
5. In a triangle ABC angle $A = 84°$, angle $C = 78°$. Points D and E lie on sides AB and BC such that angle $ACD = 48°$, angle $CAE = 63°$. Find angle CDE.

SOLUTIONS ON PAGE 53

Challenges in physics and math

Math

M21
Square root of seven. A regular hexagon with side 1 is drawn on the plane. Construct a segment of length $7^{1/2}$ using only a straightedge. (A. Aliayev)

M22
Wire cube. What's the shortest length of a piece of wire that can be bent so as to make the framework of a cube with an edge 10 cm long? (The wire can pass the same edge twice, can be bent through 90° or 180°, but can't be broken.)

M23
Patches on jeans. A pair of jeans with a total area of 1 have five patches on them. The area of each patch is not less than 1/2. Prove that there are two patches such that the area of their common part is not less than 1/5. (E. Dynkin)

M24
Power calculating. To find the value of x^8 given x, you need three arithmetic operations: $x^2 = x \cdot x$, $x^4 = x^2 \cdot x^2$, $x^8 = x^4 \cdot x^4$; to find x^{15} five operations will do: the first three of them are the same, then $x^8 \cdot x^8 = x^{16}$, and $x^{16}/x = x^{15}$. Prove that (a) x^{1000} can be found in 12 operations (multiplications and divisions); (b) x^n for any positive integer n can be found in no more than $(3/2) \log_2 n + 1$ operations. (E. Belaga)

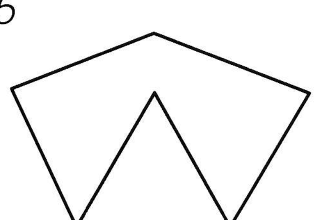
M25


Figure 1

Equilateral triangle and pentagon. Prove that from any convex equilateral pentagon (which isn't necessarily regular) it's possible to cut out an equilateral triangle, one of whose sides coincides with a side of the pentagon (fig. 1). (S. Konyagin)

Physics

P21
Fox and dog. A fox running along a straight line with velocity v_1 was chased by a dog whose velocity v_2 was constant in absolute value and always directed at the fox. When the velocities v_1 and v_2 were perpendicular to each other, the distance between the fox and the dog was l. What was the dog's acceleration at that moment? (I. Slobodetsky)

P22
Floating vessels. A large number of cylindrical vessels containing water are immersed in one another so that each vessel floats in the next one. The bottom area of the smallest vessel is s_0, which is much smaller than that of the largest vessel. A volume of water V_0 is added to the smallest vessel. What is the difference between the old and the new positions of the bottom of the smallest vessel with respect to the ground? (All the vessels continue to float.) (S. Krotov)

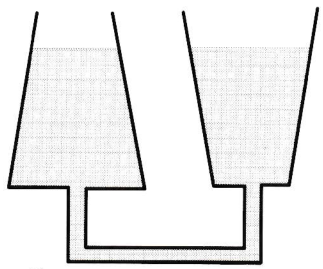


Figure 2

P23
Right or left? Two connected vessels are of the shape shown in figure 2. In what direction will the water flow if one of the vessels gets heated?

P24
Double-size battery. A lamp connected to a battery glows for three hours, then the battery runs down. Another battery is made of the same materials but is twice as large as the original one (in length, width, and height). How long will the new battery last if connected to the same lamp? (The internal resistance of the battery is much less than the lamp's resistance.) (K. Bedov)

P25
Whispering gallery. The phenomenon of a "whispering gallery" is well known in architectural acoustics. In large cathedrals (for example, in St. Peter's Basilica in Rome) tourists are invited to visit a circular gallery at the base of the main dome. A word spo-

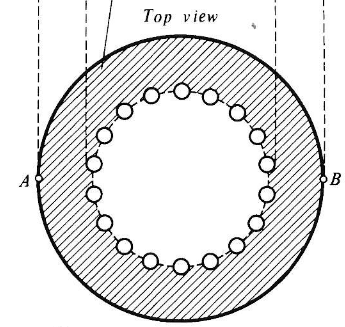
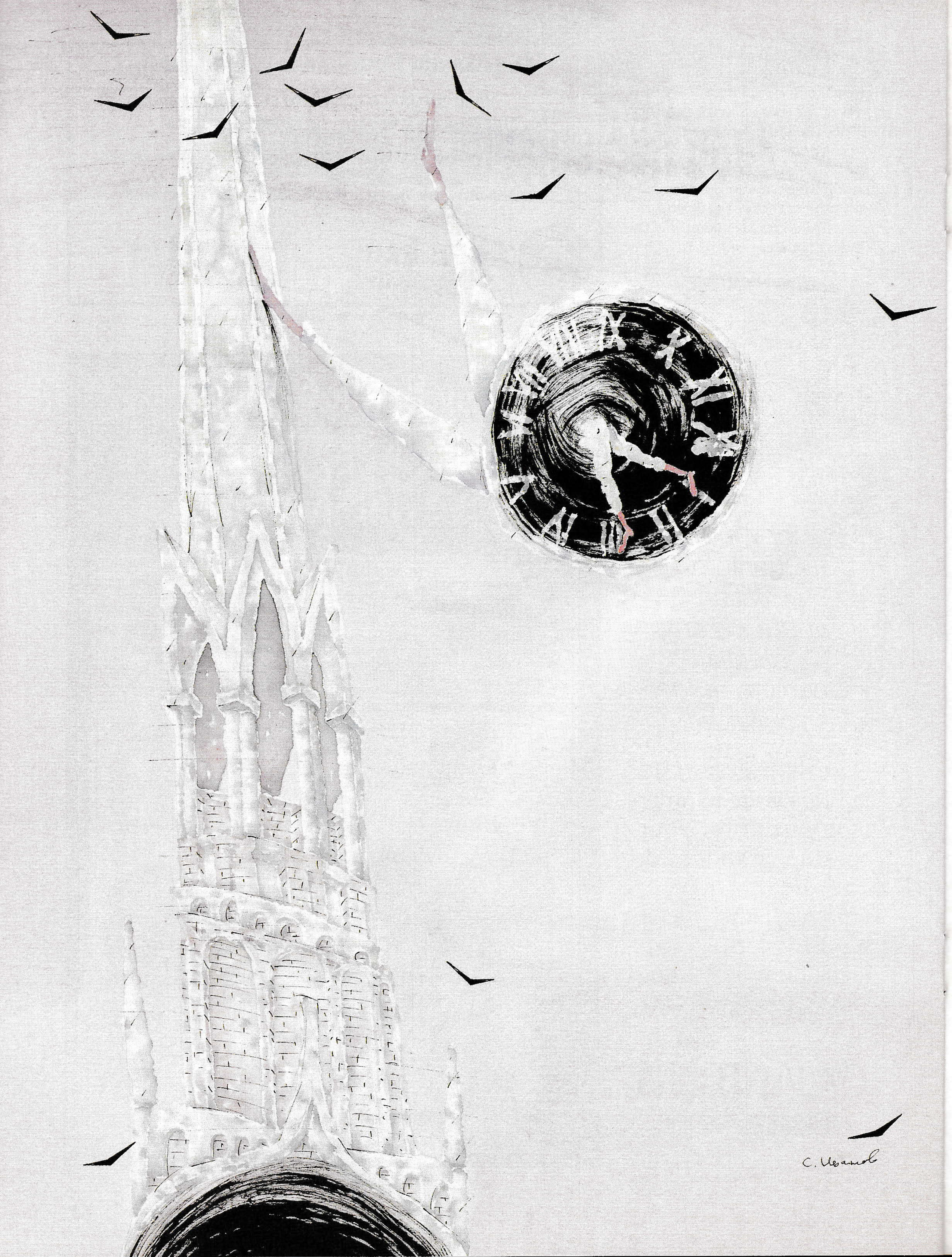


Figure 3

ken quietly at point A (fig. 3) of the gallery is distinctly heard at point B if the speaker looks along the wall. If, however, the speaker looks directly at point B, the listener hears nothing. How can you explain this?

At point A a poorly directed acoustic source emits a relatively loud impulse of duration τ. What's the duration of the impulse received at point B? (The gallery's diameter is d.) (B. Klyachin)

SOLUTIONS ON PAGE 50



Art by Sergey Ivanov

Walking on water

The curious ways we all, creatures great and small, get around

by K. Bogdanov

WE'RE USED TO DIVIDING creatures into those that live in water and those that live on land. Several types of insects, however, spend just about their entire lives on the air–water interface. One of them is the water strider.

This insect skates swiftly along the water's surface like human skaters on ice. Its long legs are splayed widely and support a long, thin body.

The upper layer of water has a high surface tension, which provides a reliable "skating rink" for the water strider. If it runs into an area with substantially lower surface tension, the insect sinks into the water and helplessly flounders in it. Such an area can be created by a few drops of hexane on the surface of the water. The surface tension of hexane (if it's in contact with air) is σ = 18 mN/m, just about one fourth that of water (70 mN/m). (Close observation, however, reveals that water striders can unerringly tell an "acceptable" surface from a dangerous one. If a few drops of water fall on a smooth surface, they rush to the shore and wait until the water calms down again.)

Let's try to estimate the water strider's buoyancy. The insect is a little more than five millimeters long. It has two pairs of long legs keeping it on the surface and a pair of shorter legs, whose main use is to manipulate its prey. The mass of a big water strider doesn't exceed 0.05 g. The force of surface tension supporting the insect on the surface can be estimated from the product $\sigma_w L$, where L is the contour length of the distorted surface. The equality $mg = \sigma_w L$ (the floating condition) yields L equal to 0.5 mN/70 mN $\cdot$ m$^{-1} \cong$ 7.1 mm. And what do we have in reality?

Look at figure 1, which gives two views of the leg–water contact area. The secret of the amazing ability of this insect to skate along the water's surface is in the ends of its legs. They're densely covered with water-repellent hair. Since a water strider has only six legs, the maximum length of the contour line, to which the force of surface tension (T) acts perpendicularly, is about 12 mm.

It's interesting that in conflict situations some insects that live on the water eject a jet of liquid with a surface tension lower than that of water. As they flee, they leave behind a kind of "no-man's land" in which their pursuers sink and start to drown.

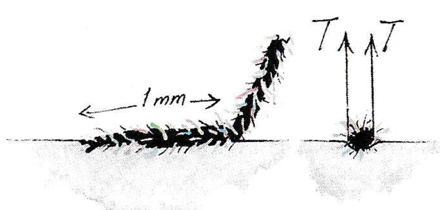


Figure 1
Two views of a water strider's leg on the water's surface.

Art by Sergey Ivanov

Those fearless "window climbers"

How can a fly walk up and down the vertical surface of a windowpane? This question intrigued the great 17th-century scientist Robert Hooke, who supposed it was because of tiny nails that flies have at the ends of their six legs. (In 1665 Hooke gave a detailed description of these nails in his book *Micrography*.) This explanation seems quite reasonable for walking on a rough surface. But in the case of smooth glass, this approach leaves something to be desired.

British scientists used the most advanced scientific equipment to demonstrate that the ability of flies to walk on an extremely smooth surface also is related to surface tension. The scientists discovered that there is hair growing between the nails at the end of each leg. The hair forms a dense brush, and each separate hair ends in a disk-shaped suction cup with an area of $2 \cdot 10^{-12}$ m^2.

An examination of a fly's footprints on a clean surface revealed that their shape is identical to that of the suction cups. The footprints don't evaporate, and a chemical analysis of their content showed them to be fats. Of course, a fat is usually a very slippery substance, but in this case it facilitates the adhesion of the hair to the glass. This is because the surface tension of the fat is high. If a fly's legs are "defatted" by immersing them briefly in hexane, the fly temporarily loses its ability to walk on a glass surface.

To measure the force keeping a fly on a horizontal pane of glass, scientists tied one to a special scale and measured the force needed to lift it. A fly weighs about 0.72 mN. When it stands on only four legs, a force of 1.03 mN is enough to lift it up, but when the fly stands on all six legs, a greater force is necessary—2.4 mN. This experiment reveals that the coupling force derives mainly from surface tension. (If the force is plotted as a function of the number of legs in contact with the surface, it turns out to be a nonlinear dependence, perhaps because there are a different number of hairs on a fly's fore and hind feet.)

That's using your head

If you've never had a chance to travel to Africa or South Asia, you've surely seen on TV or in the movies how women there carry huge loads on their heads. Sometimes the load's weight is 70% of a human being's. Sometimes the help of two men is needed to lift such a load onto the woman's head. But once the load is there the woman easily carries it away. Why is it easier to carry a load on your head than to lift it upwards?

