QUANTUM

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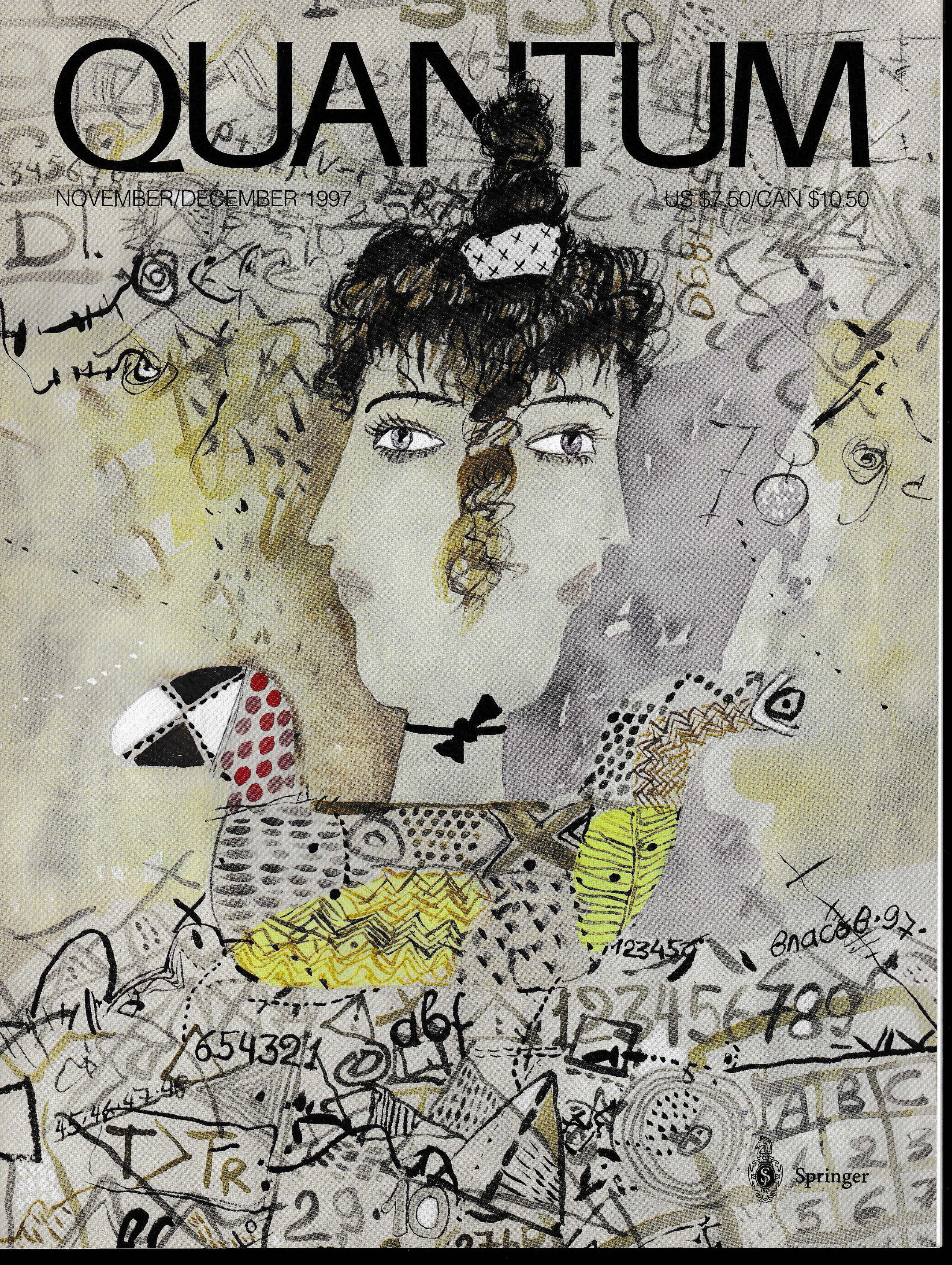

Springer

The Adoration of the Magi (c. 1508/1519) by Juan de Flandes

THE HOLIDAYS ARE A TIME WHEN MANY reflect on the biblical account of the birth of Jesus. According to the Bible, it was the Star of Bethlehem that led the Magi to the newborn Messiah. How would such a celestial phenomenon be seen today? What would a Hubble-enhanced view of such a star reveal? Oddly enough, Juan de Flandes's interpretation of the star could be more insightful than the artist ever intended. If what the Magi tracked across the desert was actually a star that had supernovaed, as some biblical scholars have suggested, then the halos that appear around the star could represent a bubble in the interstellar gas created by the solar wind emitted by the exploding star. To learn more about current thinking on the creation of these bubbles, turn to page 14.

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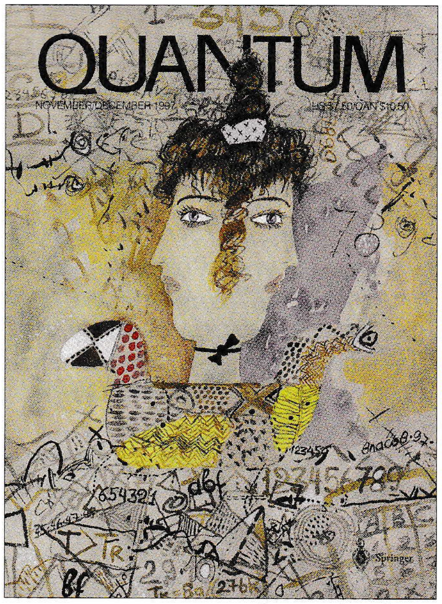

Cover art by Vasily Vlasov

There is often more than one way of looking at things, and those who come up with new ways of looking at things are often the most successful. Johannes Diderik van der Waals' contribution to how the world looked at gases earned him a Nobel Prize in physics.

To learn more about how he improved on our "ideal" view of the world, turn to page 36. Then take another look at the cover illustration to contemplate how the van der Waals equation helps scientists accurately describe transistions in the natural world around us.

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FEATURES

DEPARTMENTS

Anniversaries

Satellites and science reform

THIS PAST MONTH WE CELebrated the 40th anniversary of the launching of a number of initiatives. The first, and very obviously the initiator, was *Sputnik*. On October 4, 1957, the former Soviet Union put the first Earth-orbiting device into space. I was a junior in high school. If my memory serves me correctly, I recall at least one article that referred to it as "a mouse." It's interesting how metaphors change over time. Now our mouse is connected to our computer.

But *Sputnik* signaled not only the start of the space age, it also marked the beginning of a new wave of educational reform. For some time, scientists and mathematicians had been pleading for reform in the way science and mathematics were taught. The complaints were that the content was out-of-date and tended to be presented in an encyclopedic style—blocks of information to be memorized. The coherent wholeness of the disciplines was missing.

Sputnik energized this reform movement in science and mathematics education that had begun several years earlier. That next summer I attended a special advanced studies program for boys. (Yes, I'm embarrassed to say that back then it was only for boys!) We were allowed to choose one subject. I chose physics. For six weeks, twelve of us covered most of the material in a new course called the Physical Science Study Committee, or PSSC for short. Early in my career, I returned to PSSC as an enthusiastic new high school teacher. Armed with huge teacher's guides that highlighted the goals of all the labs and homework assignments, I began to appreciate the beauty of the course that had not been apparent when I first encountered the material as a student.

As the federal government and private foundation monies continued to flow, numerous education projects would be launched that would change the face of education. Within a couple years, I was back at graduate school and working with the Elementary Science Study (ESS) group in Newton, Massachusetts. It's interesting to me now, in hindsight, that many of the same people that started PSSC were behind the ESS efforts. It was an exciting time. A heady time. We were making materials that would change the way science was taught in the world. There was a constant stream of scholars, scientists, and educators passing through our offices at ESS. Countless workshops were given to visitors that demonstrated "real inquiry" experiences. Summer institutes were planned to train alpha teachers, who would then in turn teach others.

But it didn't "stick." The reform's promises were not fulfilled. As I sat in the lecture hall at the National Academy of Sciences last month, listening to many of the leaders of the '60s efforts reflect on *Sputnik* and the times, I wondered why the reform efforts weren't more successful. We had funding. We had scientists talking to each other and teachers about education. We had industry involved. We had the nation's attention. But still they didn't stick.

What about current efforts to excite the nation about establishing standards and developing curriculum that reflects those standards? I'm actually more optimistic than the above paragraphs would seem to indicate. I think we're smarter now. I think we realize now that progress in science and mathematics is different than in education. As one valued colleague from that time told me, "In education, the problems don't stay solved." In education, our solutions need to have an iterative character to them—sort of a self-correcting feedback loop. I believe we also need to think about the scale of the problem. There are over a million teachers of science in this country. Without a reform effort that reaches the majority of those teachers, nothing can change significantly.

This past month marked another anniversary. I joined NSTA and became publisher of *Quantum* just two years ago. Anniversaries are times to reflect as well as celebrate. I would very much enjoy hearing your opinion of *Quantum*. What does it do well for you? Where do we need to improve? How does the magazine fit in with the current reform efforts? Send me your thoughts/reflections via e-mail at gwheeler@nsta.org.

Thanks.

—Gerry Wheeler

Gerald F. Wheeler *is the Executive Director of the National Science Teachers Association and the Publisher of* Quantum.

QUANTUM

THE MAGAZINE OF MATH AND SCIENCE

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Hands-on (or -off?) science

Why is thermal sensitivity a touchy subject?

by Alexey Byalko

How would you check the temperature of an object that you were certain wasn't extremely hot or cold? You'd probably touch it with your fingers. Within a fraction of a second, the nerve endings in your fingers would tell you if the object was either warmer or cooler than your skin (34–36°C).

However, our perception of temperature is also affected by the type of material we are touching. At room temperature, wooden objects seem warmer than those made of glass or stone, and metal objects are perceived as colder. What is it that our fingers are perceiving? First, let's consider the most simple case, when the two materials making contact are identical. When a mother touches her sick child's forehead, her fingers can perceive an elevation in temperature as small as 1°C. When the two areas of skin come into contact, the cooler area begins to warm and the warmer area begins to cool. Eventually, the temperature at the point of contact will equal the average of the two temperatures:

$$T_0 = {}^1/_2(T_1 + T_2).$$

At a depth of 0.3–0.5 mm the skin is laced with several types of nerve receptors. They generate signals (nerve impulses) that transmit information on temperature, pressure, and other conditions. Some receptors are polymodal—for example, the pain receptors, which respond to very strong stimuli of a mechanical, thermal, or chemical nature. In this paper, we are only interested in the sensations of hot and cold, so we shall focus only on the thermal receptors, which "measure" not the absolute value of a temperature, but the difference between the body's temperature before contact and the temperature it has adjusted to after contact.

When perceiving an object's temperature, the first few instants after contact are the most important. This is when the skin's temperature (and that of the sensory nerve endings) begins to change. At first touch, a small but discernible period passes before your nerve endings register a difference in temperature. Let's evaluate this delay, or period of time required for a temperature change to reach the thermoreceptors.

First, we'll review the basics of thermal physics and the definitions of specific heat and thermal conductivity. The specific heat c of an object gives the rise in internal energy of a unit mass of the body when its temperature is elevated by one degree. Therefore, heating a body of mass m by ΔT degrees increases its internal energy by

$$\Delta Q = cm \cdot \Delta T.$$

The units for specific heat are J/(kg · K).

