NATIONAL CENTER FOR CASE STUDY TEACHING IN SCIENCE

Grandpa's Flying Hammer: Rotation Around the Center of Mass

by
Joel Hernandez and Canan Karaalioglu
Department of Science
Borough of Manhattan Community College / City University of New York

Introduction

Jane, a freshman in college, is chatting with her neighbor, a retired physics college professor, when she suddenly sees a rotating flying hammer thrown by her grandpa. Jane notices that as the hammer rotates in the air, there appears to be a point located in the head that does *not* rotate. After checking that her grandpa is fine, Jane asks the professor for an explanation of this observation. Rather than offering Jane a straight answer, the professor, who is an experienced educator, leads Jane to analyze the problem and reach an explanation by herself, applying knowledge acquired in her undergraduate physics class. Remarkably, Jane is able to prove that the hammer rotates around its center of mass.

Your role is to follow along with Jane and help her to complete a series of tasks that lead to this conclusion.

Part I – Essential Facts

Prof: How are you liking college so far, Jane?

Jane: I love it! I plan to study biotechnology and work as a scientist in that field.

A hammer flies by through the air.

Jane: Hold on a moment. Did you just see that hammer rotating and flying across my backyard? It seems Grandpa isn't happy; he's supposed to be repairing the back shed. Excuse me a moment while I check on him.

After a couple of minutes Jane returns to the professor.

Jane: I checked and he's fine. He was mad that the wood board split as he was driving nails through it.

Prof: Well, as you know, your Grandpa can be a little temperamental.

Jane: Was that real or was it just an illusion? I mean, it seemed to me that a point in the head of the hammer didn't rotate as the hammer was spinning in the air. Can you explain that to me?

Prof: It was not an illusion, Jane. Are you currently taking your first physics course in college?

Jane: Yes, we're now covering the chapter devoted to rotational dynamics, but I don't see how it explains this observation.

Prof: Ok, no problem. By now you should be in possession of the tools needed to explain it and even go a bit further.

Jane: Sounds good; it seems as though I'm going to get a chance to think like a real scientist.

Prof: Indeed you will. First, let's start by isolating some essential facts of the phenomenon and system to analyze. So give it a try; what are the essential facts about the system that you want to analyze, Jane?

Task 1: Identify the essential facts of the phenomenon and system.

Part II – Model

Jane: Hmm… I'm not so sure. Let me start with the phenomenon. I noticed a lot of things: that the hammer was flying, following an overall parabolic trajectory; that it was rotating at the same time; and that the center of rotation appeared to be located in the head. But if I had to pick just the bare minimum relevant to my problem, I'd drop the parabolic trajectory motion because it has nothing to do with rotation. I'll stick with the last two facts concerning rotation and the perceived center of rotation.

Prof: Great analysis, Jane! I totally agree with you. Now concentrate on your system (the hammer) and try to identify its essential characteristics.

Jane: That's tougher. I can say that the head is made of metal while the handle is made of wood. I can state that the head is smaller than the handle but that the head is heavier than the handle. I can also state that both the head and the handle are rigid objects.

Prof: Agreed, this part is tougher but the answer is included in your statements. You have to concentrate on the hammer as a rigid object. That's the essential fact about the system.

Jane: Yes, I get that it's a rigid object, but I still don't see the usefulness of these essential facts for the analysis of the problem.

Prof: You'll see it very soon. As you know, the hammer is a physical object composed of many particles, trillions and trillions of atoms. Analyzing such a multi-particle system is challenging. The next step is to develop the simplest model of the same object that is able to rotate and is rigid. Can you come up with such a model, Jane? Be creative!

Task 2: Develop a model to simulate the hammer.

Part III – Diagram

Jane: Let me think. A simplification associated with the large number of particles seems straightforward to me. I can use a system composed of a single particle.

Prof: That looks to me like good thinking so far.