The loss of energy by a man or woman performing some work can be measured by the oxygen consumption rate. One liter of consumed oxygen corresponds to 20.1 kJ of expended energy. Experiments with volunteers have demonstrated that the oxygen consumption increases in proportion to the weight of the load if the load is carried in the usual way (on the carrier's back). For instance, if the weight is equal to 50% of the carrier's own weight, the energy consumption is increased by 50%. The same situation was observed when untrained persons were asked to carry loads on their heads.

You can imagine how surprised the scientists were when they found that African women carrying a load equal to 50% of their weight increased their oxygen consumption by only 30%! How did they do that? Further observation provided the answer.

A woman going home with a vessel on her head filled to the brim with water is a common sight in Africa. However puzzling it may seem, the water never gets spilled. This means there is no (or almost no) vertical acceleration. Consequently, the center of mass of a woman carrying water doesn't oscillate in the vertical direction.

It's known that normal walking causes a noticeable displacement of the body's center of mass. Figure 2 shows two consecutive phases of walking. Assuming that the leg coming into contact with the road isn't bent (is straight at the knee joint), the center of mass is at its lowest point when both legs touch the ground. The highest point of the center of mass is reached when the leg standing on the ground is in the vertical position. This suggests that the center of mass moves along a circular arc whose radius is equal to the length of the leg, periodically going up and down several centimeters.

Figure 2
The curve followed by the center of mass as a person walks.

Of course, this up and down movement of the center of mass is utterly useless, but it consumes energy. The way we're accustomed to walking may be compared to an inexperienced driver alternately pressing the brake and gas pedals, trying to maintain a constant velocity and using up a lot more gas in the process.

Unlike Europeans, many Africans and South Asians developed a walking style that keeps their center of mass at a constant level, thus substantially reducing their energy consumption.

And now for some aerobics

Maybe you've come across a lesson in your physics textbook in which two pendulums are suspended from the same crossbar. If you kick one of them, it starts oscillating alone, but after several cycles the other begins to move synchronously with the first. A similar phenomenon can be observed in the body of a running animal when two "pendulums" interact—the periodic motion of the animal as a whole and that of its lungs.

Figure 3 shows a kangaroo's lung "pendulum." It operates in the following way: during inhalation, when the lungs fill with air, the abdomen's center of mass shifts to the left; during exhalation it moves to the right. The elastic properties of the diaphragm and other tissues are depicted in the drawing as a spring, and the organs damping the oscillations are shown as a shock absorber. Thus, the abdominal organs act as a kind of piston, oscillating in phase with the breathing.

Obviously the energy consumption of a running animal is minimal when inertial forces caused in its body by its periodic acceleration and deceleration help (rather than hinder) the breathing process. Such thinking leads us to suggest that the breathing rate should be close to the animal's stride frequency.

Special experiments performed with kangaroos, horses, rabbits, and dogs verified this idea. It was found that the most convenient ratio of stride frequency to breathing rate (especially at a brisk gallop) is 1:1. In humans this relationship is more complicated, covering a range of values (4:1, 3:1, 2:1, 1:1, 5:2, 3:2), although the ratio 2:1 seems most likely.

The apparent independence of the human breathing rate relative to the speed of running may be explained by the vertical position of the body. In humans breathing is accompanied by vertical displacement of the abdominal organs, whereas inertial forces act in the horizontal direction. So the "breath pendulum" in humans is affected by inertial forces much less than that in animals.

Going back to the textbook experiment with the two pendulums, we

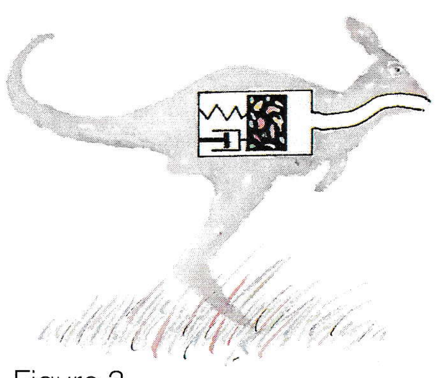


Figure 3
The "breath pendulum" in an animal's body simulated by a sliding piston attached to a spring and shock absorber.

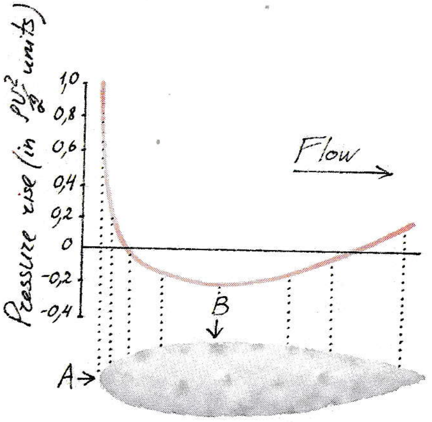


Figure 4
Distribution of pressure in a fluid flowing around a symmetrical drop-shaped body. (Since the variation in pressure is always proportional to $\rho v^2/2$, it can be expressed in dimensionless units $2p/\rho v^2$.)

notice that induced oscillations of the second pendulum have the maximum amplitude if both of them have the same period of oscillation. So we can expect that the natural oscillations of the breath pendulum should be close to the stride rate. And experimental evidence supports this conjecture. Experiments with dogs have shown that the frequency of natural oscillations of the breath pendulum was about 4 Hz, and the frequency of strides when the dogs ran at a gallop was 3.2 Hz.

Since the frequency of natural oscillations of the breath pendulum is constant for a given animal, the stride rate should be kept constant to minimize energy consumption even if the speed varies. Indeed, both four-legged and two-legged animals (for instance, the kangaroo) keep their gallop rate constant, while the length of their strides can vary by a factor of two or three.

Bernoulli is for the birds

According to Bernoulli's principle ($\rho v^2/2 + p$ = constant, where ρ is the density of a gas or liquid, v is the linear velocity of its motion, and p is the pressure) a variation in the velocity of a flow alters the pressure inside the flow. This is a direct consequence of the law of conservation of mechanical energy, since $\rho v^2/2$ is numerically equal to the kinetic energy of the fluid and p is the potential energy of its compression (the pressure energy).

Bernoulli's principle predicts that if a body moves inside a fluid (or a fluid flows around it in a streamlined flow), the pressure in the fluid adjacent to the body is different at different points of the flow. Figure 4 illustrates this variation for a drop-shaped body.

At points where the fluid encounters the body (point A), its velocity falls and the pressure in the fluid increases. Moving further along the body's contour, the fluid accelerates and in some areas adjacent to the body (point B) it moves with a greater velocity than that of the rest of the flow, which by Bernoulli's principle is accompanied by a decrease in pressure. So high pressure affecting the part of the body facing the flow tends to compress the body, and the low pressure in the vicinity of its sides tends to flatten it.

People have been able to make effective use of Bernoulli's principle. Figure 5 shows the cross section of an airplane wing. Let's consider the motion of two air particles. Suppose the particles were close to one another before striking the wing's leading edge. Then they separate and travel along the upper and lower parts of the wing, respectively, until they finally meet at the rear edge. Particle A, however, makes a longer trip than particle B, which means that the average velocity of the first particle is greater. So, by Bernoulli's principle, the average pressure on top of the wing is lower than that under it. It is this pressure difference that accounts for the upward lift force (which, of course, depends on the surface area and shape of the wing).

How is Bernoulli's principle used in the animal world? The most vivid example is the soaring flight of birds. Although the aerodynamics of such flight isn't completely understood even now, its main features are similar to that of the human imitation, the airplane. But nature has many more mysteries that can be unraveled by means of Bernoulli's principle.

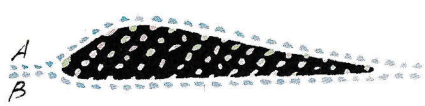


Figure 5
Air flow around an airplane wing.

Most of you surely know that a squid uses jet power to get away from a predator, expelling water out of its mantle cavity. But it's only recently that scientists understood how the cavity fills with water.

Figure 6 presents a schematic view of a squid and shows the direction of its motion caused by the jet of water expelled from the tube (siphon) near the mollusk's head. The inlet valves through which water enters the mantle cavity are in the middle of the animal's side.

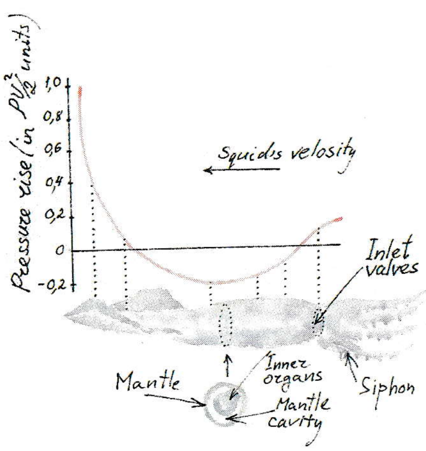


Figure 6
Distribution of water pressure around the body of a moving squid and the squid's cross section.

All our notions about the drop in pressure of a fluid flowing around a body can be readily applied to the squid. The water pressure is lowest at the middle of a swimming squid, near the mantle cavity (see figure 6). The inlet valves are located farther back so that the pressure there is greater than the average pressure inside the mantle cavity. It is this drop in pressure that causes the water to be drawn into the cavity.

The extent to which Bernoulli's principle contributes to the mechanism by which the mantle cavity fills with water has been evaluated by numerical modeling. The pressure gradient (that is, the variation in pressure over unit length) given by Bernoulli's law depends on the squid's velocity and is responsible for 50% to 90% of the water intake when its

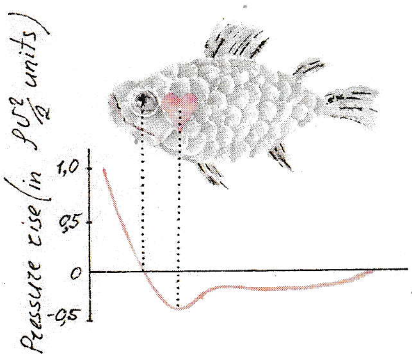


Figure 7
Distribution of pressure around the body of a swimming fish.

velocity increases from 3 to 9 meters per second. This indicates that Bernoulli's principle is indeed very important in the squid's activity, since normally its velocity ranges from 5 to 10 meters per second.

But of all animals, perhaps fish make the most effective use of Bernoulli's principle. The long course of evolution optimized their bodies to such an extent that the total drop in pressure near the middle of a swimming fish just about equals the increase in front (compare figures 4 and 7). Scientists believe that a significant drop in pressure near a fish's heart may help the heart's activity, since lower pressure in the heart's ventricles must increase the influx of blood.

Not only that, a fish's body is built in such a way that there's an area where pressure doesn't depend on the velocity of motion and always equals the hydrostatic pressure. This is where a fish's eyes are located. So the eyes—the organs with the least protection against deformation—never experience the increase in water pressure caused by an increase in its swimming speed.

Some species of beetles are known to spend most of their life under water. In so doing they breathe air from a bubble they always carry with them. *Potamodytes tuberosus*, which lives in the rivers of West Africa, is one such "submarine" beetle. Usually this beetle, together with its attached air bubble, anchors itself to a stone lying in flowing water. As soon as the beetle finds itself in still (standing) water, the bubble starts to shrink and disappears completely in a couple of hours. This forces the beetle to look for another bubble. So, when it finds a bubble, the beetle prefers to stay in flowing water.

The fact that the air bubble is more stable in flowing water can also be explained by Bernoulli's principle. The pressure of the water flowing around the bubble, elongated in the direction of the flow, is less than the hydrostatic pressure along almost the entire length of its surface. So the air pressure inside the bubble in flowing water is lower than that in still water. If a beetle is in a shallow place, the air pressure inside the bubble is below the atmospheric pressure, so that the air dissolved in the water (at atmospheric pressure) tends to enter the bubble and it starts expanding.

Even at a depth of several centimeters an air bubble often remains stable despite the hydrostatic pressure that works against it. Calculations show that for a bubble to be stable at a depth of 1 cm, the velocity of the water flow should be greater than 1 m/s; at 4 cm, the flow should be faster than 2 m/s.

The modest experimentalist, Henry Cavendish

Are unpublished results like a tree falling in the woods?

by S. Filonovich

THE NAME HENRY CAVENdish is associated with a multitude of discoveries that didn't become known until long after they had been made. The personality of the man, who dedicated his entire life to the natural sciences, has attracted the attention of physicists, historians of science, and psychologists for many years.

A scion of the noble family of the Duke of Devonshire, Cavendish was born October 10, 1731, in Nice, where his mother was living at the time on the advice of her doctors. Her health was delicate, and the birth of her children strained it even more. Lady Cavendish died shortly after the birth of her second son, when her first son Henry was two years old. At the age of eleven, Henry was sent to one of the best public schools in London, and in 1749 he entered Cambridge University, which he left in 1753 without taking a degree. It has been argued that he left Cambridge because of his painful shyness and fear of examinations.