When two bodies at different temperatures come into contact, heat exchange commences. As a rule, thermal energy is always transferred from the warmer body to the cooler one. The physical value indicating the intensity of the thermal energy transfer is called the thermal flow. The thermal flow q is the thermal energy per unit time that passes through a unit cross-sectional area oriented perpendicular to the direction of the energy transfer. The units for thermal flow are J/(m^2 · s). The greater the temperature difference between the two bodies, that is, the greater the temperature drop along a unit length, the greater the thermal flow. If the temperature at a distance Δx is higher by ΔT, then the thermal flow, or amount of energy that passes every second through the unit area perpendicular to the flow, is

$$q = -\kappa \frac{\Delta T}{\Delta x}.$$

Collage by Vera Khlebnikova

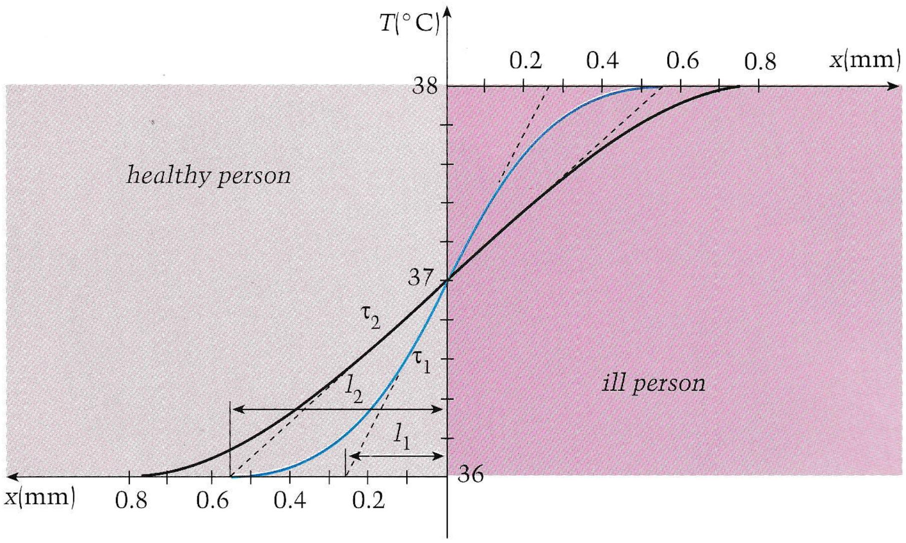


Figure 1

The temperature distribution in the near-boundary layers of the bodies during the thermal contact of a healthy and ill person in the periods $\tau_1 = 0.5$ s *and* $\tau_2 = 2$ s *after touching;* l_1 *and* l_2 *are the depths of penetration of the thermal waves in the periods* τ_1 *and* τ_2 ($l_1 \sim \sqrt{\tau_1}$, $l_2 \sim \sqrt{\tau_2}$).

Here the minus sign indicates the direction of energy flow (from higher to lower temperatures). The proportionality coefficient κ is the thermal conductivity. Its units are J/(m · s· K). The rate of heat transfer is determined by the specific heat and the thermal conductivity of the substances as well as by their densities.

Now let's estimate the rate of the temperature changes during the thermal contact of identical bodies with different temperatures. Figure 1 shows the distribution of temperature as a function of the distance from the surfaces of two bodies at time τ after first contact. Over time τ, one can see that the temperature can change markedly only in the layers of thickness l on both sides of the contact (evidently, l is dependent on time). The value for the thermal flow q at a given time is about

$$q = \kappa \frac{T_2 - T_1}{l}.$$

This means that during a time interval Δτ, the amount of energy passing through the cross-sectional area S of the thermal contact is

$$\Delta Q = qS \cdot \Delta\tau \cong \frac{\kappa(T_2 - T_1)S \cdot \Delta\tau}{l}.$$

This is the energy spent to raise the temperature of the layer of mass $m \sim \rho l S$ by an amount $\Delta T \sim T_1 - T_2$, or, $\Delta Q \sim c\rho l S \Delta T$. So,

$$\frac{\kappa S \cdot \Delta\tau \cdot \Delta T}{l} \cong c\rho l S \cdot \Delta T.$$

In this formula, the area S and the temperature drop ΔT cancel out; the time increment can be approximated by the time τ, which has passed since the beginning of the contact. Thus:

$$l(\tau) = \sqrt{\frac{\kappa}{c\rho}\tau} \qquad (1)$$

The equal sign is not a mistake: this is a precise formula, despite our ap-

	Specific heat c (J/kg · K)	Thermal conductivity κ (J/m · s · K)	Density ρ (kg/m³)	Temperature conductivity χ (m²/s)	Parameter $v = \sqrt{\frac{c\rho\kappa}{c_0\rho_0\kappa_0}}$	Contact temperature T_0 (°C)
Water	$4.18 \cdot 10^3$	0.631	$1.0 \cdot 10^3$	$1.5 \cdot 10^{-7}$	1.0	28
Air	$1.01 \cdot 10^3$	0.026	1.2	$2.1 \cdot 10^{-5}$	0.0035	35.9
Wood	$9 \cdot 10^2$	0.13	$5 \cdot 10^2$	$3 \cdot 10^{-7}$	0.15	34
Glass	$8 \cdot 10^2$	0.65	$2.6 \cdot 10^3$	$3 \cdot 10^{-7}$	0.72	29
Granite	$8.2 \cdot 10^2$	1.4	$2.7 \cdot 10^3$	$6.3 \cdot 10^{-7}$	1.1	28
Marble	$9.0 \cdot 10^2$	3.0	$2.7 \cdot 10^3$	$1.2 \cdot 10^{-6}$	1.7	26
Aluminum	38	236	$2.7 \cdot 10^3$	$2.3 \cdot 10^{-3}$	3.0	24
Iron	$4.4 \cdot 10^2$	74	$7.9 \cdot 10^3$	$2.1 \cdot 10^{-5}$	10	21.5
Gold	$1.3 \cdot 10^2$	310	$19.3 \cdot 10^3$	$1.2 \cdot 10^{-4}$	17	20.9

Table 1

Thermal properties of various materials.

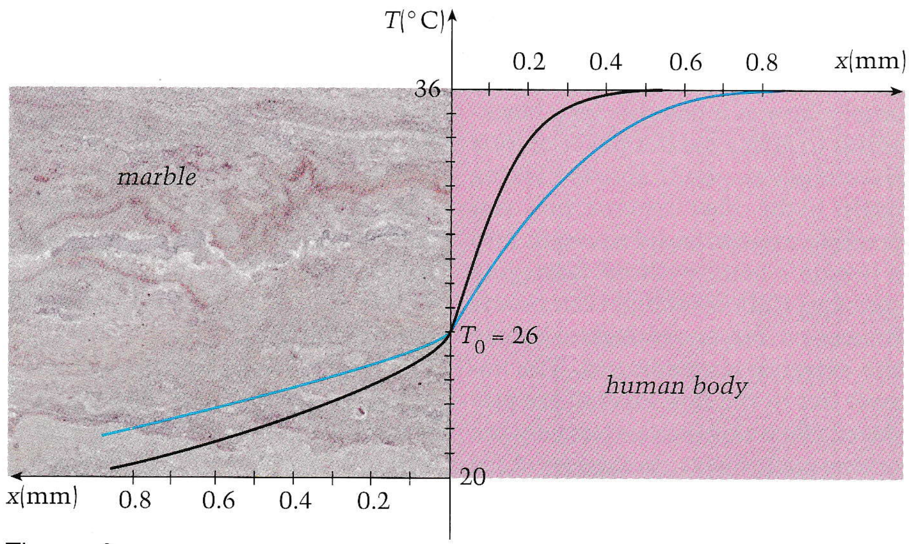


Figure 2

The temperature distribution at the contact point when human skin touches marble 0.5 s *(black curve) and* 2 s *(blue curve) after the initial contact. Temperature* T_0 *at the contact area is independent of time.*

proximations (the small errors counterbalanced each other). The curves in figure 1 are also precise.

The time necessary for the temperature at a depth l to reach the value of the contact temperature depends only on the values κ, c, and ρ, which are determined entirely by the properties of the contacting bodies. The combination of the constants under the radical sign in equation (1), $\chi = \kappa/c\rho$, is called the temperature conductivity for a given material. For example, water has the value $\chi = 1.5 \cdot 10^{-7} \text{m}^2/\text{s}$. In the examples that follow, we will use the material properties given in table 1 on the preceding page.

The structure of cutaneous tissue is rather complicated: It is not uniform, and its thermal properties depend on skin thickness. However, since biological tissue is more than 90 percent water, we will assume that tissue has the same thermal properties as water. Now we can estimate how long it takes for the thermal wave propagated at the skin's surface to reach the cutaneous receptors ($l_0 = 4 \cdot 10^{-4}$ m). The time period necessary to stabilize the temperature near the nerve terminals is

$$\tau_0 = \frac{l_0^2}{\chi_0} \cong 1 \text{ s}.$$

So, why is it that various materials of the same temperature produce such different thermal sensations when we touch them? Look at figure 2. It shows the temperature distributions for a contact with marble at two different moments in time. Note that the temperature of the contact is constant and doesn't vary with time. It is determined entirely by the thermal properties of the material we touch. Now we shall try to explain this phenomenon and find the contact temperatures for different substances.

At any given moment, the depths thermal waves have penetrated into skin and the material of an object are

$$l_0 = \sqrt{\chi_0 \cdot \tau}$$

and

$$l_m = \sqrt{\chi_m \cdot \tau}.$$

As figure 2 shows, the temperatures T_b and T_m at greater depths can be considered constant. At the point of contact, the thermal flow is identical for both adjoining surfaces. We can find this value using the thermal conductivity equation, denoting the boundary temperature by T_0:

$$q = \kappa_0 \frac{T_0 - T_b}{\sqrt{\chi_0 \tau}} = \kappa_m \frac{T_m - T_0}{\sqrt{\chi_m \tau}}.$$

The following equation is the answer to the question of why different materials generate different thermal sensations:

$$T_0 = \frac{T_b + vT_m}{1 + v}, \qquad (2)$$

where

$$v = \frac{\kappa_m}{\kappa_0}\sqrt{\frac{\chi_0}{\chi_m}} = \sqrt{\frac{\kappa_m c_m \rho_m}{\kappa_0 c_0 \rho_0}}.$$

Note that we did not assume beforehand that the contact temperature was constant—its stability is due to cancellation of time τ from the equation.

The values of T_0 for different materials are given in table 1. The table also shows the values of the contact temperatures (the temperatures of sensation) for the materials kept at room temperature (T_0 = 20°C). The temperature of the human body is assumed to be 36°C.

The values of T_0 given in the table correspond to our feelings: We do not sense the resting air at room temperature, and wooden objects seem only slightly colder than our body. On the contrary, glass and stone are perceived as cool, and metals are downright cold. Gold holds the record value of v. When we touch a bar of gold bullion, the contact temperature will be much closer to the gold's temperature than to body temperature.

Have you ever touched a metal object on a freezing cold day? Be careful! Your finger may stick. This phenomenon is caused by the freezing of the thin film of water on the surface of your fingertip, which means that the contact temperature is negative (T_0 < 0°C). Let's calculate how low the temperature has to be for this finger freezing to take effect. Formula (2) indicates that temperatures can present problems when $T < T_b/v$. For example, your fingertip will stick to iron at –3.5°C, to aluminum at –12°C, and to gold at –2°C.

At this point, let's consider a historical problem. Before the Russian revolution, when gold coins were still in circulation, why didn't the citizens' fingers stick to the coins during the severe winters? The answer is simple: the gold coins didn't stick because these small objects were warmed very quickly by the warmth of the hands. For example, a gold coin with thickness of $l \cong 2$ mm will warm up very quickly when grasped with two fingers: the warming period is $l^2/4\chi_{Au} \cong 10^{-2}$ s. The same is true for the modern coins made of less precious metals.

Figure 3
A seemingly impossible feat—a man walks barefoot across red-hot stones.

The law of thermal conduction can also explain other seemingly paradoxical phenomena. For example, how is it possible for a blacksmith to grab a red hot poker or for a firewalker to stroll across blazing coals? In the other extreme, how is it possible to pour liquid nitrogen into a person's hand without damaging the skin? Callouses might offer some protection, but not enough to handle the extreme temperatures. The answer lies in the thin layer of gas produced when skin encounters very hot or very cold objects. (In the case of the liquid nitrogen, this layer of gas is evaporated nitrogen. With the blacksmith and firewalker, the gas is created when the outer layers of skin are heated.) This layer of gas is a very poor thermal conductor, and although it is less than 0.1 mm thick, its pressure is great enough to support the weight of the human body.