Jane: The equivalent model also has to be able to rotate. A single particle can't rotate because it's like a mathematical point; so a single particle won't work. The next system in complexity is a two-particle system… *Aha!* A two-particle system *is* able to rotate around a center of rotation.

Prof: Good, you're making progress.

Jane: Now I have to make my model rigid. This looks tough again; how can I prevent the two particles from moving with respect to each other? Do I glue them? Just kidding.

Prof: I'll give you a hand. What if you join the particles with a rigid and massless rod?

Jane: Clever! But I don't understand why the rod has to be massless.

Prof: If the rod has mass, the tension at a point located between the center of rotation and a particle will have to pull not only the particle, but also the section of the rod with mass located between the point and the particle. This added rod mass depends on the position of the point, therefore the tension also depends on this position. If the rod is massless, there is no additional mass, and the tension has the same value at all points.

Jane: Understood; it's like the massless strings that are used in translational dynamics problems. Is there anything else to consider before the laws of physics are applied to this model system?

Prof: Yes, to achieve the highest level of simplicity, the only forces to consider will be the forces acting on the particles due to the rod. Any possible forces acting between the particles themselves will be neglected. This will keep the analysis concentrated on the things that matter for the essence of the problem.

Jane: That all makes sense to me.

Prof: Then it's time for you to start doing some real work. Start by making a diagram of the rotating system at some instant. Indicate the distance from particle to particle, the distance from each particle to the center of rotation, the mass of each particle, the force acting on each particle, the velocity of each particle, and the location of the center of rotation. In addition, assign variable names to each of the quantities.

Task 3: Prepare a diagram describing the model and phenomenon taking place.

Part IV – Center of Rotation

Jane: You're asking for a lot of things at the same time. Let me please divide your request into smaller parts to make sure I have things correct before I get too far. I'll designate the mass of each particle as M_1 and M_2, and the distance between them as D. The distance from each particle to the center of rotation will be D_1 and D_2. I'll assume that the center of rotation is at some point between the two particles, which is what I observed in the flying hammer. How am I doing so far?

Prof: You're doing great! Your assumption for the location of the center of rotation is reasonable.

Jane: I realize that the force acting on each particle is the tension force from the rod. The rod is massless, so the tension is the same at each point, and this means that the magnitude of the force that it applies on each particle has the same value.

Prof: That's all correct. What can you say about the linear velocities of the particles? Are they equal?

Jane: My observations indicated that they are not. I noticed that the head of the hammer was moving much slower than the end of the handle and I relate this behavior to the fact that the head was much closer to the center of rotation than the end of the handle. Therefore, I expect the particle that is located closer to the center of rotation to move with smaller linear velocity.

Prof: Excellent analysis, Jane! Now go on and implement on a diagram everything that you've stated.

Jane goes home, grabs a graph paper pad, pencil with eraser, prepares the graph, and comes back to show it to the professor.

Jane: All done and shown in Figure 1.