Cavendish travelled for some time with his younger brother throughout Europe, and then settled in London, in his father's house. Sir Charles Cavendish deserves special attention. He was a noble, though not very rich, man who was primarily interested in the natural sciences. For many years Sir Charles had been a member of the Royal Society, and for some time he was its vice president. His scientific interests were mainly concentrated in the field of electricity, which was fashionable at the time. The American scientist and statesman Benjamin Franklin wrote about Cavendish senior: "It is to be wished that this noble philosopher would communicate more of his experiments to the world, as he makes many and with great accuracy."

It's not unlikely that father and son performed some experiments together, and Cavendish's interest in science was greatly influenced by his father. But from the documents that have been preserved, it's clear that Henry performed the majority of the most important experiments on his own.

The range of Cavendish's work is so broad that it's difficult to assign him a particular place in science. During his lifetime he was famous as a chemist. Because of his pioneering work on gases, Cavendish is sometimes called the father of the chemistry of gases. He was the first to determine the nature of hydrogen as a separate gas, to verify that air is a mixture of oxygen and nitrogen, and to demonstrate that water consists of oxygen and hydrogen. He studied electrical phenomena in chemistry and found that nitric acid is generated by electric sparks in humid air.

During one of his electrochemical experiments Cavendish obtained inert gases from the air. He did this by letting electric sparks pass through oxygen-enriched air in a U-shaped glass tube. Both ends of the tube remained unsealed and each was dipped in a vessel containing a solution of caustic soda. Two metal wires, attached to the contacts of a machine that produced electricity by friction, passed through the solutions and inside the tube with the gas. When the machine was started (it was Cavendish's servant who actually rotated the disk), sparks ran between the ends of the wires and generated nitric acid, which was absorbed by the solutions. By manipulating the mixture of air and oxygen, it was possible to decrease the volume of gas inside the tube. But, as Cavendish mentioned in his laboratory journal, a small bubble of gas still remained despite all his effort. This discovery remained unknown until the end of the nineteenth century, when two British scientists, Rayleigh and Ramsay, successfully applied it to obtain inert gases from the air and study them.

Many chemical problems studied by Cavendish were also studied by his contemporaries Lavoisier, Watt, and Priestley. But Cavendish always aimed at a rigorous quantitative examination. For example, he not only proved the existence of hydrogen, he also found that this gas is lighter than air by a factor of eleven. Using the eudiometer (an instrument he modified specifically for this purpose), he studied the volumetric proportions among air, hydrogen, and water generated when such a mixture explodes.

It's often alleged that Cavendish devoted all his life to experimental science but never published any results. This is an exaggeration. It's true that, for reasons that aren't entirely clear, Cavendish seldom published his scientific findings. When he did, it was often long after the experiments were conducted, and this led to disputes about who made a discovery first. At any rate, a dozen or so of his papers in physics and chemistry were published in the *Philosophical Transactions* of the Royal Society of London. Cavendish had been a member of the society since 1760 and actively participated on a number of its committees. He took part in its meetings and dinners and helped G. Banks, the president of the Royal Society, in his work. Contemporaries said that Cavendish was reluctant to get into arguments, perhaps because of his high-pitched voice. The eminent English physicist and chemist Sir Humphry Davy wrote that Cavendish's main passion was a disinterested search for the truth and that fame and publicity repelled him.

Besides chemistry, Cavendish was interested in geology. He made several journeys across England to study the regional variations in its geology. During these trips he became interested in metallurgical processes, whose improvement required a knowledge of physics and chemistry.

Cavendish was acquainted with the most important English scientists of his time: Priestly, Davy, Watt, Young. His scientific activity continued almost up to his death, which came after a short illness on February 24, 1810. His last paper had to do with astronomical instruments.

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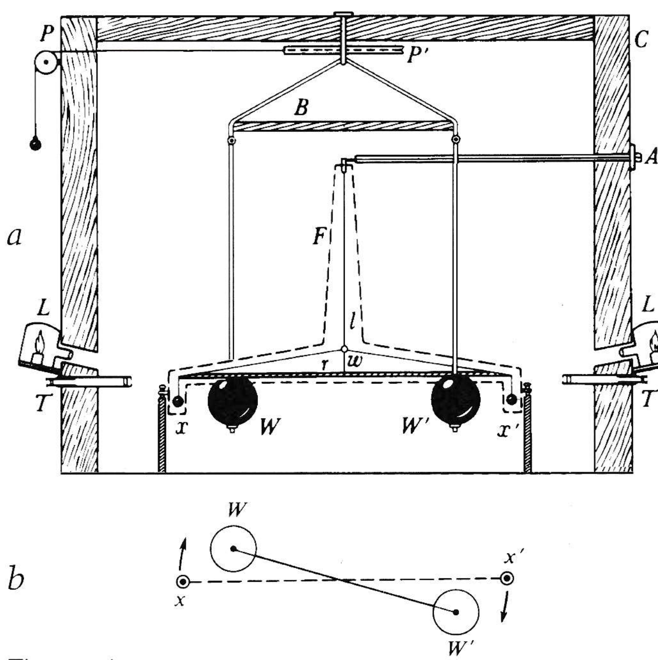


Figure 1

(a) The apparatus used by Cavendish to determine the density of the Earth. The whole apparatus is contained in an outer casing C*, which preserves it from air flows caused by movement of the experimentalist;* P *and* P' *are pulleys for rotating the beam* B *from which the two large balls* W *and* W' *are suspended;* F *(broken line) is an inner casing protecting the torsion balance from drafts and temperature changes;* A *is a thumbscrew for adjusting the torsion balance;* l *is a torsion wire;* r *is a torsion rod steadied by wires* w *supporting two small balls* x *and* x' *at the ends;* L *and* L' *are lamps;* T *and* T' *are telescopes.*
b. A view of the system of balls from above. The arrows indicate the movement of the small balls x *and* x'.

One paper by Cavendish that acquired widespread fame during his lifetime presents his work in determining the mean density of the Earth. It was published in 1798, and nowadays the experiment described in that paper is known as the Cavendish experiment. The question of determining the Earth's density arose because calculations of the Earth's deformation caused by its rotation, assuming that its density is constant, led to a disagreement with data from geophysical observations. Newton himself had suggested that the density of the Earth's inner layers could be six times that of water. But all attempts to obtain an agreement between calculation and observation failed. An exact quantitative experiment was needed.

Before the Cavendish experiment attempts had been made to determine the density of the Earth by observing the deflection of a pendulum caused by the attraction of a mountain. But the method involved a lot of errors and uncertainties, and Cavendish rejected it. Instead, he used—and substantially improved—an instrument invented by the English scientist John Mitchell.

The aim of the experiment was to determine the period of torsional oscillations of a rod with two light balls at its ends. The rod was suspended at its middle by a silver-plated copper wire (fig. 1). The period and amplitude of oscillations of the system depend on the attraction exerted on the balls attached to the ends of the rod by two larger balls outside. This attraction is caused by gravitation. Using some mathematics (and Cavendish was an expert mathematician), one can find the constant of gravity G through the measured values of the period and amplitude of oscillations. Next, one can find the mean density of the Earth by using the mean radius of the Earth and the gravitational acceleration g (whose values can be found from geophysical measurements). In fact,

$$g = \frac{GM}{R_E^2},$$

where $M = (4/3)\pi R_E^{\,3} D$ is the mass of the Earth. So

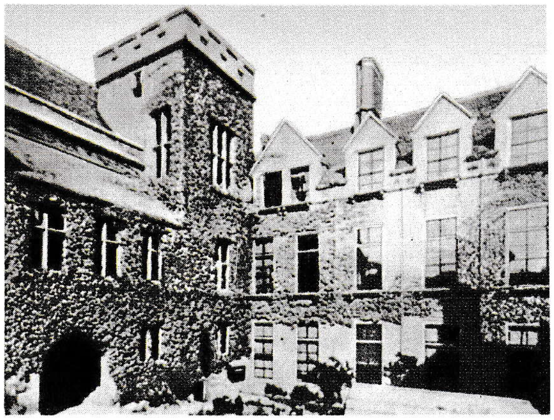


Figure 2
The Cavendish Laboratory at Cambridge University (from a photograph taken just before the turn of the century). Over the years many outstanding physicists have worked there, including Nobel laureates Lord Rayleigh, Sir Joseph John Thomson, Ernest Rutherford, Sir William Henry Bragg and Sir William L. Bragg, Charles T. R. Wilson, Sir James Chadwick, George Thomson, Sir Nevill F. Mott, and Pyotr Kapitsa.

$$D = \frac{3g}{4\pi G R_E}.$$

The important thing, of course, is to determine the fundamental constant G by using the data obtained in the laboratory, not to find the geophysical quantity D. So the Cavendish experiment is generally considered an experiment for determining G.

Cavendish showed great experimental ingenuity in constructing such an apparatus that the distance between the small and the large balls could be changed by an observer at a distance. This reduced the influence of extraneous factors on the results of the experiment. The use of a telescope for taking readings off the apparatus enabled him to make very exact measurements of the displacement of the balls (down to 1/20 inch).

To reduce experimental error Cavendish devised a special method of measurement, which he performed by observing the arm of the rod. He wrote in his paper: "I observe three successive extreme points of a vibration, and take the mean between the first and third of these points, as the extreme point of vibration in one direction, and then assume the mean between this and the second extreme as the point of rest . . . ." In this way he was able to determine the deflection of the rod from some middle position; or, to put it another way, to find the amplitude of the oscillations. To determine the time, or period, of oscillation, he proceeded as follows: "I observe the two extreme points of a vibration, and also the times at which the arm arrives at two given divisions between these extremes, taking care, as well as I can guess, that the divisions shall be on different sides of the middle point, and very far from it. I then compute the middle point of the vibration, and by proportion, find the time at which the arm comes to this middle point. I then, after a number of vibrations, repeat this operation, and divide the interval of time, between the coming of the arm to these middle points, by the number of vibrations, which gives the time of one vibration."

To determine the mean density of the Earth, Cavendish performed seventeen series of measurements. According to his data the ratio of the density of the Earth to that of water equals 5.48.

The experiments were highly appreciated by contemporary scientists. In 1820 the eminent French mathematician and physicist Laplace wrote: "On examining with the most scrupulous attention the apparatus of Monsieur Cavendish and all his experiments made with the precision and thoughtfulness that are characteristic of this excellent physicist, I see no objection to his result, which assigns 5.48 as the value of the mean density of the Earth."

At present the mean density of the Earth is taken to be 5.517 g/cm^3. A modernized version of Cavendish's torsion balance is still used for physical measurements.

The scientific heritage of Henry Cavendish isn't confined to the experiments and discoveries described above. Cavendish obtained important results in studying heat phenomena as well. He determined the specific heat of various substances, studied the process of melting, and discovered the phenomenon of latent heat of melting. Cavendish also performed important experiments in electricity and magnetism, some of which became known because of the efforts of James Clerk Maxwell.

The story of how they came to be published is interesting in its own right. In 1861 William Cavendish, the Duke of Devonshire, was elected chancellor of Cambridge University. The duke had graduated from Cambridge and had shown some talent in mathematics. In 1870 he suggested that a physics laboratory be built at the university, and he created a special fund to that effect. In 1871 a laboratory named after its founder, the Cavendish Laboratory, was established. In accordance with the recommendations of the eminent scientists Stokes, Rayleigh, and Thomson, the position of Cavendish Professor was offered to the great physicist James Clerk Maxwell. He accepted the offer and actively turned to building the laboratory, which was completed in three years. The Duke of Devonshire also placed Cavendish's manuscripts in Maxwell's hands, and Maxwell agreed to look them over. Maxwell was amazed by what he read. It turned out that Cavendish had discovered Ohm's law long before Ohm, had studied the conductivity of solutions, and had

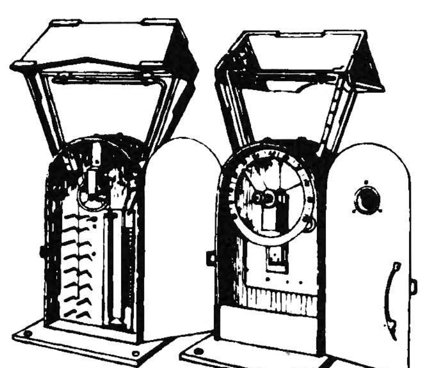


Figure 3
Cavendish constructed special measuring devices for almost all of his experiments, taking into account convenience of use as well as precision of measurement. He made this very precise thermometer for studying the thermal properties of substances. In its appearance the apparatus isn't all that different from modern high-precision thermometers.

made very precise measurements of capacitance. Maxwell devoted great care to publication of the manuscripts and even repeated some of the experiments himself. Cavendish's manuscripts were finally published in 1879, just a few months before Maxwell died.