But how long can it protect the skin from extreme temperatures? We already know that the thickness of the layer of gas is $l = 10^{-4}$ m. Let's also assume that the outer layer of skin can endure temperatures between 0°C and 100°C without becoming frostbitten or blistering.

Inside the gaseous layer the temperature is distributed linearly from the temperature of liquid nitrogen (or a red-hot object) T_1 to the contact temperature T_0. Thus, the thermal flow is

$$q = \kappa_{gas} \frac{T_0 - T_1}{l}.$$

The temperature distribution inside the skin depends on time in the same manner, as for any contact. Let's make both heat flows equal:

$$\kappa_0 \frac{T_b - T_0}{\sqrt{\chi_0 \cdot \tau}} = \kappa_{gas} \frac{T_0 - T_1}{l}.$$

You can see that in this case, the contact temperature T_0 is not constant: it rises when encountering hot objects and falls when encountering cold ones.

The skin will not be damaged at temperatures of $0°C < T < 100°C$. Therefore, these are the limiting values of the contact temperature. Now we can estimate how long it will take for the skin to reach such temperatures:

$$\tau_{max} = \frac{1}{\chi}\left(\frac{\kappa_0}{\kappa_{gas}} \cdot \frac{T_b - T_0}{T_0 - T_1} \cdot l\right)^2.$$

The values of thermal conductivity of various gases are very similar. So, in our estimation we will use thermal conductivity for air: $\kappa_{air} = 0.026$ J/(m · s · K). The calculations show that heating the skin through the layer of gas to 100°C by means of contact with an object with a temperature of 600°C will take just under 0.5 s. This is just enough time to transfer a hot iron from the fire to the anvil or take another step across the hot coals.

In the case of the liquid nitrogen ($T_1 = -196°C$, $T_0 = 0°C$), it takes 1.3 s to cool the skin below 0°C, which is more than enough time to amaze the uninitiated. A drop of liquid nitrogen can be handled safely for an even longer period of time if it is constantly moved from side to side of the palm. Of course, our hands are far too precious to risk on such stunts. But it is does give new meaning to the phrase "hands-on" science.

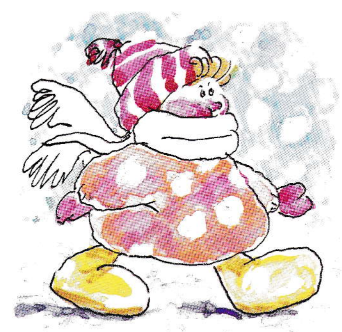
Just for the fun of it!

B216

Dicey math. You can see three faces of each of two dice. The total number of dots on these faces is 27. What is the total number of dots you can see on each die?

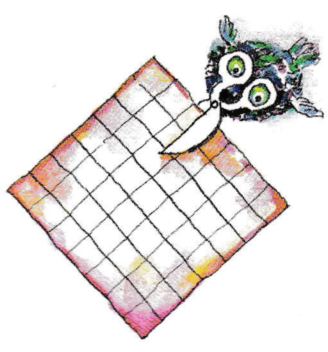
B217

All-encompassing curves. Consider two fixed points A and B, $AB = 5$, on the plane. Choose points C and D on the plane so that, in the resulting quadrilateral $ABCD$, segment $BC = 6$, $CD = 4$, and $DA = 1$. Draw the curves that contain all possible positions of C and D.



B218

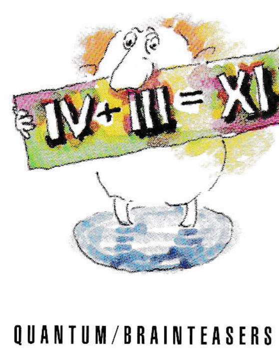
Crunchy snow. On very cold winter days, newly fallen snow crunches underfoot. Why is that?



B219

Heady calculation. Calculate the following quantity to the fifth decimal place without using your calculator:

$$\left(\sqrt[3]{2}+1\right)\sqrt[3]{\frac{1}{3}\left(\sqrt[3]{2}-1\right)}.$$



B220

Cell loss. You are given a 7×7 square consisting of forty-nine 1×1 cells. Remove one cell so that it's possible to cut all the remaining cells into 1×4 strips.

ANSWERS, HINTS & SOLUTIONS ON PAGE 51

Art by P. Chernusky

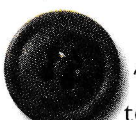
Forked roads and forked tongues

A mathematical map to the truth

by P. Blekher

OF ALL THE DIFFERENT types of brainteasers, logic problems are especially popular. As a rule, one needs no special mathematical erudition to solve logic problems. Even people who have little to do with mathematics can understand the nature of the problems.

In this article we analyze four logic problems[1]. In all of them we meet with two sorts of people: truth tellers, who tell only the truth, and liars, who tell only lies. But as we shall see, the most difficult and interesting of such problems are those incorporating cheats, people who can say anything just to confuse the inquirer.



Problem 1. A road forks. One fork leads to town A, which is inhabited by truth tellers only. The other fork leads to town B, inhabited by liars only. A mathematician meets a resident of one of these towns at the fork. The mathematician wants to know which road leads to town A. Can he find out by asking only one question?

As it happens, the mathematician can find out by asking only one question even if we impose an additional condition: The question must be phrased such that its answer will be either "yes" or "no." This condition is implied everywhere in this article.

The mathematician solves the problem by pointing at one of the roads and asking, "Does this road go to your town?" An affirmative response would mean the indicated road goes to A, and a negative answer would mean it goes to B. In fact, if the respondent lives in A, then his "yes" means the road goes to *A* and his "no" means it goes to B since he tells only the truth. On the other hand, if the respondent lives in B, then he is a liar and his "yes" means the road does not go to B (and therefore goes to A), and his "no" means that it does go to B, his hometown. Either way, the answer "yes" means the road goes to A, and the answer "no" means it goes to B.

Note that the mathematician cannot determine from the answer whether he is speaking with a resident of A or a resident of B. Still, he does not have to. The author knows some other solutions to problem 1, but they all employ the idea that the question (whether the given road goes to A) should be phrased so that a liar has to give a "double negative" answer for it. Since double negation is equivalent to a positive answer, the liar's answer would coincide with the answer of a truth teller. This very thing happens in the foregoing solution.

The second problem we consider is a complicated variant of the first. The difficulty results from the presence of a cheat among the respondents.

Problem 2. Suppose a mathematician meets three people at the fork in the road of problem 1. The mathematician knows that one of them is a resident of town A, one is a resident of town B, and the third is a cheat. However, the mathematician does not know who is who. Can he find out which of the two roads goes to A by asking only two questions?

We should specify that the mathematician can ask each question of any of the three people and that only the queried person will answer the question. Besides this, each of the three knows which person is which, and each knows which of the roads goes to A and which to B.

Art by Dmitry Krymov

[1]Problems 1 and 2 were communicated to the author by the Hungarian mathematician P. Mayor.


truth
truth
truth
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truth
Truth

The solution to problem 1 prompts the following idea: What if it is possible to discover from the first question which one of the three is not a cheat? If so, then problem 2 would be reduced to problem 1, and we could find the road to A by asking one of those who is not a cheat in the question from problem 1. It turns out that such a first question exists, although to the author's mind it is not an easy one to be guessed.

For convenience let's enumerate the three people in an arbitrary way. The first person should be asked the following question: "Suppose that each of you will go to A or to B according to the following conditions: A resident of A will go to A, a resident of B will go to B, and the cheat, provided that you are not the cheat, will go with you. If you are the cheat, you will go wherever you want. Will this person [the second person is pointed to] go to A under these conditions?"

If the answer to this question is "yes," the third man is not the cheat; if the answer is "no," the second man is not the cheat. In fact, if the question was posed to the cheat, then the third and the second men are not cheats. Furthermore, if the question was posed to the truth teller, then his "yes" means the second man is the cheat and, therefore, the third man is not. On the other hand, the truth teller's "no" means the second man is a liar (since only the liar will not go with him). And if the question was posed to the liar, then his "yes" means the second man is the cheat and the third is the truth teller because the cheat has to go to B and the truth teller goes to A. The liar's "no" means, on the contrary, that the second man is the truth teller and the third is the cheat.

Analyzing all the opportunities, we see that the third person, if the answer is "yes," and the second person, if the answer is "no," are not cheats for sure. Thus we can pick out a person who is not the cheat and problem 2 is solved.

Note that once again the main idea was to compel the liar to make a double negation so that his answers accord with those of the truth teller's and distinguish between cheats and noncheats.

problem 3. A chemistry conference is attended by N scientists—some of which are chemists and the rest of which are alchemists. It is known that there are more chemists than alchemists in attendance and that chemists answer all questions honestly and alchemists always lie. A mathematician attending the conference wants to know whether each scientist is a chemist or an alchemist. To this end he can ask any of the scientists what any other scientist is. Propose a method that would allow the mathematician to find out who is who by asking $(N-1)$ questions.

The solution for problem 3 is relatively simple. We begin by asking any one of the scientists (we shall call him the first scientist for convenience) about all the rest of the scientists. These $(N-1)$ scientists will be divided into two groups: a group of those that the first scientist called chemists and a group of those whom he called alchemists. We then assign the first scientist to the first group and choose the larger of the two groups. Then the scientists from the larger group are chemists, and the scientists from the other group are alchemists (see if you can prove it!). The problem is solved.

problem 4. Now we have come to the central problem of the article. Its conditions are the same as those for problem 3, but the alchemists are now cheats rather than liars. The task is to find out who is who at the conference with the help of no more than $3N/2$ questions.

This problem is much more complex than problem 3. Although a liar's answers are wrong, they are always wrong, which allows one to extract from them almost as much information as from the answers of a truth teller. Introducing cheats into the problem adds complexity because a cheat's answers are completely arbitrary, and gleaning any information from them is much more difficult.

The solution that follows allows one to discover who is a chemist and who is an alchemist by asking even fewer than $3N/2$ questions. In fact, the number of questions necessary is $2k$ if $N = 2k + 1$ is odd, and $3(k-1)$ if $N = 2k$ is even.

Let's first consider the case when N is odd. We shall construct the solution by induction on k. If the conference was attended by $N = 1$ scientist ($k = 0$), then he would evidently be a chemist because there are more chemists than alchemists. No questions are needed in this case: $q = 0 = 3 \times 0$.

Suppose that for all odd numbers less than the given $N = 2k + 1$ we have already constructed a method that allows one to solve the problem with the necessary number of questions. We shall then give a solution for the number $N = 2k + 1$.

Let's enumerate, for convenience, all participants of the conference and start asking the second scientist, the third scientist, and so on, whether the first scientist is a chemist or alchemist. We continue this poll until one of the next two events happens:

Event A: A majority of scientists we have asked say the first scientist is an alchemist.

Event B. The number of scientists who have said the first scientist is a chemist is equal to k.

Suppose event A has just occurred. At this moment, if t scientists said that the first was a chemist and f said he was an alchemist, then $f = t + 1$. In fact, $f > t$, and if it had happened that $f \geq t + 2$, then event A would have happened at least one question earlier. Also, the total number of questions asked during this poll is $q_1 = f + t = 2f - 1$. (The situation in which the second scientist already says that the first one is an alchemist is a particular case of event A where $t = 0$ and $f = 1$.)

If event B happened and the number of scientists who had said the first scientist was an alchemist was f, then the total number of questions would be $q_1 = k + f$.