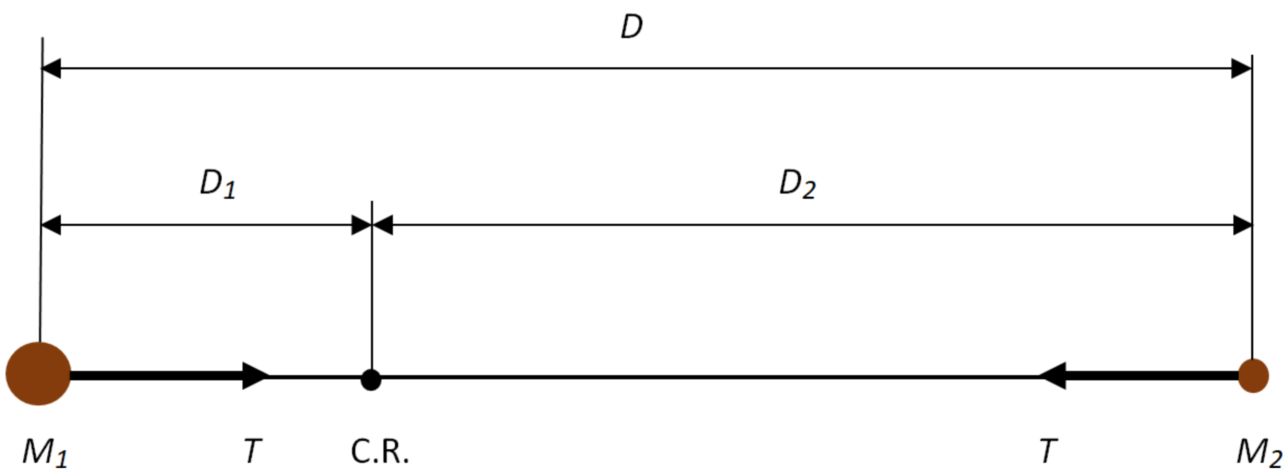

Figure 1. Jane's initial diagram.

Prof: All looks perfect to me.

Jane: That was easy!

Prof: Can you now write the expression for the centripetal force acting on each particle in terms of the angular speed?

Task 4: Determine the location of the center of rotation of the model using a centripetal force analysis.

Part V – Center of Mass

Jane: Sure. For each particle the centripetal force is:

$$F_{c1} = M_1\ \omega_1^2\ D_1 \quad \text{and} \quad F_{c2} = M_2\ \omega_2^2\ D_2 \tag{1}$$

Prof: You previously selected M_1 to be in general different from M_2, and D_1 in general to be different from D_2. What can you say about the values of ω_1 and ω_2? Are they in general different or not?

Jane: Hmm… I hadn't thought about that. Give me a moment. I think that each particle will in general be moving with different linear speeds because their radii of rotation are in general different (D_1 is in general different from D_2). However, their rotational speeds are a different story.

Prof: Let me give you a clue. Your system is rigid, correct? What is the defining characteristic of the rotational motion of a rigid object concerning the rotational speed for each point in the object?

Jane: Got it! The rotational speeds ω_1 and ω_2 have the same value ω because in a rigid body all particles move with the same rotational speed. So, the centripetal forces can be written as:

$$F_{c1} = M_1\ \omega^2 D_1 \quad \text{and} \quad F_{c2} = M_2\ \omega^2 D_2 \tag{2}$$

I previously mentioned that the strength of the centripetal forces on each particle has to be the same ($Fc_1 = Fc_2$) because the tension in the rod is the same at all points. Therefore, the two right hand parts of Equation 2 can be made equal as follows:

$$M_1\ \omega^2\ D_1 = M_2\ \omega^2\ D_2$$

Dividing each side of this equation by ω^2 yields:

$$M_1 D_1 = M_2 D_2 \tag{3}$$

Prof: You've made great progress, Jane! I'd now like you to use Equation 3 to solve for D_1 as a function of M_1, M_2, and D.

Jane: No problem. That takes only a few algebraic operations.

From the diagram: $D = D_1 + D_2$. Therefore: $D_2 = D - D_1$ and Equation 3 can be rewritten as:

$$M_1 D_1 = M_2 (D - D_1) = M_2 D - M_2 D_1$$

Solving for D_1 produces:

$$D_1 = \frac{M_2 D}{M_1 + M_2} \tag{4}$$

Prof: Hey Jane, you haven't realized it yet, but you've just obtained a very important result. Can you look at Equation 4, which you just derived, and tell me what it means physically?

Jane: Definitely. The expression provides the distance from particle 1 to the center of rotation. I see it. It allows me to know the location of the point in the system that is not showing rotational motion. With your guidance, I think I've answered the question I asked you.

Prof: Right on the money! You've solved the problem for this simple model of your hammer. But you can go a little further. I'll guide you again. You probably remember the concept of center of mass of a system. I want you to calculate the X-coordinate of the center of mass of your model by putting the origin of the X-axis at the location of particle 1. Do you know how to do this, Jane?

Task 5: Determine the location of the center of mass of the model.