Did scientists acquire a better understanding of Cavendish's work from this publication? This simple enumeration speaks volumes about the immense amount of information contained in it. He was the first to give an accurate definition of electrical capacitance and used the capacitance of a prescribed size as a unit of capacitance; he studied the dependence of the conductivity of aqueous saline solutions on concentration and temperature; and he predicted the laws of direct current long before Ohm.

Cavendish also found that the repulsion (or attraction) of electrical charges depends on distance, a discovery made more than 10 years before Coulomb. Maxwell found the paper describing Cavendish's apparatus and measurement procedure, and it seems that the paper had been prepared for publication. This experiment is of particular interest because all modern tests of Coulomb's law are based on the method proposed by Cavendish.

Here's how Cavendish described his experiment: "I took a globe 12.1 inches in diameter and suspended it by a solid stick of glass run through the middle of it as an axis, and covered with sealing-wax to make it a more perfect non-conductor of electricity. I then inclosed this globe between two hollow pasteboard hemispheres 13.3 inches in diameter, and about 1/20 of an inch thick, in such manner that there could hardly be less than 1/10 of an inch distance between the globe and the inner surface of the hemispheres in any part, the two hemispheres being applied to each other so as to form a complete sphere, and the edges made to fit as close as possible, notches being cut in each of them so as to form holes for the stick of glass to pass through. By this means I had an inner globe included within an hollow globe in such a manner that there was no communication by which the electricity could pass from one to the other. I then made a communication between them by a piece of wire run through one of the hemispheres and touching the inner globe, a piece of silk string being fastened to the end of the wire, by which I could draw it out at pleasure.

"Having done this I electrified the hemispheres by means of a wire communicating with the positive side of a Leyden vial, and then having withdrawn this wire, immediately drew out the wire which made a communication between the inner globe and the outer one, which, as it was drawn away by a silk string, could not discharge the electricity either of the globe or hemispheres. I then instantly separated the two hemispheres, taking care in doing it that they should not touch the inner globe and applied a pair of small pith balls, suspended by fine linen threads, to the inner globe, to see whether it was at all over or undercharged."

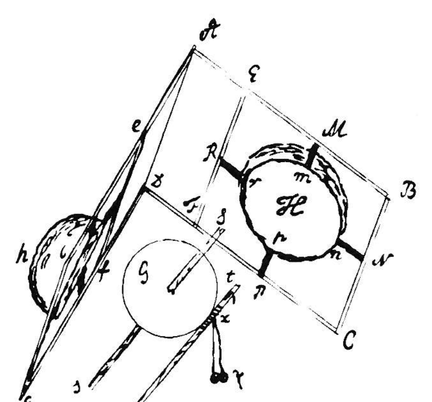


Figure 4
Cavendish's sketch for his experiment with the globe and two hemispheres.

Except for the very last phrase, the description looks very modern! Because Cavendish accepted Franklin's theory of electricity, he used the terms "over" or "undercharged" body, which simply means "electrified body."

One can easily show that if the inner globe is charged after the described procedure was used, then the electrical interaction between point charges doesn't obey the law $1/r^2$. Cavendish invented a special means of making measurements more accurate. He even calculated possible experimental error and decided that if the law of electrical force is

$$F \sim \frac{1}{r^{2 \pm q}},$$

then

$$q < 1/50.$$

Maxwell was so excited by this experiment that he asked his assistant to repeat it using a more sensitive electrometer to determine if the inner globe has any charge or not.

The result of this test was $q < 1/21600$. The progress of this physical experiment might be illustrated by the fact that modern tests give $q < (2.7 \pm 3.1) \cdot 10^{-16}$!

And to think that all these results found by Cavendish had remained unpublished! Some of his experiments were conducted anew. In most cases Cavendish's results turned out to be very accurate.

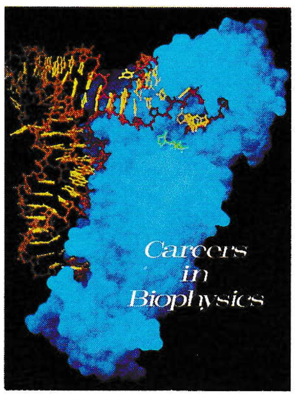
Interest in the scientific legacy of Henry Cavendish hasn't diminished. In 1927 a new edition of his papers, which contained some previously unpublished material, was published by Cambridge University Press. And again there was a sensation: his measurements of the Earth's magnetic field gave new data for the magnetic history of the Earth. It turned out that Cavendish had put forward the idea of energy conservation and considered the quantity corresponding to the potential energy. Again there were lamentations that Cavendish's results had been unknown for more than one hundred years. Not only his results but the problems themselves were largely unknown to his contemporaries, and quite often they constituted the program of research that was conducted throughout the nineteenth century.

So acquaintance with the scientific work of Henry Cavendish amazes and bewilders us by the scope of his imagination and the accuracy of his experiments. Even though many of his results were obtained anew by other scientists, who are rightly considered the authors of these discoveries, Cavendish's work has an importance all its own. ◘

Bulletin board

Careers in biophysics

The Biophysical Society is offering a 20-page full color brochure called "Careers in Biophysics." Designed for high school and college students, the booklet discusses opportunities for those interested in the physics and physical chemistry of biological processes, making great use of quantitative measurements and analysis. Biophysicists work in universities, industry, medical centers, research institutes, and government, using the methods of mathematics, physics, chemistry, and biology to study how living organisms function.



A 22-minute videotape is also available for a small fee. It shows a table discussion by three scientists and two students. To obtain information on the video, or to receive a free brochure, write to Emily Gray, Administrative Director, Biophysical Society, 9650 Rockville Pike, Bethesda, MD 20814, or call 301 530-7114.

Brandeis Summer Odyssey

For those students entering grades 10-12 who are interested in an academic experience that combines science and interdisciplinary studies with social and recreational activities, Brandeis University offers its Summer Odyssey. Two programs are available: the Academic Study Program, which offers students one innovative course in such scientific fields as biotechnology or astronomy, and one complementary course in areas such as creative writing or politics; and the Science Research Internship, which allows students to serve as research apprentices in laboratories at Brandeis University, working closely with faculty members on frontier research topics in such areas as computer science, physics, and psychology. Students in both programs take part in workshops, recreational field trips, and cultural outings during the course of the session.

This year the Science Research Internships are conducted from June 23 to August 16. The Academic Study Program takes place July 7 to August 3. Application deadlines are April 15 and May 15, respectively. For more information on the Summer Odyssey programs, costs, and financial aid availability, contact Jane Schoenfeld, Assistant Provost, Summer Odyssey, Brandeis University, PO Box 9110, Waltham, MA 02254-9110, or call 617 736-2113.

Program in Mathematics for Young Scientists (PROMYS)

Boston University and the National Science Foundation offer a PROMYS for students entering grades 10-12. Through their intensive efforts to solve a large assortment of challenging problems in number theory, the participants practice the art of mathematical discovery—numerical exploration, formulation and critique of conjectures, and techniques of proof and generalization. More experienced participants may also study algebra, combinatorics, and the theory of algebraic curves. Special lectures by outside speakers offer a broad view of mathematics and its role in the sciences. Each participant will also belong to a problem-solving group that meets with a professional mathematician three times a week.

This year's program runs from June 30 to August 10. Admissions decisions will be based on the following criteria: applicants' solutions to a set of challenging problems included with the application packet, teacher recommendations, high school transcripts, and student essays explaining their interest in the program. Applications will be accepted from March 1 to June 15, and financial aid is available. For more information or an application packet, write to PROMYS, Department of Mathematics, Boston University, 111 Cummington Street, Boston, MA 02215, or call 617 353-2560.

Principles of science in a kit

Edmund Scientific Company has introduced a new line of five kits that can be assembled into actual working models, allowing young scientists a greater understanding of science principles. Kits come complete with parts and instructions for making these projects: a working water pump, an air speed/direction anemometer, a light-flashing railroad signal, an electricity-producing generator, and a working motor. Kits retail from $12.95 each. For more information, write to Edmund Scientific Company, Department 11B1,E999, Edscorp Building, Barrington, NJ 08007.

—Compiled by Elisabeth Tobia

Calendar calculations

The "Doomsday" rule

by John Conway

HERE'S AN EASY WAY TO REMEMBER the entire calendar for any particular year.

You first find out on what day of the week the last day of February falls. I call this particular day of the week the "Doomsday" for the year. For example, in 1991, February has 28 days, and since February 28, 1991, is a Thursday, we shall say

in 1991, "Doomsday" is "Thursday."

Now the date that is four weeks earlier than February 28 is

"February 0" = January 31,

and so in 1991 (or any other year that isn't a leap year) the last day of January is also a "Doomsday."

Leaving leap years aside for the moment, let's move on to the later months in the year. We can think of February 28 as "March 0," so that the date exactly five weeks later is "March 35" = April 4, so that

the fourth day of the fourth month is a Doomsday, and similarly:
the sixth day of the sixth month is a Doomsday,
the eighth day of the eighth month is a Doomsday,
the tenth day of the tenth month is a Doomsday, and finally
the twelfth day of the twelfth month is a Doomsday.

Why do these dates all fall on the same day of the week? The reason is that the interval between any two adjacent ones is two months and two days, which amounts to 30 + 31 + 2 = 63 days, since it happens that one of the months has 30 days and the other has 31. And of course 63 days = nine weeks.

Some people (including me) have difficulty remembering (for instance) just which month is the eighth month of the year. I recommend that such people repeat the following refrain:

"April the fourth, June the sixth, August the eighth, October the tenth, December the twelfth,"

which serves the double purpose of reminding us both that August (say) is the eighth month and that August the eighth is a Doomsday in that month.

What about Doomsdays in the odd-numbered months other than January? The rule is that in the nth month, if n is odd, the $n + 4$th or $n - 4$th day is a Doomsday, namely the $n + 4$th day in a long odd month (31 days) but the $n - 4$th day in a short odd month (30 days). You don't have to pause to work out which months are long and which short if you just

remEMBER that
SeptEMBER and
NovEMBER

are the only short odd months. The Doomsday Table summarizes all this:

Month	Doomsday	Mnemonic
January	31 or 32	"last"
February	28 or 29	last
March	7	3 long
April	4	4 even
May	9	5 long
June	6	6 even
July	11	7 long
August	8	8 even
September	5	9 short
October	10	10 even
November	7	11 short
December	12	12 even

The entries for January and February differ from the others in that they are affected by the leap year phenomenon. The Doomsday for February is by definition its last day—that is, the 28th or 29th depending on whether the year is an ordinary year or a leap year. The Doomsday we pick for January is correspondingly the 31st or 32nd. Although, of course, the "32nd of January" is actually in February, we prefer to pretend that in leap years January has 32 days, so that we're taking the "last" day of January.

If you want to become an adept, you should now memorize this table. It's a good idea to find a like-minded friend to practice with: one of you names months at random, the other responds with the corresponding Doomsdays. After a time you should go on to name the other Doomsdays in these months, which of course are found by adding and subtracting multiples of 7 from those given. For instance,

"First Doomsday in July?" "the Fourth of July" (11 – 7);
"Last Doomsday in December?" "December 26" (12 + 14);
"Doomsday in mid-August?" "August 15" (8 + 7).

The first of these examples is easily remembered by Americans!

Now you can go out and startle your friends by telling them the day of the week for any given date in 1991. What you do when faced with a given date is to quickly figure out a nearby Doomsday (Thursday in 1991) and express that date as a few days "on" (after) or "off" (before) that Doomsday. The rest is easy.

Examples:

June 9 = "3 on" (June 6) = 3 on Thursday = Sunday;
Christmas day (Dec. 25) = "1 off" (Dec. 26) = 1 off Thursday = Wednesday.

Just how does one work out that "3 on Thursday" is Sunday? English-speaking readers might find my mnemonics (memory devices) helpful:

NUNday, ONEday, TWOsday, TREBLESday, FOURSday, FIVEday, SIXerday, SE'ENday

for SUNday, MONday, TUESday, WEDNESday, THURSday, FRIday, SATurday, SUNday.

These help identify the days of the week with numbers, and so it becomes trivial to see that "3 on FOURSday" is "SE'ENday."