We can see that the poll will stop before all the scientists attending the conference are asked. Indeed, let's suppose the opposite: that neither A nor B had happened before the last scientist was asked. Suppose that by this moment, t scientists had said the first scientist was a chemist and f had said he was an alchemist. Because event A had not happened, $f \le t$. And because event B hadn't happened either, $t \le k - 1$. Therefore, the total number of scientists polled is $f + t \le 2(k - 1)$. If we add the first and the last scientists to them, we find that the total number of scientists at the conference is not greater than $2k$, while in fact it is $2k + 1$. This contradiction proves that either event A or event B must happen before the last scientist is asked.

Now let's assume event A happened. Then we can say that in the group composed of the first scientist and all the polled scientists, the number of chemists does not exceed the number of alchemists.

In fact, if the first scientist is a chemist, then those f scientists who said he is an alchemist are alchemists themselves. In this case—since the total number of scientists in the considered group is $1 + t + f = 2f$—the number of alchemists in the group is not less than the number of chemists. And if the first scientist is an alchemist, then t scientists who have said he is a chemist are alchemists, too. Thus, the number of alchemists is not less than $1 + t = f$, that is, not less than one half.

Furthermore, according to the conditions of the problem, the total number of chemists is greater than the total number of alchemists. Thus, in the remaining group of $N - 2f = 2(k - f) + 1$ scientists, the number of chemists must be greater than the number of alchemists, too. The number $N - 2f$ is less than N, and therefore, according to the inductive hypothesis, there exists a method to find out who is a chemist and who is an alchemist by asking $q_2 = 3(k - f)$ questions. Let's choose an arbitrary chemist from this group (there must be a chemist in it) and ask him who the first scientist is (it would take us $q_3 = 1$ more question).

If the first scientist is an alchemist, then those t scientists who said he is a chemist are alchemists. Therefore, we need only ask the chemist we've chosen to identify chemists and alchemists among the f scientists, saying that the first one is an alchemist (it will take another $q_4 = f$ questions). So we can find the complete distribution of chemists and alchemists at the conference by $q = q_1 + q_2 + q_3 + q_4 = 2f - 1 + 3(k - f) + 1 + f = 3k$ questions—just what we've proposed.

And if the first scientist is a chemist, then those f scientists who said he is an alchemist are alchemists themselves. Consequently, we need only ask the chemist we've found to identify chemists and alchemists among the t scientists, saying that the first one is a chemist. We shall spend $q_4 = t$ more questions for it. The total number of questions in this case is $q = q_1 + q_2 + q_3 + q_4 = 2f - 1 + 3(k - f) + 1 + t = 3k - 1$. This is even less than the number of questions we can use, according to the problem statement. Therefore, the case when event A happens is considered completely.

Now we shall consider the case when event B happens. We state that the first scientist must be a chemist in this case. If he were an alchemist, the k scientists who stated he was a chemist would be alchemists and the total number of alchemists would be not less than $k + 1$, which is more than one half, and would therefore contradict the statement of the problem.

Thus the first scientist is a chemist, and the f scientists saying he is an alchemist are alchemists themselves. Now we ask the first scientist to identify chemists and alchemists among the k scientists who said that he is a chemist (it will take $q_2 = k$ questions). Next we ask the first scientist to identify chemists and alchemists among the scientists that didn't take part in the poll (it will take $q_3 = N - (1 + k + f) = 2k + 1 - 1 - k - f = k - f$ questions). So we shall find out who is who at the conference by $q = q_1 + q_2 + q_3 = k + f + k + k - f = 3k$ questions. Thus we have considered both cases—when event A happens and when event B happens. The problem is solved completely for odd N.

If N is even the solution is almost a word-for-word repetition of the solution given for odd N, and we leave it as an exercise for those who want to achieve a deeper understanding of our reasoning. ◘

Interstellar bubbles

A passing phase in the life cycle of stars

by S. Silich

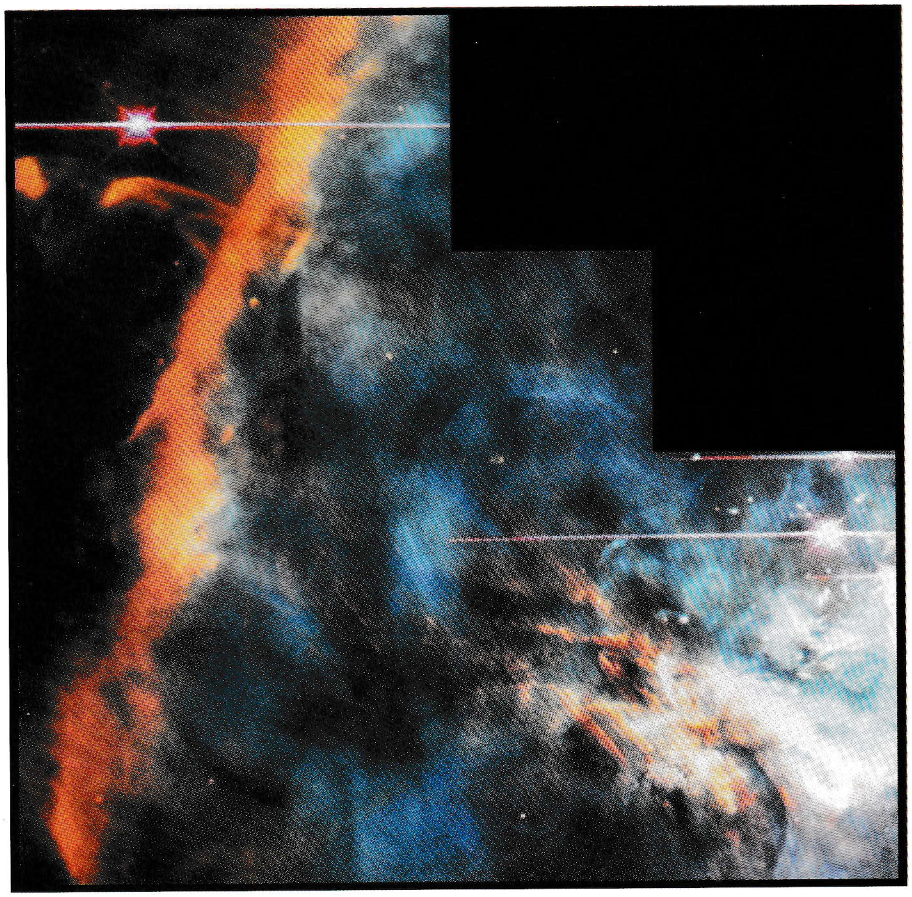
ASTRONOMERS HAVE DIScovered that the space between stars is filled with an extremely rarefied interstellar gas composed of hydrogen, a small amount of helium, and other chemical elements. This gas becomes superheated when in the vicinity of a star, causing the gas to emit light. These regions of space are known as emission nebulas. In the winter sky, the famous Orion Nebula can be seen just below the three bright stars that form Orion's belt. It's visible to the naked eye if you live in the country, but can also be seen with binoculars by city dwellers.

Away from the influence of the stars, interstellar gas cools down to only a few degrees Kelvin and condenses into cold opaque clouds that light cannot pass through. In summer, in that part of the Milky Way that passes through the constellation Cygnus (the Swan), you can see a dark wedge splitting the Milky Way in two. This wedge is actually a dark interstellar cloud that is blocking out the stars behind it.

Does our galaxy look like foam?

Although our Galaxy contains only a small amount of interstellar gas (about 5% of its total mass), the gas plays an important role in the creation of stars. When enough interstellar gas collects in one place, a star is born. But where does this gas collect? Just like water, interstellar gas is drawn into potential wells by gravitational forces. However, when Earth's gravity draws gas toward its center, the gas is blocked by the surface and forms our atmosphere.

In contrast, the only thing that stops interstellar gas from falling toward the center of our Galaxy is



Photos courtesy of NASA

Figure 1
Very recent star formation (300,000 years ago) within the Great Nebula of Orion. The plume of gas at the lower left is the result of the ejection of material from a recently formed star.

the pressure of the gas, and the "centrifugal force" created by the Galaxy's rotation. This has resulted in the formation of a layer of interstellar gas in the plane of the Galaxy. This thin, rotating pancake—30 kiloparsecs in diameter and only 150 to 200 parsecs in thickness (1 parsec = $3 \cdot 10^{16}$ m)—is the place where stars are born.

Once a star is born, it immediately affects the interstellar gas that surrounds it. A solar wind of radiation and particles emitted by the new sun pushes the interstellar gas away from the vicinity to form a small (on the galactic scale) bubble. The interstellar bubbles around our Sun and similar ordinary stars are rather small. However, the bubbles "blown" by stars that are brighter, more massive, and more active are markedly larger. This heterogeneous bubbling gives the interstellar gas a foamlike appearance, with large bubbles surrounding large stars and small bubbles surrounding small stars. This description, however, is far too simple once you understand the true nature of interstellar gas.

Is it more like a tunnel?

At the end of its life cycle, a massive star explodes as a supernova, throwing off its outer layers at a velocity of thousands of kilometers per second. The energy of its explosion is about 10^{44} J. The shell thrown off expands violently, bulldozing its way through the interstellar gas in front of it. After 12,000 years or so, when the shell's expansion has slowed, a large cavity will have formed around the site of the explosion (astronomers call such cavities *supernova remnants*). One can easily imagine an analogous situation; blow a large soap bubble with a straw and compare it to the small bubbles in a foam.

Each bubble is preserved while the inner gas remains hot and can withstand pressure of the surrounding gas. When the gas cools, the bubbles collapse. Supernovas are common in our Galaxy, occurring once or twice each century. And, because it takes many thousands of years for the bubbles of hot gas to cool and collapse, the lifespans of bubbles overlap. In fact, when a new bubble is formed in the vicinity of an older bubble, the new bubble will often pierce the shell of the old bubble and inject hot gas into it. This infusion of hot gas prolongs the life of the old bubble. Eventually, a chain of interconnected bubbles is formed that resembles the winding tunnel system of a mole. However, this notion of an interstellar medium rife with tunnels didn't last long.

Or Swiss cheese?

Radio-astronomy observations of the last decade have showed that, in addition to the fine-foamed structure and crossing tunnels, the interstellar medium also has cavities with diameters hundreds of parsecs, or even kiloparsecs, in length. Our Solar System is located just at the edge of one such giant cavity with a diameter of about 300 parsecs. Similar observations have been made in neighboring galaxies whose gaseous disks resemble pieces of Swiss cheese with their holes protruding outward.

What is the force that pushes interstellar gas away from the galaxy's disk? To create a cavity 1 kiloparsec in diameter, 10^{17} J of energy is needed, which is much more energy than is available from a supernova explosion. Astrophysicists have suggested a number of energy-release mechanisms that might provide the energy needed to create giant interstellar bubbles. The two most promising theories involve the impact of a massive gaseous cloud on a disk-shaped galaxy (similar to a meteor impact on a planet) or a series of supernovas taking place at the center of a star-production site (similar to a series of explosions at an ammo dump). This latter mechanism seems to provide a better explanation of the observed phenomena.

Stars drive away the interstellar gas

The larger the mass, the higher the temperature of a star, and the more intense the burning of the nuclear fuel in its interior. Consequently, massive stars burn out quickly and have a relatively short life span. A star's life expectancy t_L depends upon the initial mass M of a star:

$$t_L = 5 \cdot 10^7 \left(\frac{M}{10 M_S} \right)^{-1.6} \text{ years,}$$

where $M_s = 2 \cdot 10^{30}$ kg is the mass of the Sun. At the end of their life span, stars with a mass of $M \geq 7M_s$ become unstable and explode as supernovas, losing most of their matter. Let's consider in detail how the matter thrown out by the blast interacts with the surrounding interstellar gas.

After the explosion, a shock wave moves with supersonic velocity through the interstellar gas, causing almost instantaneous changes in temperature, pressure, density, and velocity of the matter in a narrow layer about as wide as the atomic mean free path. During this process, a large amount of the kinetic energy generated by the moving gas is converted into heat. After the wave front passes, the interstellar gas is much hotter. Then, gradually, the gas cools as its energy escapes as radiation and is expended during the expansion. The velocity of the shock wave decreases, and finally the expansion of the hot bubble around the exploded star stops entirely.