Part VI – Locating the Center of Rotation

Jane: Sure, I know. With this selection of the origin of the *X*-axis, the position coordinates for particles 1 and 2 will be $X_1 = 0$ and $X_2 = D$ (shown in Figure 2).

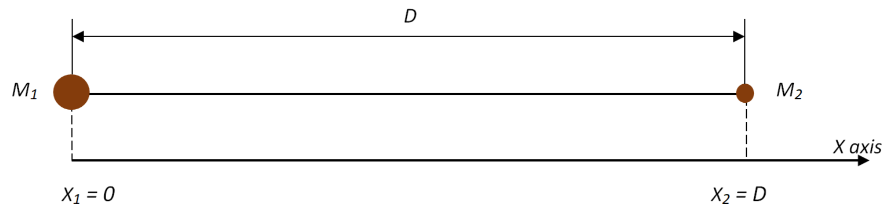

Figure 2. Jane's second diagram.

Therefore, the expression for the *X*-coordinate of the center of mass of the system is:

$$X_{C.\ mass} = \frac{M_1 X_1 + M_2 X_2}{M_1 + M_2} = \frac{M_1\, 0 + M_2\, D}{M_1 + M_2} = \frac{M_2\, D}{M_1 + M_2} \qquad (5)$$

Eureka! I've just discovered that the location of the center of mass of the system coincides with the location of the center of rotation of the system; therefore, I can equally state that the system rotates around its center of mass!

Prof: Great, Jane! This result is very general. In other words, you've found a way to locate the center of rotation of the flying hammer. Can you state it more generally to be clear that you fully understand the importance of this result?

Jane: Sure; the location of the center of rotation of the flying hammer is the location of the center of mass of the hammer.

Prof: Excellent! Now I want you to apply your knowledge to a planar version of the hammer as described in Figure 3. The metal head has a mass of 1 kg, and the handle has a mass of 0.1 kg. With this information, calculate the *X*-coordinate for the location of the center of rotation of the hammer.

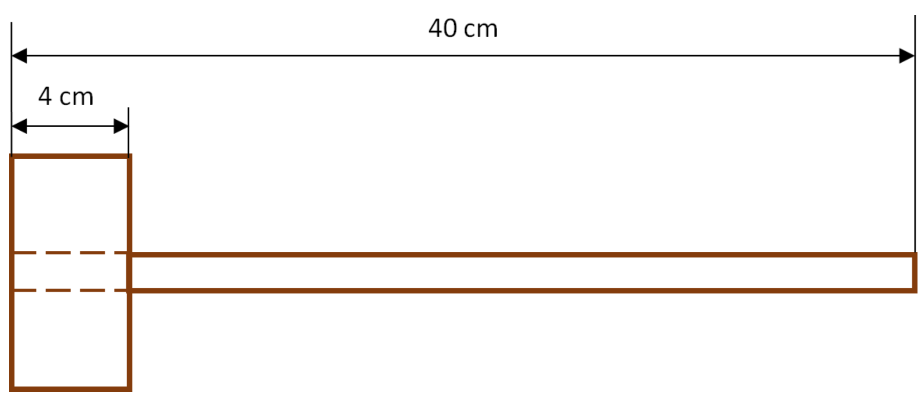

Figure 3. Planar version of a hammer.

Task 6: Determine the location of the center of rotation of the planar hammer.

Part VII – Conclusion

Jane: This now looks doable. The only thing that I have to do is calculate the *X*-coordinate of the center of mass of the hammer. To achieve this I'll first find the location of the center of mass of the head and the handle separately. Then, I can consider each as a point mass located at their respective centers of mass and apply the formula for the center of mass of a system of two particles. Here goes:

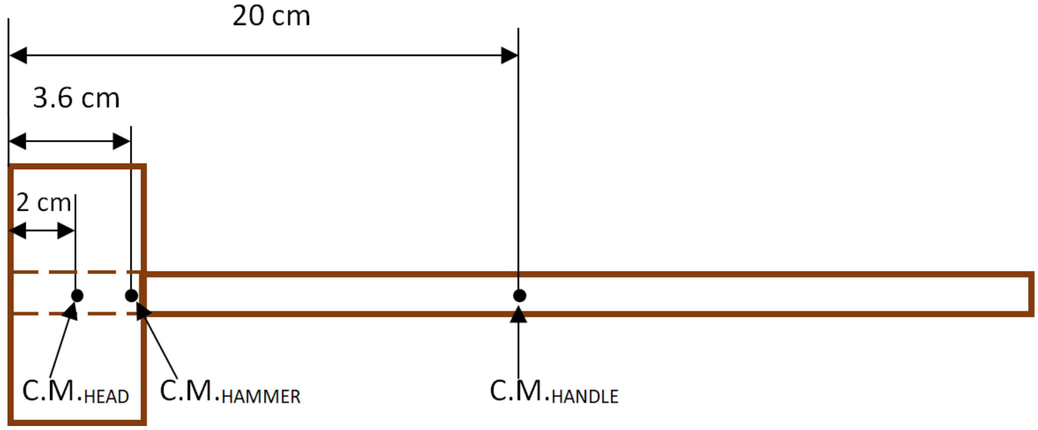

Figure 4. Planar version of a hammer with center of rotation.

By obvious symmetry, the *X*-coordinate of the center of mass for the head and the handle are $X_{C.M.head} = 2$ cm and $X_{C.M.\ handle} = 20$ cm, respectively. The *X*-coordinate for the center of mass of the hammer is:

$$X_{C.M.\ hammer} = \frac{M_{head}\ X_{C.M.\ head} + M_{handle}\ X_{C.M.\ handle}}{M_{head} + M_{handle}}$$

$$X_{C.M.\ hammer} = \frac{(1\text{ kg})(2\text{ cm}) + (0.1\text{ kg})(20\text{ cm})}{1\text{ kg} + 0.1\text{ kg}} = 3.6\text{ cm}$$

The location of the center of mass is inside the head and close to the edge facing the handle as shown in Figure 4. This is also the location of the center of rotation of the hammer as discussed above. I now fully understand why the flying hammer exhibited a point inside the head that did not perform any rotational motion.

Prof: Your result, as I mentioned before, is very general. It also applies to non-rigid objects, to interacting particles, to systems under the action of gravity, and others. Interesting examples are the rotational motion of figure skaters, divers, and of our planet Earth around its axis. You have the traits of a scientist. I think you've got a bright future ahead of you; you've got what it takes to make great contributions to biotechnology research and development.

Jane: Very encouraging words! I wish I had more college professors like you who are able to motivate students and lead them to actively find solutions to problems rather than to just recite boring material.

Questions

1. Write the general expression used to calculate the X-coordinate for the center of mass of a discrete system of N particles with a total mass M.

2. Do you expect a rotating system composed of two spheres connected by an elastic rubber band (not massless, with uniform mass density, and constant cross section) to have a center of rotation that coincides with the center of mass of the system? How would you calculate the position of the center of mass in this case?

3. Can the center of mass of a rotating object be located at a point which is outside of the object? Explain.

4. Write the expressions for centripetal force in terms of linear and angular speeds respectively.

5. Explain why the rod in the case study is in tension and not in compression.

6. Consider a particle of mass M moving in a circle of radius R around a center of rotation O as shown in Figure 5. Two forces of strength F_1 and F_2 act on the particle. Write the mathematical relation between the centripetal force and forces F_1 and F_2.

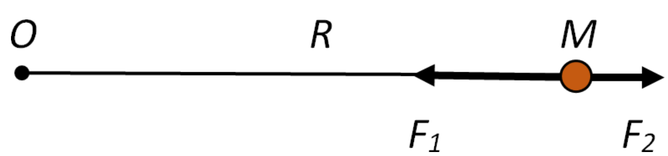

Figure 5. Mass moving in a circle.