Doomsdays in other years

If you have learned all this, you might like to know how to work out dates in others years as well. To find any Doomsday in a give century, all you need to know is what the Doomsday was for the century year. For example,

1900 = Wednesday,

and how Doomsday changes from year to year. The rule is that Doomsday normally advances by one day a year, but by an additional day in leap years. It follows that as 12 years in a century roll by, Doomsday advances by 12 + 3 days, since three of those years will be leap years. Since this is one day more than two weeks, we see that, as far as Doomsday is concerned,

"A dozen years is but one day."

This gives us an easy rule to find the Doomsday for any year in any century. Add together the century day, the number of whole dozens thereafter, the remainder, and the number of fours in the remainder, casting out multiples of 7 whenever you like. For example, for 1991 we say

"Wednesday, 7 dozen, 7, and 1 equals Thursday,"

because we can ignore those 7's, and because Doomsday in 1900 was a Wednesday, "91" is "7 dozen and 7," and there is just one "4" in the last "7." For the year 1969 we should say

"Wednesday, 5 dozen, 9, and 2 = Wednesday + 2 = Friday,"

because we can cancel 5 + 9 = 14, and because 69 is "5 dozen and 9," and there are two 4's in 9.

In practice it's best to combine this calculation with the calculation within the given year, as in this example:

"What day of the week was August 10, 1946?"
"2 on Wednesday, 3 dozen, 10, and 2 = Wednesday + 3 = Saturday."

This is because August 10 is "2 on" a Doomsday, and we can cancel 2 + 10 + 2 = 14 days. You'd be wise to assemble all the things to be added before trying to add any of them since there will probably be lots of cancellations, which will mean that in the end you hardly have to add anything!

Doomsdays in other centuries

Our last table gives all the Doomsdays you are likely to need for the "century years":

Julian Doomsdays				Gregorian Doomsdays				
000	700	1400	Sunday					
100	800	1500	Saturday		1600	2000	2400	Tuesday
200	900	1600	Friday		1700	2100	2500	Sunday
300	1000	1700	Thursday		1800	2200	2600	Friday
400	1100	1800	Wed.	1500	1900	2300	2700	Wed.
500	1200	1900	Tuesday					
600	1300	2000	Monday					

In actual fact there was no "year 0," since 1 B.C. was immediately followed by 1 A.D. I just remember that in the Julian system the multiples of 700 were Sundays, and moving a century backwards adds one day. In the Gregorian system, I remember that 1900 was a Wednesday and that each century backwards to 1600 adds *two* days, while the entire period is 400 years.

In the Julian system, as instituted by Julius Caesar, every multiple of 4 was a leap year. In the Gregorian system, instituted by Pope Gregory IV, the multiples of 100 are not leap years unless they're also multiples of 400. The Julian system was used up to October 4, 1582, in Italy, France, and Spain; September 2, 1752, in Britain (and the North American colonies); and 1919 in Russia. So, for example, January 1, 1901, was

"2 off Wednesday, 0 dozen, 0, and 1 = Tuesday"

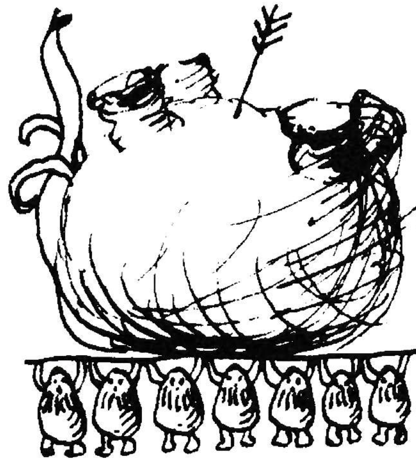
in America, but

"2 off Tuesday, 0, 0, 1 = Monday"

in Russia.

The simplicity of mathematics

And troglodyte fractions



Art by Pavel Chernusky

AT THE END OF THE 1940s the great mathematician John von Neumann gave a report on the future of computers. He told his listeners that mathematics was only a very small and very simple part of life. The shuffling and coughing in the hall indicated that the audience wasn't in complete agreement. Sensing this, von Neumann added, "If you don't believe that mathematics is simple, it's only because you don't realize how complicated life is."

RECENT ARCHAEOLOGICAL FINDINGS reveal that an understanding of fractions as parts of the whole had arisen way back in the Stone Age, when none of our hearty ancestors could manage to eat an entire wooly mammoth. When the mammoths became extinct and there were only little animals to hunt, fractions were no longer needed and gradually fell out of use.

This just in . . .

THE *WASHINGTON POST* recently introduced a new column called "Why Things Are," consisting of questions we've all thought of but were afraid (or too prudent) to ask. Among such queries as "Why do we remember the middle names of assassins?", "Why are some quarters red?", and, of course, "Why is this column here?", Joel Achenbach poses the following question of interest to all of us who think about physical laws (or at least obey them conscientiously): **Why do objects fall at the same rate toward the Earth regardless of their weight?**

"You would think," writes Achenbach, "that Marlon 'The Refrigerator' Brando, if dropped from the top of the Empire State Building, would hit the ground before a paper clip that was dropped simultaneously." As you all know from your elementary physics textbook, add the condition of a vacuum and you can safely predict the two objects will land at the same time.

Now, there's a skeptic in every classroom (and certainly several out in readerland), so Achenbach proposes an experiment: "Pick up a paper clip. Now pick up Marlon Brando. Marlon Brando is definitely heavier. What do we mean by 'heavier'? We mean that, holding the 300-pound Oscar-winning actor by the lapels, we can detect that he is subject to greater gravitational force. But Brando has another distinct feature: He is hard to move. The more massive an object, the greater the force needed to move it from a state of rest. This is true whether you are rolling someone down the sidewalk or dropping him from a skyscraper.

"The hard thing to realize is that the objects don't fall 'because there's nothing underneath them.' Objects fall because they are being *moved* by gravity. You do not, in fact, 'drop' Marlon Brando; you just place him in a point in space. Since there is no structure to support him, gravity can move him without encumbrance.

"The point here is that heaviness is a two-sided coin. As you get heavier, gravity pulls harder, *but it is also that much harder to budge you*. So weight doesn't make you fall faster or slower. That's your answer."

Unfortunately, Achenbach couldn't leave well enough alone. He went on to elaborate: "Theoretically, if you had unbelievably sensitive instruments, and if you dropped Marlon Brando and the paper clip in separate experiments instead of simultaneously, you might be able to show that the cinematic giant hit the ground a fraction of a microsecond more quickly than the paper clip. This is because the star of 'The Godfather' exerts his own gravitational attraction and pulls the Earth toward him as he descends. So does the paper clip, but not as dramatically. This is worthless cogitation, though. Gravity is the weakest of the four known forces in the universe (except perhaps on Mondays) and even the porcine Brando exerts an infinitesimally slight pull."

His point about gravity on Monday mornings is certainly well taken. But an alert reader named Michael Page fired off a letter to the editor taking issue with the very last phrase. "Three cheers for the new column 'Why Things Are,'" he writes. "While I enjoyed most of the questions and answers, particularly the two on physics, I am compelled to note that one statement violates Newton's third law of motion."

When Achenbach says that Marlon Brando exerts only an infinitesimally slight pull on the Earth, he "destroys a fundamental and beautiful symmetry in nature: that forces come in equal and opposite pairs; that every action has an equal and opposite reaction; that no matter how hard you try, you can't lift yourself and the chair you're sitting in by pulling on the sides of the chair. Indeed, by using a bathroom scale and Newton's third law, we can easily measure the strength of Brando's gravitational pull on the Earth. It isn't infinitesimal; it's about 300 pounds!"

Perhaps Achenbach could have spared himself the irritation of a nettling letter by specifying the scale he used in so blithely dismissing the gravitational pull of this great, albeit cosmically rather small, actor.

Math

M21

Two possible constructions are shown in figure 1. We leave it to you to show that the dark segments are indeed $7^{1/2}$ long.

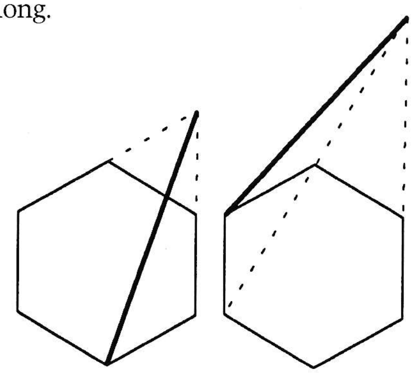


Figure 1

M22

The answer is 150 cm. Figure 2 shows how to make the required cube out of a piece of wire this long. Let's prove that a shorter wire won't do.

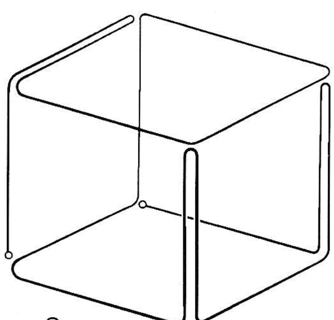


Figure 2

Consider first the framework of a cube made so that the wire begins, ends, and is bent only at the vertices of the cube. Let's count the number of segments of the wire that join adjacent vertices (each of them is 10 cm long). The number of segments issuing from any vertex is obviously not less than 3. Moreover, if a vertex isn't an end point of the wire, then the corresponding number of segments is even (it's twice the number of times the wire passes through this vertex), so it's not less than 4. A cube has 8 vertices and at most 2 of them can be the end points of the wire; therefore, the sum of the number of segments issuing from each vertex is not less than $2 \cdot 3 + 6 \cdot 4 = 30$. In this sum each segment is counted twice because it has two ends. Thus, the number of segments is not less than 15, and the length of the wire is not less than 150 cm.

Now consider an arbitrary wire cube. We're going to get rid of bends that are interior to the edges and bring the ends and turns of the wire to some vertices without extending the wire. By the end of these transformations we'll get a framework of the sort considered above, so the original piece of wire had to be at least 150 cm long.

The transformations can be performed edge by edge. Skipping the details, we just show in figure 3 (top row) five essentially different patterns of a portion of wire between two neighboring vertices of the cube. In the middle row you see the process of transformation, and at the bottom, the results. (V. Dubrovsky)

M23

Let x_k be the area of the part of the jeans that is covered by exactly k patches, $k = 0, 1, \dots, 5$. Then the area of the jeans is

$$A_0 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 1,$$

the sum of the areas of patches is

$$A_1 = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \geq 5/2$$

(of course, the area of the n-fold intersection of patches is counted here n times), and the sum of the areas of the 10 paired intersections is equal to

$$A_2 = x_2 + 3x_3 + 6x_4 + 10x_5$$

(the factors 1, 3, 6, and 10 here are the numbers of pairs of patches chosen from 2, 3, 4, and 5 patches). Since

$$\begin{aligned} A_2 &\geq -3x_0 - x_1 + x_2 + 3x_3 + 5x_4 + 7x_5 \\ &= 2A_1 - 3A_0 \geq 2, \end{aligned}$$

at least one of 10 possible paired intersections has an area not less than $2/10 = 1/5$.

To tackle more general questions of this sort, one should use the so-called *formula of inclusions and exclusions:* if A_1 is the sum of the areas of some figures and A_k is the sum of the areas of their k-fold intersections ($k = 2, 3, \dots$), then the area of their union equals $A_1 - A_2 + A_3 - \dots$. (N. Vasilyev)

M24

The main tool needed to solve this problem is the calculation of x^n for $n = 2^k$ in k multiplications:

$$x^2 = x \cdot x,\ x^4 = x^2 \cdot x^2, \dots, x^{2^k} = x^{2^{k-1}} \cdot x^{2^{k-1}},$$

in the course of which we also obtain all the powers

$$x^{2^m},$$

$m = 1, \dots, k - 1$. Evidently 2^k is the highest exponent of x that can be reached in k operations, so x^n for an arbitrary n can't be found in less than $\log_2 n$ operations.

(a) Based on the representation $x^{1000} = x^{1024}/(x^{16} \cdot x^8)$, we can proceed as fol-

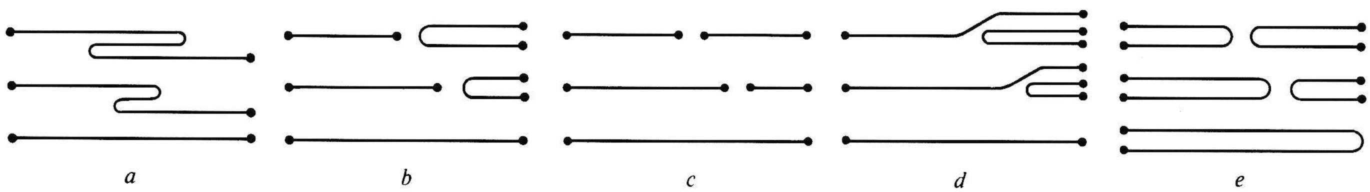


Figure 3

lows: find the values of x^n for $n = 2, 4, 8, 16, \ldots, 1024 = 2^{10}$ in 10 multiplications and then complete the calculation with one more multiplication and division. It can be proved that 12 is the minimum number of operations needed to obtain x^{1000}, though the rough estimate given above yields 10 as the lower bound for this number ($\log_2 1000 = 9.96\ldots$).