As previously stated, the energy released during a single supernova is insufficient to push away enough interstellar gas to create the "holes in the Swiss cheese." Therefore, it is thought that the energy from a series of supernovas must combine to create these bubbles. But why does this happen? As a rule, stars are born as part of large stellar groups in the interior of cold clouds. Simultaneously, thousands of "species" of stars (stars of the same mass) are formed. And, because the life span of a star depends on its mass, stars of the same species within a stellar group tend to supernova at the same time. First the most massive stars explode, then the smaller ones.

At the onset of a stellar cluster's life, the medium is heated by the powerful radiation of the hot massive stars. Then, later in their life span, the massive stars begin to discharge matter in the form of stellar winds, which push away the surrounding interstellar gas. Small cavities surrounding individual stars interact with each other and produce a common shell. After this stage, the more powerful process begins—the blasts of the supernovas. The stars explode one after another, beginning with the most massive. First in this succession are the stars with masses of 30–50M_s, which explode four to five million years after the birth of the stellar cluster. Next in turn are the smaller stars in the cluster with masses of 7–8M_s, which explode 40 to 60 million years after their birth. As the hundreds of blasts go on one after another, the entire process can be considered continuous. The constant rate of energy being produced by the process is expressed as

$$L_{SN} = 6.3 \cdot 10^{35}\, NE_{44}\ \text{J/s},$$

where N is the number of massive stars in a cluster, and E_{44} the energy of one supernova's blast expressed in the units of 10^{44} J. After about 50 million years, the source of energy switches off when the last supernova in the group has exploded. However, the common shell still expands for some period of time due to the accumulated heat and kinetic energy, which eventually leads to the formation of a giant interstellar bubble. Now let's take a look at the evolution of an interstellar bubble.

The simplest bubble is a sphere

Problems involving the propagation of a shock wave after a point explosion are very difficult and can be solved only by integrating a system of nonlinear equations of gas dynamics. This problem has no general analytical solution. In the late1940s, after the development of the first nuclear weapons, scientists became interested in analyzing shock waves.

First, let's consider the most simple case—a powerful point explosion in a homogeneous medium. This type of explosion is characterized by a spherical, symmetrical shock wave and the shell that it forms. This problem was solved by L. I. Sedov and J. Taylor. The solutions make two assumptions. First, almost all the gas driven away by the shock wave accumulates in a narrow layer immediately in front of the shock wave. Second, the pressure inside the cavity is constant almost everywhere. The latter is caused by the high temperature of the gas heated by the shock wave, as well as by the quick dispersion of possible heterogeneities. These two properties radically simplify the problem of a powerful explosion in a homogeneous medium. They underlie the *thin-layer approximation* that is widely applied in astrophysics and in plasma physics. This theory is also known as the snow plow model because the gas is thought to collect in front of the wave just as snow collects in a thin layer in front of a plow blade.

The thin-layer approximation is based on two postulates. It is thought that all the gas is collected in the infinitely thin layer immediately ahead of the shock wave front, and that the pressure inside the cavity is homogeneous and depends only upon time. The same approximation can be used to model an explosion in a heterogeneous medium, resulting in the formation of a nonspherical shell.

It should be pointed out that, when this approximation is used, information about the distribution of hydrodynamic values inside a bubble is lost, and only an average value of the inside pressure is considered. Nevertheless, this method makes it possible to describe such

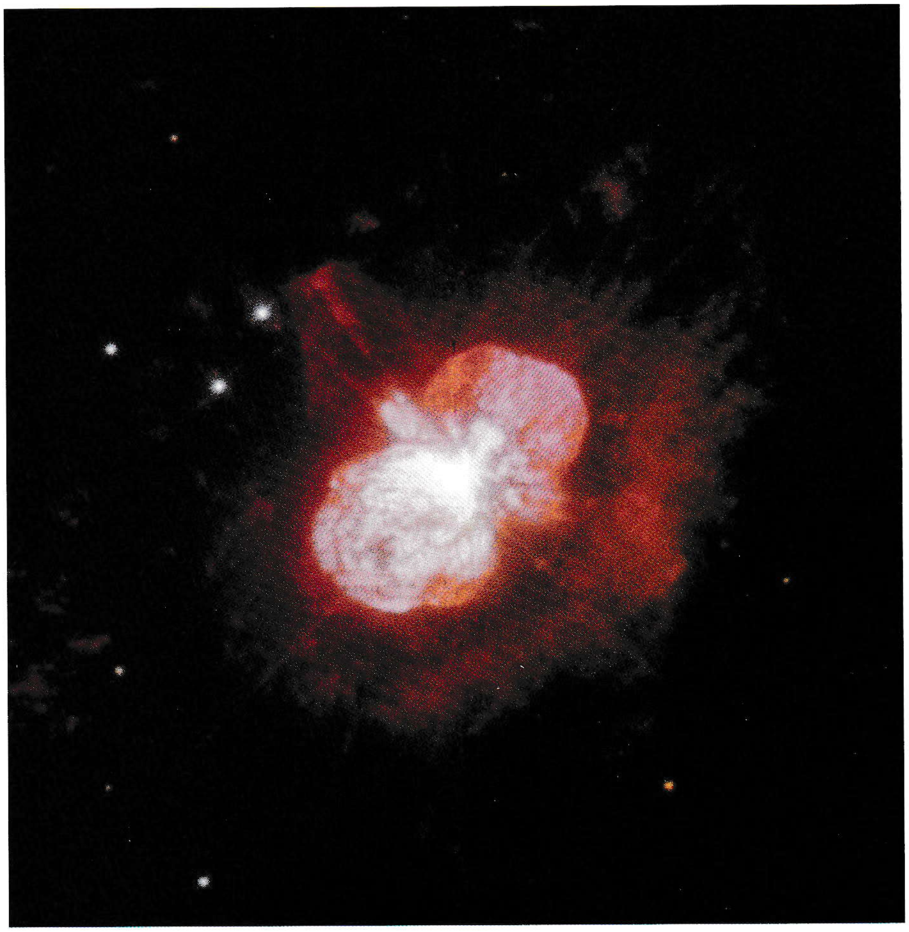


Figure 2

Billowing gas and dust clouds erupting from the supermassive star Eta Carinae. This giant outburst occurred approximately 150 years ago, producing two polar lobes and a large thin equatorial disk that are moving out at about 150 million miles per hour.

important properties of interstellar shells as their shapes, their velocities of expansion, and their distributions of surface densities.

It is interesting that the dependence of the spherical shell radius R_s on time and initial conditions can be obtained with the very simple method of dimensional analysis. Clearly, an explosion is characterized only by its energy E_0 and the density ρ_0 of the medium in which it spreads since its pressure and temperature are rather small. So, the dependence can be expressed as

$$R_s = AE_0^{\alpha}t^{\gamma}\rho_0^{\beta},$$

where A is a dimensionless constant. The same equation is valid for the dimensions of the corresponding variables:

$$\text{m} = \text{J}^{\alpha} \cdot (\text{kg/m}^3)^{\beta} \cdot \text{s}^{\gamma} = \text{m}^{2\alpha - 3\beta} \cdot \text{kg}^{\alpha + \beta} \cdot \text{s}^{\gamma - 2\alpha}.$$

From the indices of the exponents we have the system of equations

$$2\alpha - 3\beta = 1,$$
$$\alpha + \beta = 0,$$
$$\gamma - 2\alpha = 0,$$

from which we get $\alpha = 1/5$, $\beta = -1/5$, and $\gamma = 2/5$. Therefore,

$$R_s \sim \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}. \qquad (1)$$

Taking the time derivative of R_s, we find that the velocity of the shell expansion decreases with time:

$$v_s \sim \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{-3/5}. \qquad (2)$$

Such is the evolution of an interstellar bubble formed by an instantaneous release of energy in a homogeneous medium—for example, as a result of a supernova.

Stellar wind

At different stages of their evolution, and particularly at the final stage, stars lose some of their mass. In massive stars, the rate of outflow of matter reaches 1,000 km/s, so the total energy released as stellar wind into space over thousands of years is quite comparable to the energy released during a supernova explosion. Because the energy release associated with stellar wind is so gradual, it lacks the destructive force of a supernova. Still, the cavities formed by stellar winds in the interstellar medium have much in common with those formed by supernovas. However, due to the persistent inflow of energy into the cavity, the inner structure of a bubble "blown up" by stellar wind differs from the inner structure of a bubble created by a supernova.

A cavity created by stellar wind has four zones, as shown in figure 3. The internal zone (a) is the region of freely expanding stellar wind moving with a constant velocity; it is limited by the internal shock wave (SW). The radius of this zone is small in comparison with the size of the entire cavity. The next zone (b) is filled with hot and almost isobaric gas, evaporating mainly from the inner border of the cold and dense shell (c), which is separated from

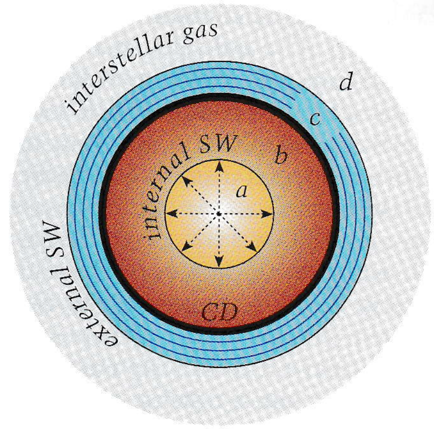


Figure 3

Scheme of a cavity formed by the stellar wind. (1) interstellar gas; (2) internal shock wave (SW); (3) external shock wave (SW); (4) contact discontinuity (CD).

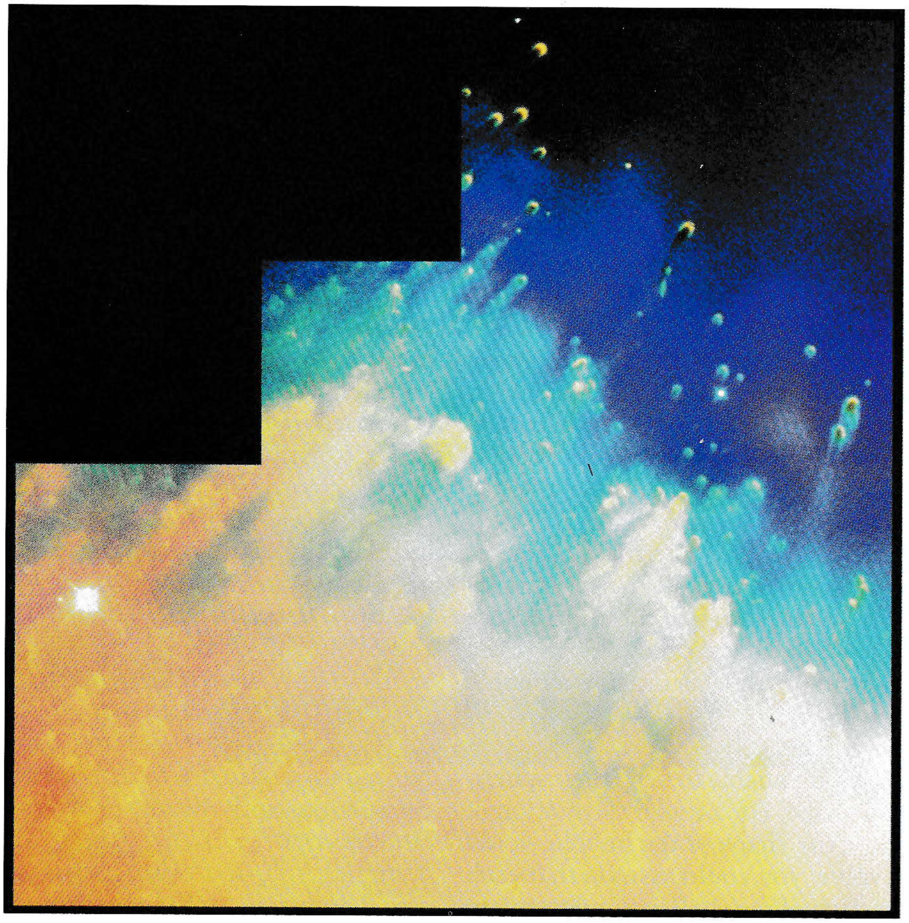


Figure 4

The collision of two gases near a dying star. Astronomers theorize that the "cometary knots" in the upper right-hand corner were formed when gas spewed from the surface of the doomed star later collided with cooler gas thrown off by the star some 10,000 years earlier.

zone (b) by the contact discontinuity (CD). The cold dense shell (c) consists of interstellar gas (d) compressed by the external shock wave.

Similar to point expansion, the motion of such a shell is described by the laws of conservation of mass, momentum, and energy. In the case of stellar wind, however, the total energy of the remnants is not conserved. At the front of an internal shock wave, the kinetic energy of the stellar matter flowing away from the surface of a star turns into heat. This results in the heating of the gas in zone (b), which acts as an elastic bumper that pushes the external dense shell. The change in the thermal energy of this bumper is defined by the power of the stellar wind L. Using the dimensional technique, we easily obtain the dependence of the shell radius on time, the density of the surrounding gas, and power of the stellar wind:

$$R_s \sim \left(\frac{L}{\rho_0}\right)^{1/5} t^{3/5}. \qquad (3)$$

As we see, the persistent influx of energy into the cavity makes its expansion more uniform.

From sphere to the real shape

Formulas (1) through (3) show how the rate of the shell's expansion depends on the density of the surrounding gas: When the other conditions are equal, the lower the gas density and the faster the rate of expansion. In our previous example, we assumed that all the gas outside the bubbles was of a uniform density. But what would happen if the density of the gas outside a bubble varied from point to point? Giant bubbles that encounter these conditions are unable to keep their spherical shape. For a rough estimation of how the shape of the bubble would evolve, you could assume that each section of the shell was independent of the others and apply the previous formulas to predict their rate of expansion. In areas where the external gas is less dense, the shell will expand more quickly. This leads to elongation of the shell in areas where the gas is less dense.

In a disk-shaped galaxy, the attraction to the nucleus is almost entirely counterbalanced by the centrifugal force, and only the acceleration out of the galaxy's plane is left unbalanced. Both the pressure and density of the interstellar gas decrease with distance from the galaxy's plane, which is similar to what takes place in Earth's atmosphere at high altitudes. Consequently, large shells expand more quickly in directions away from the galaxy's plane.

Bubble expansion is also affected by the uneven, or differential, rotation of the various elements in a galaxy. Just as the planets in a solar system revolve around a sun at different speeds, different elements in a galaxy revolve around its center at different speeds. When an interstellar bubble expands to a large enough size, it causes the surrounding gas along the galaxy's plane to lag behind in its rotation or to speed ahead. Consequently, bubbles tend to elongate in the direction of galactic rotation where the gas has been pushed away and become less dense.

To accurately describe the expansion of large galactic shells, you need to solve three-dimensional problems of gas dynamics. Solving such problems led to the development of new numerical methods and their application to astrophysics. However, the precise algorithms for modeling the large galactic shells are extremely difficult and require a huge amount of calculation time and effort. For example, the calculations to predict the evolution of two-dimensional shells in only one variant—made in the 1980s by American astrophysicists MacCray, MacLow, and Norman—required 6 to 12 hours of calculations with the most powerful supercomputer of that time. The complete solution of three-dimensional problems runs into enormous calculating troubles even when very fast computers are used.

Therefore, for all practical purposes, we can consider only the general approaches to solving the multidimensional problems based upon thin-layer approximations described previously. There are two methods to solve the problems of gas dynamics. In Euler's approach, the changes in all physical values at given points in space are tracked. The flow of fluids is described by the time-varying fields of velocity, density, and temperature. In Lagrange's method, the motions of individual elements of the medium are tracked. It is Lagrange's approach that is used to study motion of multidimensional shells by means of thin-layer approximations. For calculations, the

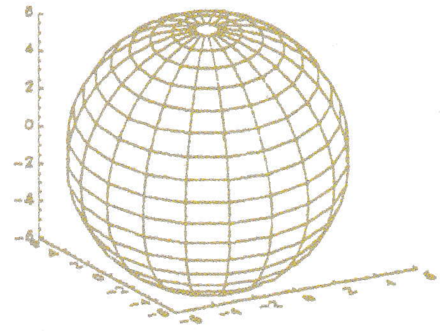


Figure 5

Division of a spherical shell in Langrangean elements at the beginning of the calculations.

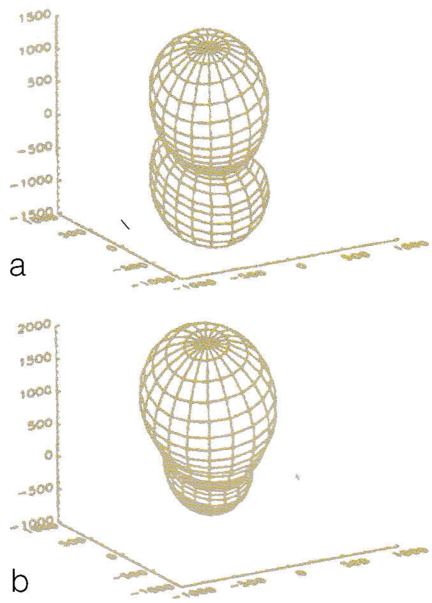


Figure 6

(a) Shell formed by the explosion of about 100 supernovas located in the galactic plane at a distance of 8.5 parsecs from the galaxy's center. (b) The same initial conditions as in (a), but the exploding supernovas are located 50 parsecs above the galactic plane.

entire shell is divided into N Lagrangean elements (fig. 5). The motion of each individual element is described by the laws of mass and momentum conservation, just as for the entire spherical symmetrical shell in a homogeneous medium. However, instead of one system of equations describing the change of radius of the spherical shell, we have a large number of equations describing the motion of individual shell elements.

To obtain a suitably accurate calculation, we divide the shell into 1,600 Lagrangean elements. This division produces a system of 11,200 ordinary differential equations because the motion of each surface element is described by seven equations: one for conservation of mass, three for momentum components, three for velocity components, plus one equation they all share, conservation of energy. This problem can be solved with the help of a good computer and special equations developed by applied mathematics. Figure 6 shows the results of calculations of the evolution of shells in our Galaxy. Shown are shells located at the same distance from the galactical center as our Sun. The energy influx into the cavity is provided by a cluster containing about 100 massive stars. Figure 6a shows a shell formed around a cluster located in the galactical plane. In figure 6b, the star cluster is shifted by just 50 parsecs above the galactic plane. It is clear that the shell's shape depends strongly upon the location of the cluster. As can be seen, the shapes of the shells are distorted due to differential rotation of the disk-shaped galaxy.

As we can see, the giant interstellar bubbles assume the shape of an hourglass. During the later stages of development, a neck is formed on the shell near the galactical plane, which is a narrow layer containing most of the collected interstellar matter. Within this layer, just the right conditions exist to give birth to large interstellar clouds. And, it is from within these clouds that new stellar groups emerge. Thus, the giant interstellar bubble born as a result of stellar explosions becomes the place where a new generation of stars begins its life. The superloop of cosmic evolution is complete.

The article was edited for students by V. Surdin.

Unidentical twins

Using conjugate numbers to tame irrationalities

by V. N. Vaguten

MANY DIFFICULT PROBLEMS IN GEOMETRY can be solved by noticing various types of symmetry in the given figure. Considerations of symmetry can also prove useful in problems involving algebra.

In the several situations considered in this article, it turns out to be helpful to substitute a number of the form $a + b\sqrt{d}$ by its *conjugate* $a - b\sqrt{d}$. This simple device—changing the sign before a radical—helps solve various problems in algebra and calculus, from straightforward estimates to sophisticated olympiad problems. Most of our examples serve as introductions to some profound mathematical theories.

Pairs of conjugate numbers appear when we solve a quadratic equation whose discriminant is not a perfect square. For example, the equation $x^2 - x - 1 = 0$ has the pair of conjugate roots

$$x_1 = \frac{1-\sqrt{5}}{2},\ x_2 = \frac{1+\sqrt{5}}{2}.$$

We'll return to this later. We'll begin with examples of another type, where we'll be "tossing" things ...

... From numerator to denominator (and the reverse)

Suppose you are solving a problem from a textbook and find the answer $1/(3 - \sqrt{7})$, but in the book they give the answer $(3 + \sqrt{7})/2$. Don't hurry to look for a mistake in your solution—these numbers are equal, because

$$(3 + \sqrt{7})(3 - \sqrt{7}) = 3^2 - 7 = 2.$$

Here are several examples in which it proves beneficial to shift the "irrationality" from numerator to denominator or vice versa.

1. Calculate the sum

$$\frac{1}{(1+\sqrt{2})}+\frac{1}{(\sqrt{2}+\sqrt{3})}+\ldots+\frac{1}{(\sqrt{99}+\sqrt{100})}.$$

The sum falls out immediately if we rewrite it as

$$(\sqrt{2}-1)+(\sqrt{3}-\sqrt{2})+\ldots+(\sqrt{100}-\sqrt{99})=-1+10=9.$$

(The sum "telescopes": after repeated cancellation of the intermediate terms, only two terms at the extremities survive, -1 and $\sqrt{100}$.)

2. Prove that for all natural m and n

$$\left|\frac{m}{n}-\sqrt{2}\right|\geq\frac{1}{\alpha n^2}, \qquad (1)$$

where $\alpha = \sqrt{3} + \sqrt{2}$.

First note that the inequality

$$\left|\frac{m-n\sqrt{2}}{n}\right|=\frac{|m^2-2n^2|}{(m+n\sqrt{2})n}\geq\frac{1}{(m+n\sqrt{2})n} \qquad (2)$$

always holds because the number $|m^2 - 2n^2|$ is a nonzero integer (the equality $m^2 = 2n^2$ is impossible). We prove inequality (1) by contradiction. Suppose that there exist two natural numbers m and n such that the inequality is not true. Then

$$\frac{-1}{\alpha n^2}<\frac{m}{n}-\sqrt{2}<\frac{1}{\alpha n^2}.$$

If we take the inequality on the right and multiply it by n, we obtain $m < n\sqrt{2} + 1/\alpha n$. Adding $n\sqrt{2}$ to both sides and multiplying by n once again, we get

$$n\left(m+n\sqrt{2}\right)<n\left(2n\sqrt{2}+\frac{1}{\alpha n}\right)=2n^2\sqrt{2}+\frac{1}{\sqrt{3}+\sqrt{2}} \qquad (3)$$
$$=2n^2\sqrt{2}+\sqrt{3}-\sqrt{2}.$$

Now we can show that $2n^2 + \sqrt{3} - \sqrt{2} \leq n^2(\sqrt{3} + \sqrt{2}) = \alpha n^2$. Indeed, this inequality is equivalent to $n^2\sqrt{2} + \sqrt{3} \leq n^2\sqrt{3} + \sqrt{2}$, or $\sqrt{3} - \sqrt{2} = n^2(\sqrt{3} - \sqrt{2})$, which is certainly true, since $\sqrt{3} - \sqrt{2}$ is positive, and n^2 is a natural number.

So we have $n(m + n\sqrt{2}) \leq \alpha n^2$. Taking reciprocals, we have $1/\alpha n^2 < 1/(m + n\sqrt{2})n \leq |(m - n\sqrt{2})|/n$ (by inequality (2)), which means that inequality (1) is true after all for any m and n.