(b) Multiplications and divisions of powers of x are reduced to additions and subtractions of their exponents. So it suffices to prove that, *starting with the number 1, we can obtain any positive integer n in no more than* $(3/2)\log_2 n + 1$ *additions or subtractions*. We'll describe two methods of calculating n based on the binary number system. The shorter of the two will give us the desired estimation.

It's known that any positive integer n can be uniquely represented in the form

$$n = a_l \cdot 2^l + a_{l-1} \cdot 2^{l-1} + \ldots + a_1 \cdot 2 + a_0, \quad (1)$$

where a_k is 0 or 1 for $k = 0, 1, \ldots, l-1$, $a_l = 1$. The notation

$$(n)_2 = a_l a_{l-1} \ldots a_0$$

is the *binary representation* of n. Let $s(n)$ be the number of nonzero terms in the sum (1)—that is, $s(n) = a_l + a_{l-1} + \ldots + a_0$.

The first method is to calculate all these terms in l additions ($1 + 1 = 2$, $2 + 2 = 4$, $\ldots$, $2^{l-1} + 2^{l-1} = 2^l$) and then add them up, which will take $s(n) - 1$ more additions. The total number of operations according to this method is $N_1 = l + s(n) - 1$.

The second method is to find first the complement $\overline{n}$ of n with respect to 2^{l+1}:

$$\begin{aligned}\overline{n} &= 2^{l+1} - n \\ &= 2^l + 2^{l-1} + \ldots + 1 - n + 1 \\ &= (1 - a_l) \cdot 2^l + (1 - a_{l-1}) \cdot 2^{l-1} \\ &\qquad + \ldots (1 - a_0) + 1.\end{aligned}$$

As we already know, it will take $l + s(\overline{n}) - 1$ additions. Now we need one more addition to obtain $2^{l+1} = 2^l + 2^l$ (2^l has been found in the course of calculating $\overline{n}$), and one subtraction: $n = 2^{l+1} - \overline{n}$. Since $s(\overline{n}) \leq (1 - a_l) + (1 - a_{l-1}) + \ldots + (1 - a_0) + 1 = l + 2 - s(n)$, here the total number of operations equals

$$N_2 = l + s(\overline{n}) + 1 \leq 2l - s(n) + 3.$$

The smaller of the numbers N_1 and N_2 doesn't exceed half of their sum:

$$(N_1 + N_2)/2 \leq (3/2)l + 1.$$

It remains to notice that $l \leq \log_2 n$.

Of course, replacing the smaller of the numbers N_1 and N_2 with $(3/2)\log_2 n + 1$, we usually lose accuracy in the estimation: in problem (a), x^{1000} was obtained by the second method in $N = 12$ operations, when $(3/2)\log_2 1000 + 1 \cong 15.95$. Moreover, sometimes neither of our methods is the shortest possible. For example, x^{1000} can be found in 9 multiplications (think how!), though from $(170)_2 = 10101010$ and $(\overline{170})_2 = (86)_2 = 1010110$ it follows that $l = 7$, $s(n) = s(\overline{n}) = 4$ for $n = 170$, and so the first method needs $N_1 = 10$ and the second $N_2 = 12$ operations. (E. Belaga)

M25

Suppose that for some convex equilateral pentagon $ABCDE$ the statement of the problem isn't true. We can assume that all the sides of the pentagon are of length 1 and AD is its longest diagonal. In figure 4,

$$\begin{aligned}2 &= AB + CD < AO + OB + CO + OD \\ &= AC + BD \leq 2AD,\end{aligned}$$

so $AD > 1$. Since side AD is the longest one in triangle ABD ($AD \geq BD$, $AD > AB = 1$), the opposite angle ABD is greater than 60° because it's greater than the other two angles of the triangle. If angle BAD were also greater than (or equal to) 60°, then the equilateral triangle constructed on AB would go in triangle ABD and all the more in the pentagon. This would contradict our initial supposition, so angle BAD

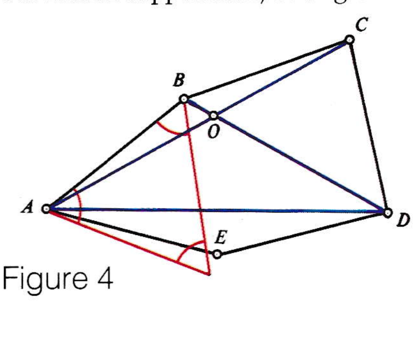


Figure 4

< 60°, which means that point B is located on the arc B_1B_2 of angle measure 60°, radius 1, and center A (ends B_1 and B_2 excluded; see figure 5). Similarly, the locus of point C is the open arc C_1C_2 with center D, congruent with B_1B_2. The arcs intersect because $AD < AE + ED = 2$.

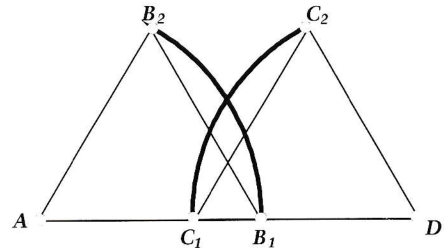


Figure 5

To complete the proof, let's show that the distance between any two points B and C of our open arcs is less than 1 (in contradiction to the assumption that BC, like the other sides of $ABCDE$, is of length 1).

Fix B and let C slide along the arc C_1C_2 (fig. 6). By the law of cosines for triangle BDC, BC increases with the increase of angle BDC, so $BC < BC_1$ or $BC < BC_2$.

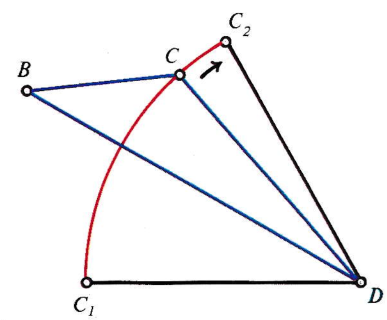


Figure 6

The same is true for the segment BC_i ($i = 1, 2$) as it slides along the arc B_1B_2: its maximum length is achieved when B coincides with B_1 or B_2. Thus, the segment BC is shorter than one of the segments B_kC_i ($i, k = 1, 2$). But the lengths of all these segments are obviously less than 1.

This solution can be developed further to prove the statement of the problem for any convex equilateral polygon with an odd number of sides. But for polygons with an even number of sides, the statement is false: such a polygon can be made arbitrarily narrow by moving its two opposite vertices apart. (N. Vasilyev, V. Dubrovsky)

Physics

P21

Since the absolute value of the dog's velocity is constant but its direction is different at different moments, its acceleration is perpendicular to the velocity. The trajectory of any material point over a short period of time can be approximated by the arc of a circle. The dog's acceleration is then equal to the centrifugal acceleration

$$a = \frac{v_2^2}{R},$$

where R is the radius of the circle approximating the real trajectory of the running dog.

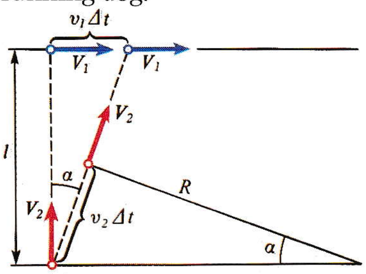


Figure 7

Consider now the displacement of the dog over a short interval of time Δt. During this time the vector of the dog's velocity rotates by an angle α such that $\alpha = v_2\Delta t/R$ (fig. 7). On the other hand, over the same interval of time the fox covers the distance $v_1\Delta t = \alpha l$, since the vector of the dog's velocity is constantly aimed at the fox. Consequently, $v_2\Delta t/R = v_1\Delta t/l$. And so

$$R = \frac{v_2}{v_1} l$$

and

$$a = \frac{v_1 v_2}{l}.$$

P22

Consider the floating condition of any of the vessels: the force of gravity acting on this vessel is compensated by the difference between the inside and outside pressure. So both before and after the addition of water into any vessel, the difference between the outer and inner water levels remains the same. This means that the position of all the water levels remains fixed with respect to the ground.

Thus, the water level in the smallest vessel doesn't change with respect to the ground. Consequently, the bottom of this vessel goes down by the distance

$$h = \frac{v_0}{s_0},$$

that is, by the height of the added layer of water.

P23

Whichever vessel gets heated, the water flows to the right. Let the right vessel be the one heated. Then the water in it expands, acquiring a greater volume. If the vessel were cylindrical (fig. 8) the water pressure at the bottom wouldn't have changed, since the decrease in water density would have been compensated by an increase in the water level. This follows from the fact that the total force of pressure applied to the bottom equals the weight of the water contained in the vessel. On the other hand, it's equal to the product $F = pS$, where p is the pressure at the bottom and S is the bottom area. Since neither the weight nor the bottom area alters upon heating up, the pressure at the bottom of a cylindrical vessel doesn't change.

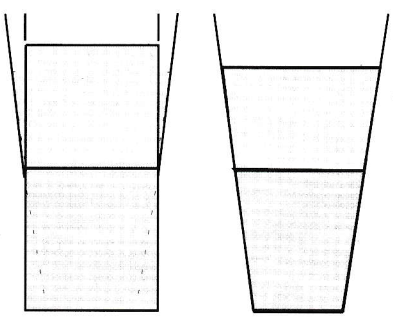


Figure 8

In a conical vessel the same decrease in water density is accompanied by a smaller increase in the water level. There are two reasons for this. First, the conical vessel contains less liquid, so the variation in its volume is also less. Second, the expanding water fills the volume shaded in the figure, whose upper level is lower. So the pressure at the bottom decreases. At equilibrium the pressure of the connecting tube must be the same, so the liquid starts flowing from left to right.

The case in which the left vessel is the one heated is treated in the same way. The pressure at the bottom increases upon being heated, which means water again starts to flow from left to right.

P24

Since the electromotive force (EMF) of a chemical battery is determined only by its chemical composition, the EMF of the second battery is the same as that of the first. Denote it by E, the lamp's resistance by R, and the internal resistance of the battery by r. Then the total power released in the first battery circuit is $P_1 = E^2/(R + r_1)$, and in the second battery circuit it's $P_2 = E^2/(R + r_2)$. Since $r_1 << R$ (and, consequently, $r_2 << R$), we have

$$P_1 \cong P_2 \cong \frac{E^2}{R}.$$

A chemical source of current performs work, releasing the energy stored in its chemical components. The larger version contains $2^3 = 8$ times the reagents in the original, so at the same power level it's capable of producing eight times the work of the first. This implies that the lamp connected to the bigger battery will glow for 24 hours.

P25

Imagine a *directed* sound source placed at point A (this corresponds to the case of a speaking person). Then, strictly speaking, of all the beams directed at point B only one beam reaches this point (fig. 9a). The rest of the beams arrive at other points close to B. If, however, the same beams are emitted from point A along the gallery's wall, several beams end up at point B (fig. 9b). So it's much more

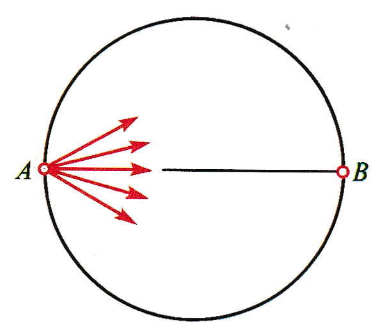


Figure 9a

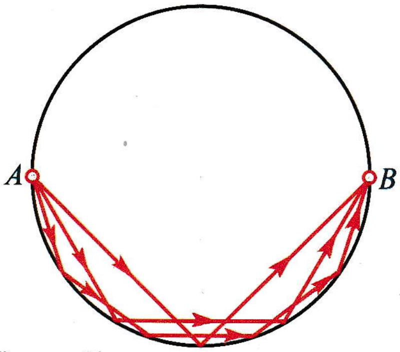


Figure 9b

efficient to speak along the wall so that the sound "glides" along the whispering gallery.

If an *undirected* intensive acoustic source is placed at point A, the sound can reach point B along many routes. The shortest way is the straight line AB. The signal traveling by this route arrives at point B first, the propagation time being

$$\tau = \frac{d}{v},$$

where v is the speed of sound. Then the two signals that were reflected once from the wall arrive at point B (from the left and from the right—see figure 9c). They are followed by signals that underwent two, three, or more reflections off the wall. The last to arrive will be the two signals emitted from A practically along the tangent to the gallery wall at point A in two opposite directions. Each of them will be reflected from the wall many, many times and cover a route that's practically half the gallery's circumference; the time they take to arrive will equal

$$\tau = \frac{\pi d/2}{v} = \frac{\pi d}{2v}.$$

The difference between the propagation times of the first and the last signals is

$$\Delta\tau = \tau - \tau_0 = \frac{d}{v}\left(\frac{\pi}{2} - 1\right).$$

This is how long the duration τ_1 of the acoustic signal emitted at point A is prolonged when heard at point B. Consequently, the duration τ_2 of the signal detected at point B is

$$\tau_2 = \tau_1 + \Delta\tau = \tau_1 + \frac{d}{v}\left(\frac{\pi}{2} - 1\right).$$

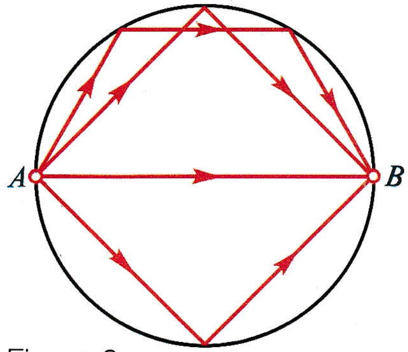


Figure 9c

It's interesting that for any other pair of emission and detection points, the time increment $\Delta\tau$ is greater. (Prove it yourself!)

Brainteasers

B21

No, he can't. One of the two triples of numbers must contain an even number and two odd ones. Their sum is even, so the product of the two sums must be even.

B22

See figure 10.

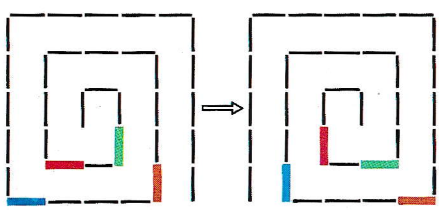


Figure 10

B23

If we denote by D and E the operations of doubling and erasing, then one of the possible sequences of operations is D, E, E, D, D, D, E, D, resulting in the sequence of numbers 458, 916, 91, 9, 18, 36, 72, 7, 14.

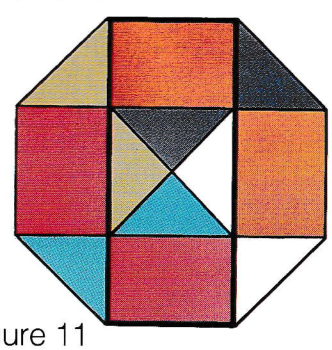


Figure 11

B24

See figure 11, in which equal figures have the same color.

B25

Unburned particles (smoke) are lifted by an upward flow of hot air. When the surrounding air cools down, the particles begin to drop and eventually settle to the ground.

Circumcircles

1. Angle $CDE = 40°$. Hint: see figure 12. $AK = KB$, so angle $ABK = 20°$; angle KDE = angle $KBE = 60° - 20° = 40°$.

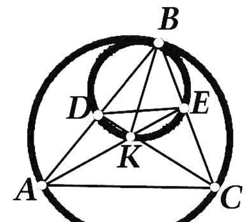


Figure 12

2. Angle $BMC = 110°$. Hint: see figure 13. Triangle BOC is equilateral, BM is the perpendicular bisector of OC; angle $BMC = 180° - 30° - 40°$.

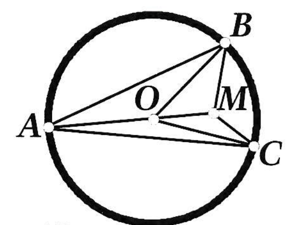


Figure 13

3. Angle $BKC = 60°$. Hint: see figure 14. Triangle AOB is equilateral, OK is the perpendicular bisector of AC, BK is the perpendicular bisector of AO.

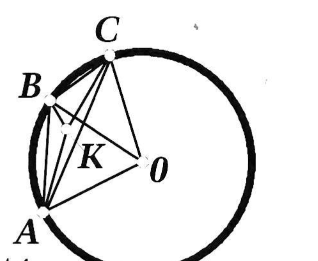


Figure 14

4. Angle $BCM = 80°$. Hint: according to property IV, point B is the

circumcenter of triangle AMC (draw a picture). Denoting angle BCM by x, we can find from isosceles triangle BCM that angle $MBC = 180° - 2x$; angle $ABM = 280° - 2x$; angle $ACM = x + 40°$. The condition that angle ABM = angle ACM implies that $x = 80°$.

5. Angle $CDE = 81°$. Hint: see figure 15. If O is the circumcenter of triangle CDE, then triangle DOE is equilateral, OA is the perpendicular bisector of CD, triangles AOE and ADE are congruent, and angle CDO = angle $EAO = 21°$, so that angle $CDE = 60° + 21° = 81°$.

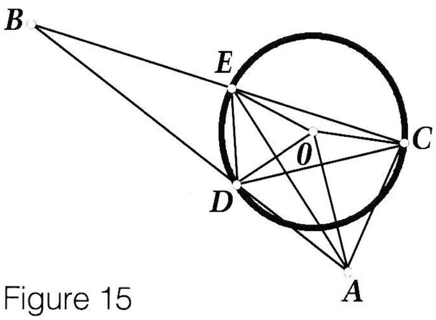


Figure 15

Kaleidoscope

1. $\pi \cong (1 + 2^{1/2}/2)^2 = 2.914...$

2. Construct a right triangle ABC with right angle at A, $AB = 1$, $AC = 1/2$, mark point D on the extension of BC so that $CD = CA = 1/2$. Then $BD = \tau$.

3. See figure 16. Also prove that each rectangle left after cutting off the next square is a "golden" one and that the diagonals in the figure meet at right angles at the point of intersection of all these golden rectangles.

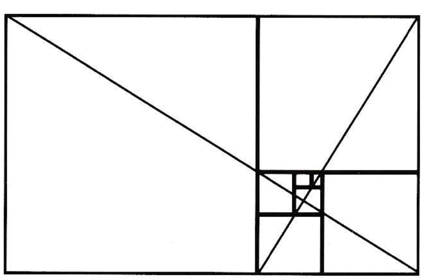


Figure 16

4. Obviously quadrangle $AEDN$ (see figure 2 in the Kaleidoscope) is a rhombus, so segments AN, ND, and all the sides of the pentagon are of equal

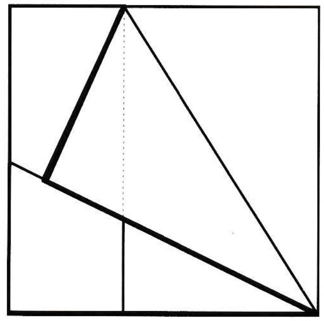


Figure 17

length. In addition, triangles ACD, DNC, BNM, and ODF are all similar to one another. So it follows that all the ratios in question are equal. The equality $AC{:}AN = AN{:}NC$ means by definition that N divides AC in the golden ratio τ. The last equality in the problem follows from, say, triangle ODF, in which $OD = OF = \tau \cdot DF$ and angle $DOF = \pi/5$.

5. The area of the cuboid's surface is equal to $2(\tau + 1/\tau + \tau \cdot 1/\tau) = 4\tau$, the diameter of the sphere equals 2, so the area of its surface equals 4π.

Waves

1. The wave speed depends on the time delay mentioned.

2. $\Delta t = T/2$; $\Delta\varphi = \pi$.

3. Estimate the collision time as the time τ necessary for the deformation wave (sound wave) to travel the distance of the ball's diameter:

$$\tau \sim \frac{d}{v} \sim d\sqrt{\frac{\rho}{E}} \sim 10^{-6}\ \text{s}.$$

4. When the rotor is stopped abruptly, a deformation wave starts to propagate along the concrete floor and at some point reaches the coil with the sample under investigation. To ensure that it doesn't affect the measurement, one must take care that the wave reaches the sample only after the experiment is over. The electromagnetic field travels at the speed of light, which is much greater than the speed of the deformation wave (which is a sound wave). We can assume that the magnetic field in the sample is created instantaneously. So the minimum distance between the generator and the coil equals $l = v\Delta t = 50$ m (where $v = 500$ m/s is the speed of sound in concrete).

5. Denote the rigidity of the springs by k, the mass of an oxygen atom by M, and the mass of a carbon atom by m ($m/M = 12/16$). For case (a) the oxygen atoms oscillate about the immobile carbon atom synchronously. Therefore, their frequency is

$$\omega_a = \sqrt{\frac{k}{M}}.$$

For oscillations of type (b) the carbon atom is affected by two forces equal to m in absolute value and acting in the same direction. If the ball representing the carbon atom is split into two equal parts, their movements are identical, both of them having the same acceleration, speed, and coordinates. So the problem is reduced to finding the oscillation frequency of two balls of mass M and $m/2$ connected by a spring.

The system oscillates about its immobile center of mass located at a distance $l' = lm/(m + 2M)$ from the ball of mass M (where l is the spring's length in the undeformed state).

So we can assume that the oxygen atom (the ball of mass M) is connected to the center of mass by a spring of length l'. The rigidity of this part of the spring is greater than the rigidity of the whole spring,

$$k' = \frac{kl}{l'} = \frac{k(m+2M)}{m},$$

and the oscillation frequency of the ball of mass M connected to the spring of rigidity k' equals

$$\omega_b = \sqrt{\frac{k'}{M}} = \sqrt{\frac{k(m+2M)}{mM}}.$$

Thus, the desired frequency ratio is

$$\frac{\omega_a}{\omega_b} = \sqrt{\frac{m}{(m+2M)}} = \sqrt{\frac{3}{11}}.$$

Corrections

Maybe you were sharp enough to catch these errors in the November/December issue:

p. 20, col. 2: It is obviously point A that "makes a circular arc with radius

OA and center at point O," not point E.

p. 28, col. 1: A "k" was dropped from in front of the Δt in the middle of the column.

p. 28, col. 3: Each instance of the intermediate term "$(kt)^m/m!$" should be followed by three dots: $\ldots + (kt)^m/m! + \ldots$.

p. 30, col. 1 and 2: 4321 – 1234 = 3087, not 3089; when you do the Kaprekar transformations, you reach the magic number 6174 in just two more steps. (The error jumped out at our advisory board member Peg Kenney because it takes at most seven subtractions to arrive at 6174; the incorrect initial subtraction led to a total of eight steps.)

p. 51, col. 1: Our publishing software mysteriously dropped a denominator "2" in the displayed equation.

p. 57, col. 2: For "$\log_2 n + 1$" read "$[\log_2 n] + 1$," and for "$\log_2$" in the next line read "$[\cdot]$" (that is, the expression in brackets stands for the greatest integer function).

p. 61, col. 2: To render the solution in black and white, it's necessary to pretend that black stands for "blue" and cross-hatching stands for "red."

Finally, to encourage good spelling, we acknowledge the typo in column 2 of page 2!

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Index of Advertisers

A collapsible saddle

Now all you need is a collapsible horse

by Vladimir Dubrovsky

WITH THIS ARTICLE WE inaugurate a new department—*Quantum*'s Toy Store! We hope you enjoy building the models and wrestling with the puzzles we plan to offer in this space at the back of the magazine.

Those of you who have been with us from the start probably remember the brilliant article by Dmitry Fuchs in the very first issue of *Quantum* (Jan. 1990). Fuchs told us about surfaces obtained by bending a sheet of paper. Such a surface can be sliced into a family of straight lines, or "rulings," and is, therefore, a ruled surface. There are, however, ruled surfaces that can't be obtained by bending a sheet of paper. These were also mentioned in Fuchs's article, and one of them appeared anew in the May issue in connection with—population genetics!

Now we'll teach you how to make a model of this beautifully curved surface out of plain cardboard or stiff construction paper. You can see the finished product in figure 1.