Inequality (1) shows that the number $\sqrt{2}$ is badly approached by fractions with small denominators. An analogue of this inequality (just with some other α) holds for each "quadratic irrationality" and not just for $\sqrt{2}$. Inequality (1) holds for all $\alpha > \sqrt{3} + \sqrt{2}$, but this is still not the best possible constant. Questions concerning the approximation of quadratic irrationalities with rational numbers make up an elaborate and important part of number theory. In the following problems we shall once again encounter approximations to $\sqrt{2}$.

3. Find the limit of the sequence $a_n = (\sqrt{n^2+1} - n)n$. (We assume that you are familiar with the concept of limits, at least on the intuitive level.)

Let's rewrite a_n as

$$(\sqrt{n^2+1}-n)n=\frac{n}{\sqrt{n^2+1}+n}=\frac{1}{1+\sqrt{1+1/n^2}}.$$

Now it is evident that a_n increases and approaches 1/2.

4. Consider the two sequences $a_n = \sqrt{n+1} + \sqrt{n}$ and $b_n = \sqrt{4n+2}$. Prove that

(a) $[a_n] = [b_n]$, where $[x]$ denotes the greatest integer not exceeding x;

(b) $0 < b_n - a_n < 1/(16n\sqrt{n})$.

We can easily check that $a[n]^2 < b[n]^2$. Indeed, this would mean that $2n + 1 + 2\sqrt{n(n+1)} < 4n + 2$. A short computation will show that this inequality is equivalent to the inequality $4n^2 + 4n < 4n^2 + 4n + 1$, which is certainly true. Thus $a[n]^2 < b[n]^2$, and (since both are positive) $a[n] < b[n]$. Next we show that $a[n]^2 > 4n + 1$.

Again, the inequality $2n + 1 + 2\sqrt{n(n+1)} > 4n + 1$ is equivalent to $\sqrt{n(n+1)} > n = \sqrt{n \cdot n}$, which is certainly true. Also, the number $b[n]^2 = 4n + 2$ gives a remainder of 2 when divided by 4, and thus it cannot be the square of an integer (the reader is invited to check this directly). Therefore, the square of the integer $[b_n]$ is not greater than $4n + 1$.

We now have the following inequalities: $[b_n] \le \sqrt{4(n+1)} < a_n < b_n$. This means that a_n is "squeezed between" $[b_n]$ and b_n, an interval of length less than 1. This means that $[a_n] = [b_n]$, which proves (a).

Now we need only find the upper bound for the difference $b_n - a_n$. In the following algebraic argument, notice how we shift conjugate numbers from numerator to denominator twice:

$$b_n - a_n = \sqrt{4n+2} - \left(\sqrt{n} + \sqrt{n+1}\right)$$

$$= \frac{\left[\sqrt{4n+2} - \left(\sqrt{n} + \sqrt{n+1}\right)\right]\left[\sqrt{4n+2} + \left(\sqrt{n} + \sqrt{n+1}\right)\right]}{\sqrt{4n+2} + \left(\sqrt{n} + \sqrt{n+1}\right)}$$

$$= \frac{2n+1-2\sqrt{n(n+1)}}{\sqrt{4n+2} + \left(\sqrt{n} + \sqrt{n+1}\right)} \cdot \frac{2n+1+2\sqrt{n(n+1)}}{2n+1+2\sqrt{n(n+1)}}$$

$$= \frac{1}{\left(\sqrt{4n+2} + \sqrt{n} + \sqrt{n+1}\right)} \cdot \frac{1}{\left(2n+1+2\sqrt{n(n-1)}\right)}$$

(luckily $\left[(2n+1) \cdot 2\sqrt{n(n+1)}\right]\left[(2n+1) + 2\sqrt{n(n+1)}\right] = 1$).

Now,

$$\sqrt{4n+2} + \sqrt{n} + \sqrt{n+1} > \sqrt{4n} + \sqrt{n} + \sqrt{n} = 2\sqrt{n} + \sqrt{n} \mp \sqrt{n}$$

and

$$2n+1+2\sqrt{n(n+1)} > 2n+1+2\sqrt{n \cdot n} = 4n+1 > 4n,$$

So, this last product of two fractions is not greater than

$$\frac{1}{\left[2\sqrt{n} + \sqrt{n}\right](4n)} = \frac{1}{16n\sqrt{n}},$$

which is what we wished to prove.

Note that this estimate is also not very precise. But to prove it (and to investigate functions with multiple radicals), one should use the methods of calculus.

Exchanging plus for minus

If an expression involving $\sqrt{d}$ is equal to $p + q\sqrt{d}$, and we substitute $-\sqrt{d}$ for $\sqrt{d}$ everywhere in the expression, the resulting expression must equal the conjugate number $p - q\sqrt{d}$. We shall often use the following case of this principle: If a and b are rational numbers, and $\sqrt{d}$ is not, then

$$(a + b\sqrt{d})^n = p + q\sqrt{d} \to (a - b\sqrt{d})^n = p - q\sqrt{d}. \quad (4)$$

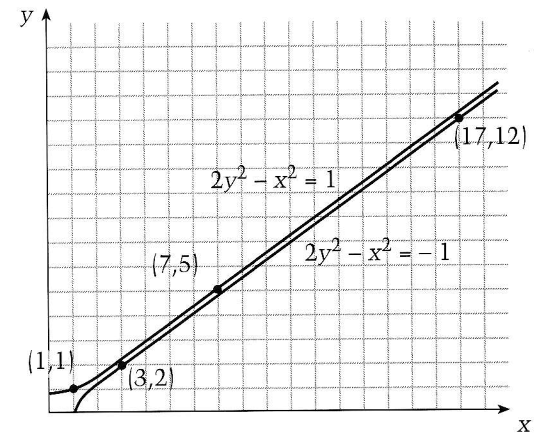


Figure 1

5. Prove that the equation

$$(x + y\sqrt{5})^4 + (z + t\sqrt{5})^4 = 2 + \sqrt{5},$$

in which x, y, z, and t are rational numbers, has no solutions.

We can, of course, try to individually find the sum of terms in the left that do not contain $\sqrt{5}$ and set it equal to 2, then the coefficient of $\sqrt{5}$ and set it equal to 1. But it's not clear what we should do with the cumbersome system of equations obtained in this way. Instead, we shall use principle (4) and switch the sign before $\sqrt{5}$:

$$(x - y\sqrt{5})^4 + (z - t\sqrt{5})^4 = 2 - \sqrt{5}.$$

The number on the left is positive, while the number on the right is negative! This contradiction proves that the rational numbers x, y, z, and t cannot exist.

6. Prove that there exist infinitely many pairs (x, y) of natural numbers for which

$$|x^2 - 2y^2| = 1. \quad (5)$$

It is easy to find several pairs of this sort with small x and y. They are (1, 1), (3, 2), (7, 5), (17, 12), ... (fig. 1). But how can we continue this process? Can we find a general form to write down these solutions?

The number $1 + \sqrt{2}$ will help us answer these questions. The table below reflects the law that allows us to

n	$(1+\sqrt{2})^n$	x_n	y_n	$x_n^2 - 2y_n^2$	$(1-\sqrt{2})^n$
1	$1+\sqrt{2}$	1	1	1 − 2 = −1	$1-\sqrt{2}$
2	$3+2\sqrt{2}$	3	2	9 − 8 = 1	$3-2\sqrt{2}$
3	$7+5\sqrt{2}$	7	5	49 − 50 = −1	$7-5\sqrt{2}$
4	$17+12\sqrt{2}$	17	12	289 − 288 = 1	$17-12\sqrt{2}$
5	$41+29\sqrt{2}$	41	29	1681 − 1682 = −1	$41-29\sqrt{2}$
...	...	...	...	...	...

keep finding new solutions (x, y). What should go in the sixth line of the table?

We see that coefficients x_n and y_n of the number

$$x_n + y_n\sqrt{2} = (1+\sqrt{2})^n$$

give the necessary pair. In order to prove this, we look in the column of conjugate numbers in this table (we use principle (4) once again):

$$x_n - y_n\sqrt{2} = (1-\sqrt{2})^n.$$

Multiplying the last two equations, we get

$$x_n^2 - 2y_n^2 = (1+\sqrt{2})^n(1-\sqrt{2})^n = [(1+\sqrt{2})(1-\sqrt{2})]^n = (-1)^n,$$

and thus the expression we are interested in alternately equals 1 and –1. Adding up the same two equations and then subtracting the second from the first, we finally obtain an explicit expression for x_n and y_n:

$$x_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2},\quad y_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Is it possible to solve this problems without resorting to the irrational numbers $1+\sqrt{2}$ and $1-\sqrt{2}$? Now, when we know the answer, it is not difficult to express the pair (x_{n+1}, y_{n+1})(by means of the previous pair (x_n, y_n): we have $x_{n+1} + y_{n+1}\sqrt{2} = (x_n + y_n\sqrt{2})(1+\sqrt{2})$, and thus

$$x_{n+1} = x_n + 2y_n,\quad y_{n+1} = x_n + y_n. \qquad (6)$$

One could probably guess this recurrence relation by considering few first few solutions and then checking that

$$|x_n^2 - 2y_n^2| = |x_{n+1}^2 - 2y_{n+1}^2|.$$

If we set the initial conditions $x_1 = 1$ and $y_1 = 1$, we can conclude by induction that $|x_n^2 - 2y_n^2| = 1$ for all n. Further, expressing conversely (x_n, y_n) via (x_{n+1}, y_{n+1}), and using the "method of descent," we can prove that the set of solutions for equation (5) is exhausted by the series we have found. In the same way we can solve any of "Pelle's equations," that is, a diophantine equation of the form $x^2 - dy^2 = c$ (and every square equation in integer numbers can be reduced to this form), although this equation can have several different series of solutions.

Recurrence relations similar to (6) appear not only in number theory but also in various problems of analysis and probability theory. Here is a characteristic example of this type of combinatoric problem, given to the participants of the 1979 International Mathematical Olympiad in London:

7. A frog is sitting at vertex A of a regular octagon. The frog can jump from any vertex of the octagon except for E, the vertex opposite A, to any one of two neighboring vertices. When the frog gets to E, it stops there forever. Find the number e_m of different ways the frog can get from A to E by jumping exactly m times.

If we paint the vertices of the octagon alternately black and white (fig. 2), it becomes clear that $e_{2k-1} = 0$. That is, the frog cannot jump from A to E in an odd number of moves, because after each jump the color of the vertex under the frog will change, but the colors of A and E are the same.

Suppose we denote by c_n the numbers of ways the frog can get from A to either vertex labeled C in exactly $2n$ jumps (symmetry shows that it does not matter which vertex C we choose). Similarly we denote by a_n the number of ways the frog can get from A back to point A in $2n$ jumps. It is easy to check that $a_1 = 2$ and $c_1 = 1$ (see figures 2a–2d).

We can also derive a recurrence relation for a_{n+1} in terms of a_n and c_n. Suppose the frog has moved from A back to A in $2(n+1)$ moves. Where was the frog two moves ago? Either at point A or at one of the vertices marked C. If it was at A, then there are two ways to get back to A (via the left-hand vertex marked B, or via the right-hand vertex marked B). And the frog could have made it to A in the first $2n$ moves in a_n different ways. Hence

$$a_{2n+1} = 2a_n + 2c_n. \qquad (7a)$$

A similar argument will show that

$$c_{2n+1} = a_n + 2c_n. \qquad (7b)$$

And the number the problem asks for, e_m, is just equal to $2a_{n-1}$.