The model is assembled from two sets of parallel "slices" that interlock by means of slits cut in them. To prepare the components of our model, take seven paper rectangles shaped as in figure 2, cut them along the oblique lines in congruent halves, and slice

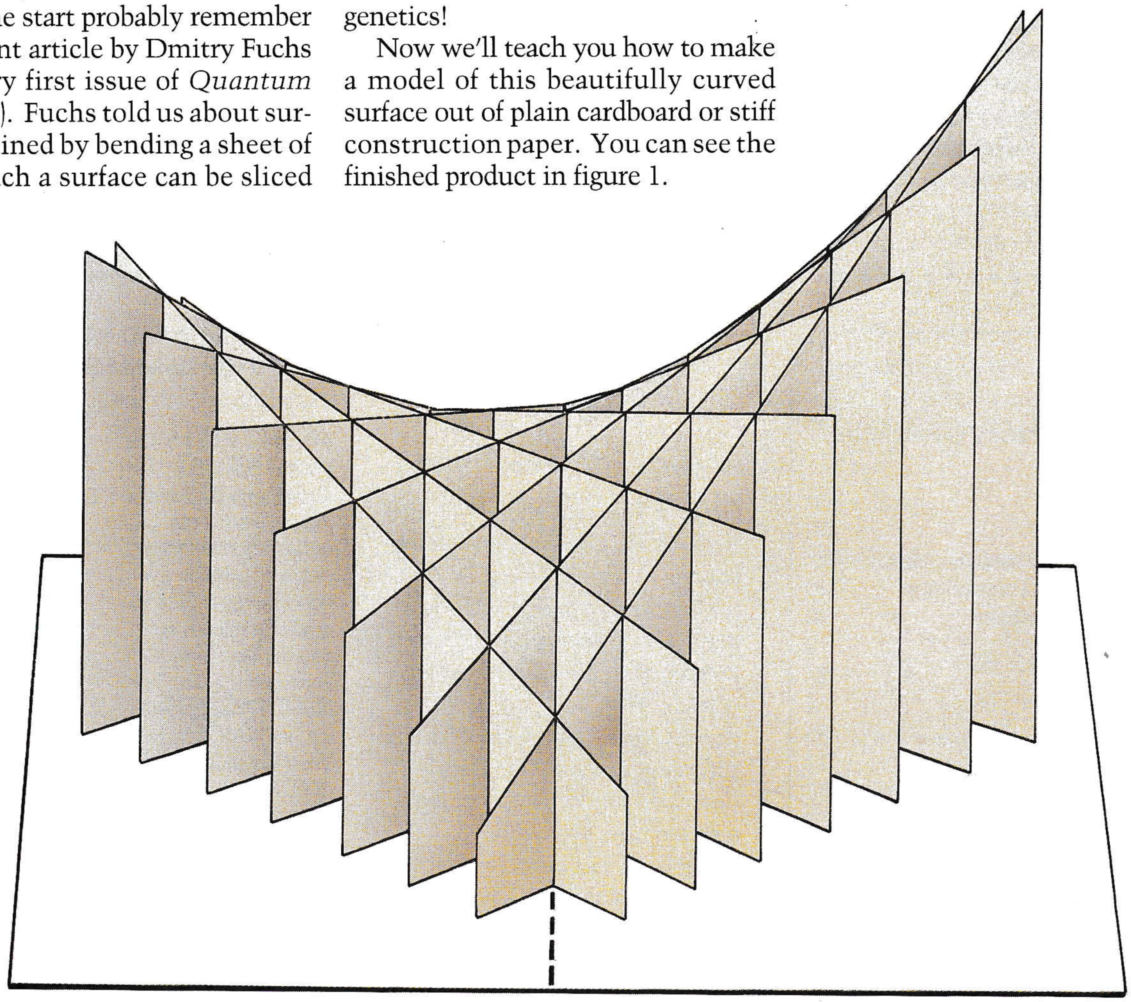


Figure 1

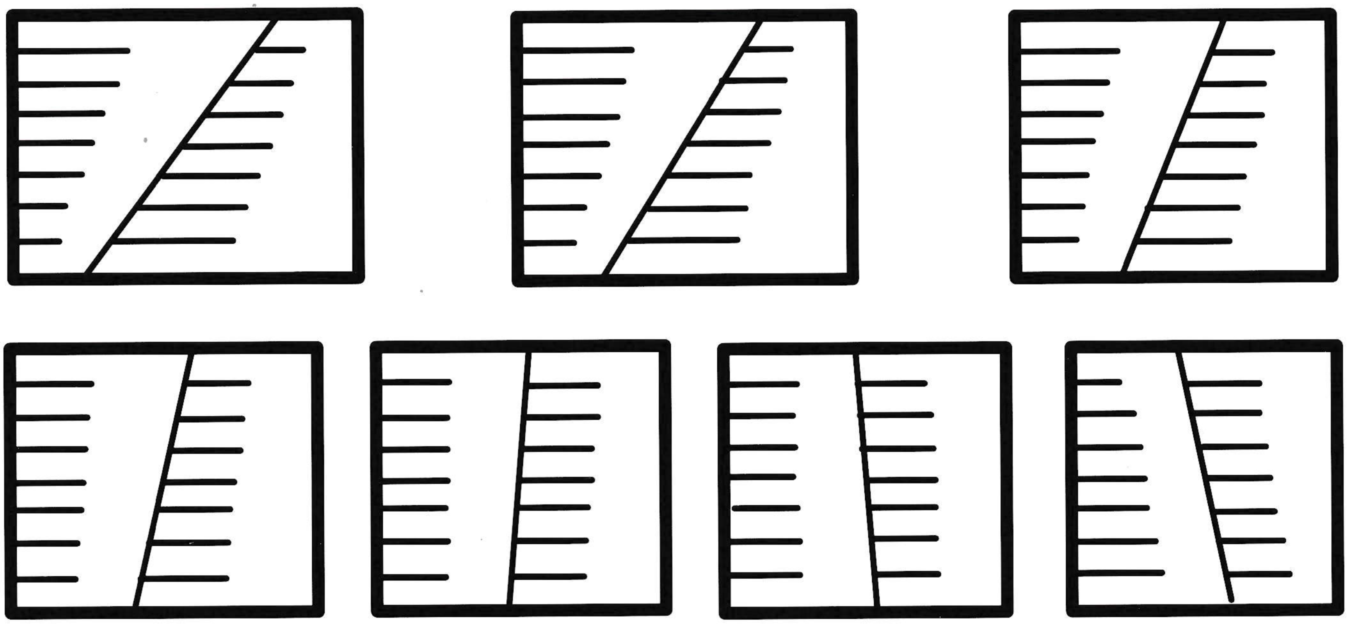


Figure 2

the trapezoids thus obtained parallel to their bases. The left halves of the rectangles make up one set of slices, the right ones make up the other set. Our model, for aesthetic reasons, was made slightly nonsymmetrical, so if you want to reproduce it be sure to adhere strictly to the shapes and relative sizes in the figure. (Of course, you can choose to redesign it on your own.) When the slices are meshed, their oblique edges form a saddlelike surface, which mathematicians call a *hyperbolic paraboloid*. Because it's such a refined shape created by the simplest construction elements (straight rods), this surface is often used in architecture. But an architect designing a saddle roof must be careful: if the rods are fixed so that they can turn at their joints, the whole lattice becomes mobile, even collapsible. This interesting property is perfectly well observed in our model. You can fold it up and spread it out again, and if you glue its two opposite bottom corners inside a cardboard folder, you'll get a nice collapsible toy.

The name of the surface deserves some comment. Why the hyperbola and parabola? Let's derive the coordinate equation of our saddle. We can choose the coordinates so that the slices of our model are parallel to the xz- and yz-planes; the x-axis belongs to one family of rulings and the y-axis together with the line $l = \{(x, y, z): x = 1, y = z\}$ belongs to the other family (fig. 3). The entire first family of rulings consists of all the lines parallel to the xz-plane and intersecting the y-axis and line l. Let $P(x, y, z)$ be an arbitrary point of the surface, A and B the points where the ruling of the first family that passes through P meets the y-axis and line l. It follows from the equations of l that the coordinates of B are $(1, y, y)$. If P_1 and B_1 are the projections of P and B onto the xy-plane, then by the similarity of triangles APP_1 and ABB_1 we have $PP_1:BB_1 = AP_1:AB_1$, or $z/y = x/1$. We then get the equation

$$z = xy. \qquad (1)$$

Cutting the surface by the planes $x = a$ and $y = a$, we can verify at once that it really has two families of linear rulings: $\{x = a, z = ay\}$ and $\{y = a, z = ax\}$. The cross section by the horizontal plane $z = a$ yields a curve $\{xy = a, z = a\}$ in which you surely recognize a hyperbola (when $a \neq 0$). Finally, consider a vertical plane $y = ax$, $a \neq 0$. It cuts our surface along the curve whose projection onto the xz-plane has the equation $z = ax^2$ (substitute ax for y in equation (1)). So the projection, and thus the curve itself, is a parabola. Now find these curves on your model! (Indeed, it's more difficult not to find them, since any cross section of a hyperbolic paraboloid that isn't a ruling or pair of rulings is either a hyperbola or a parabola. Can you prove that?)

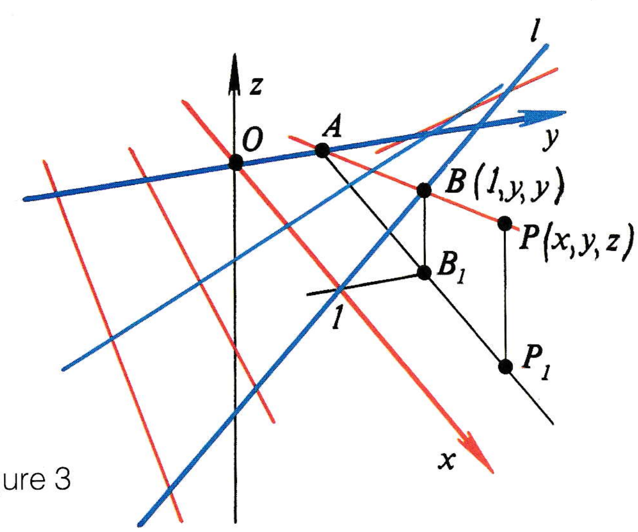
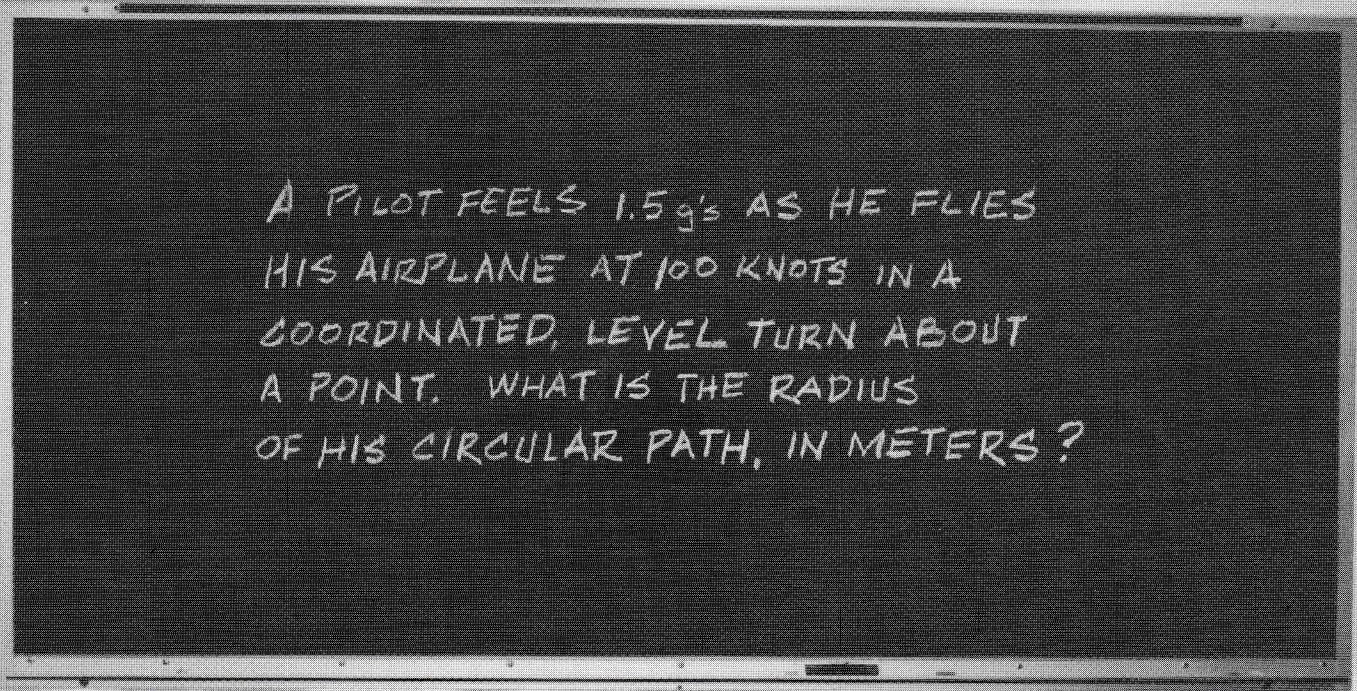


Figure 3

If you like solving this problem, then Embry-Riddle Aeronautical University has a program for you.



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