But how can we find explicit formulas for a_n and c_n? Let's rewrite relations (7a) and (7b) in the following form:

$$a_{n+1} + c_{n+1}\sqrt{2} = (a_n + c_n\sqrt{2})(2+\sqrt{2}). \qquad (8)$$

Then, according to our principle of exchanging plus for

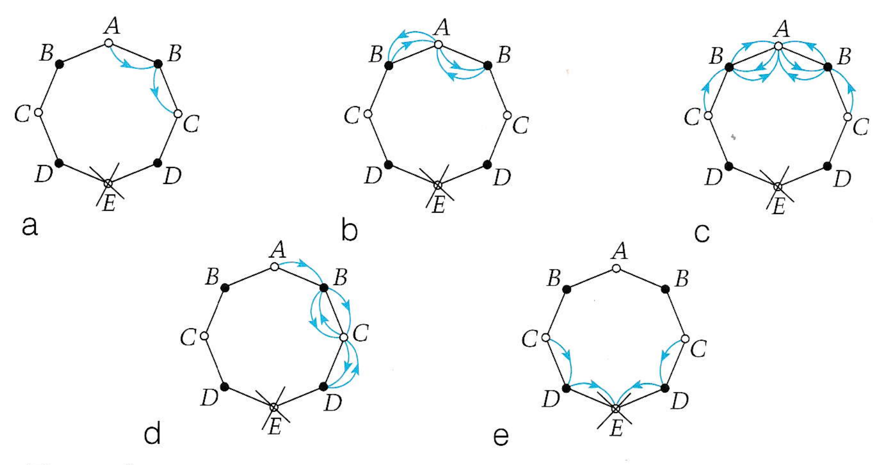


Figure 2

minus, we have

$$a_{n+1} - c_{n+1}\sqrt{2} = (a_n - c_n\sqrt{2})(2 - \sqrt{2}). \qquad (9)$$

It is not difficult to see, from the above formulas, that

$$a_2 + c_2\sqrt{2} = (2 + \sqrt{2})^2,$$
$$a_3 + c_3\sqrt{2} = (2 + \sqrt{2})^3,$$

and, in general, that

$$a_n + c_n\sqrt{2} = (2 + \sqrt{2})^n.$$

(A more formal proof would involve a simple application of the principle of mathematical induction.) Similarly, we have

$$a_n - c_n\sqrt{2} = (2 - \sqrt{2})^n.$$

Therefore,

$$c_n = \frac{\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n}{2\sqrt{2}},$$

and, since $e_{2n} = 2c_{n-1}$, we have, at last,

$$e_{2n} = \frac{\left(2+\sqrt{2}\right)^{n-1} - \left(2-\sqrt{2}\right)^{n-1}}{\sqrt{2}}; \; e_{2n-1} = 0.$$

The problem is solved. And yet it isn't clear how we were supposed to come up with the idea of using formulas containing $\pm\sqrt{2}$ in this problem (and in the previous problem), when the problem statements mention only integers!

It turns out that the appearance of conjugate numbers in the solution of recurrence relations like (7a) and (7b) is a result of the application of a standard method of solution from linear algebra. This method allows us first to find all geometric progressions ($a_n = a_0\lambda^n$, $c_n = c_0\lambda^n$) that satisfy the recurrence. The values of λ for which such progressions exist are called *characteristic values* or *eigenvalues* and are determined by a certain *characteristic equation*. For the system (7a, 7b) this equation is a quadratic with integer coefficients whose roots are just $2 + \sqrt{2}$ and $2 - \sqrt{2}$. It turns out that knowing these roots allows us to express any solution of the recurrence relation (as what is called a "linear combination" of the corresponding geometric progressions). Then the "initial conditions" (here $a_1 = 2$, $c_1 = 1$) determine the solution uniquely.

It turns out also that many simple recursive sequences of integers have characteristic equations that are quadratic, with integer coefficients. Thus their characteristic values are conjugate quadratic irrationalities. The reader is invited to research the solution that this method gives for the famous Fibonacci sequence.

Note that the greater eigenvalue determines the rate of growth of the sequence: When n is large in problem 7, we have $e_n \cong (2 + \sqrt{2})^n/\sqrt{2}$. The other way to say it is $\lim_{n\to\infty} e_{n+1}/e_n = 2 + \sqrt{2}$. An analogue of this observation for problem 6 ($\lim_{n\to\infty} x_n/y_n = \sqrt{2}$) shows that both terms of the sum $x_n + y_n\sqrt{2}$ almost equal each other when n is large.

An algebraic epilogue

We have investigated several examples connected with bordering branches of algebra, calculus, and number theory. (In fact, each of the subjects we've considered could be the subject of a separate article in *Quantum*!) In conclusion, let's look at conjugate numbers from the point of view of pure algebra.

Suppose we have a set P of numbers, or symbols, or algebraic expressions that can be combined using the four ordinary arithmetic operations. Such a set is called a *field*. For example, the rational numbers form a field, as do the real numbers. If d is an element of a field P and the equation $x^2 - d = 0$ has no solution in P, then one can *extend* P by "inventing" a new object $\sqrt{d}$, which yields d when multiplied by itself. We can then consider all expressions of the form $p + q\sqrt{d}$, where p and q are elements of P. Thus we write $(p + q\sqrt{d})(p' + q'\sqrt{d}) = (pp' + qq'd) + (pq' + qp')\sqrt{d}$. It is easy to check that the new set P_1, consisting of all elements of the form $p + q\sqrt{d}$ again forms a field. This new field contains a "copy" within it of the old field P—namely, the elements $p + 0\sqrt{d}$. For example, if we take P to be the field of real numbers and consider the equation $x^2 + 1 = 0$ (so that $d = -1$), then P_1 is the field of complex numbers.

The new field P_1 (called the "quadratic extension of P") is equipped with an interesting mapping:

$$\lambda = p + q\sqrt{d} \to \bar{\lambda} = p - q\sqrt{d}.$$

It is called a *conjugation*, and its main properties are

1. All elements of the old field P map into themselves;
2. All equations containing arithmetic operations stay under this mapping:

$$\overline{\lambda + \mu} = \bar{\lambda} + \bar{\mu}; \; \overline{\lambda\mu} = \bar{\lambda}\bar{\mu}. \qquad (10)$$

This mapping is a particular case of *Galois automorphisms* of the extension P_1 of field P, after the French mathematician Evariste Galois.

[1]More precisely, a *field P* is a set equipped with two operations (we call them addition and multiplication) satisfying the following axioms:

1. P is closed with respect to both operations.
2. Both operations are commutative.
3. Both operations are associative.
4. Each operation has an identity element (0 for addition, 1 for multiplication).
5. Each element p of P has an inverse with respect to each operation ($-p$ for addition, $1/p$ for multiplication), except that the additive identity has no multiplicative inverse.
6. Multiplication distributes over addition.

We can also consider "double extensions" of a field. For example, we can extend the field of rational numbers with the symbols $\sqrt{2}$ and $\sqrt{3}$. We then obtain a field with a greater number of Galois automorphisms. In addtion to the indentity mapping (which is included in this set), there are three others:

$$(\sqrt{2} \to -\sqrt{2},\ \sqrt{3} \to \sqrt{3}),$$
$$(\sqrt{2} \to \sqrt{2},\ \sqrt{3} \to -\sqrt{3}),$$
$$(\sqrt{2} \to -\sqrt{2},\ \sqrt{3} \to -\sqrt{3}).$$

The composition of any two of these four mappings is another of these four. In fact, combining them by composition forms a *group*, which is essentially the same group as the one formed by the symmetries of a rectangle.

It turns out that the roots of *any* polynomial can be added to the basic field P. Automorphisms of the new field that appear in this way constitute the subject of one of the most interesting branches of algebra of the 19th and 20th centuries: Galois theory. This theory allows us, in particular, to address the question of the solvability of equations by radicals.

To reinforce the themes of this article, we offer the following problems for your enjoyment.

Exercises.

1. Which is greater: $\sqrt{1996} + \sqrt{1997}$ or $\sqrt{1995} + \sqrt{1998}$?

2. Prove that for all positive x

$$\left|\sqrt{x^2+1} - x - \frac{1}{2}x\right| < \frac{1}{8}x^3.$$

3. Find the first hundred decimal digits in the decimal notation of the number $(\sqrt{50} + 7)^{100}$.

4. Eliminate the irrationality in the denominator of the following fraction: $1/(1 + \sqrt{2} + \sqrt{3})$. ◘

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M216

Switch places. Solve the equation

$$x2^{1/x} + \frac{1}{x}2^x = 4.$$

M217

Bike sharing. Town A is 30 miles from town B. Three friends want to travel from A to B. They have two bicycles with them: one is a racing bike, which any of them can ride at a speed of 30 mph; the other is a mountain bike, which they ride at a speed of 20 mph. Each of them walks at the rate of 6 mph. Any of the friends can leave a bicycle at the side of the road, where it will lie undisturbed until another friend comes and uses it. The three friends want to minimize the time they spend on their journey (we'll say that the trip is over when the last friend arrives at B). Find the shortest possible time for this joint trip.

M218

Meeting of circles. Let M be the point where the diagonals of a parallelogram $ABCD$ meet. Consider three circles passing through M: the first and the second circles are tangent to AB at points A and B, respectively, and the third circle passes through C and D. Denote the points, other than M, where the third circle intersects the first and the second ones by P and Q, respectively. Prove that PQ is tangent to the first and second circles.

M219

Developing a polyhedron. Let $ABCD$ be a regular triangular pyramid (which means that ABC is an equilateral triangle and the edges AD, BD, and CD are equal), and let the plane angles at vertex D be equal to α. A plane parallel to ABC meets AD, BD, and CD at points A_1, B_1, and C_1, respectively. Now we cut the surface of polyhedron $ABCA_1B_1C_1$ along five of its edges A_1B_1, B_1C_1, C_1C, CA, and AB, and spread it on the plane. Find all values of α where the development must overlap itself.

M220

Graph rotation (for calculus fans). The graph of $y = x^3 + ax^2 + 19x + 97$ is rotated 45° about some point. The resulting graph is the graph of some function $y = f(x)$ (that is, each value of x corresponds to a unique value of y). Find all a for which this can be true.

Physics

P216

Bead on a ring. A small bead moves along a stationary thin wire ring. The coefficient of friction between the bead and ring is $\mu = 0.1$, and the force of gravity is absent. How much will the velocity of the bead have been slowed after the bead has completed three revolutions around the ring? If you can't provide a precise solution, try to come up with an approximation. (M. Yermilov)

P217

Gas in a vessel. The temperature of the walls of a vessel filled with a gas is T, and the temperature of the gas itself is T_1. Will the pressure exerted by the gas on the walls of the vessel be greater when the walls are cooler ($T < T_1$) or warmer ($T > T_1$) than the gas? (G. Myakishev)

P218

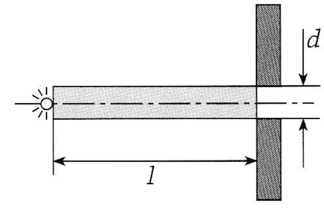
Charged ball and a probe. A small uncharged conducting object (a probe) is brought near an isolated charged conducting ball. Before inserting the probe into the ball's electric field, the potential of the point where the probe is to be inserted is $\phi = 10{,}000$ V. After inserting the probe, the ball's potential changes by a value of $\Delta\phi = 1$ V. Find the force acting on the ball at this moment. The probe is much smaller than the diameter of the ball. (A. Zilberman)

P219

Ring in a magnetic field. A thin wire ring with a diameter d and a resistance R is placed in a magnetic field B that is parallel to the plane of the ring. A voltage source with a potential difference of V_0 is to be connected to two points on the ring. What two points on the ring would you choose to generate the maximum force acting on the ring from the magnetic field? (A. Zilberman)

P220

Light conductor. A point source of light is located at a distance $l = 1$ m from a screen that has an opening with a diameter $d = 1$ cm opposite the source. How will the amount of light passing through the opening change if a transparent glass cylinder with an index of refraction $n = 1.5$ is placed between the source and the screen as shown in the figure? The cylinder's length $l = 1$ m, its diameter $d = 1$ cm, and the light source is located on the cylinder's axis. (A. Butov)



ANSWERS, HINTS & SOLUTIONS ON PAGE 48

Jingle bell?

If a bell rings in a vacuum, does it make a sound?

by N. Paravyan

HERE'S AN INTERESTING experiment you can do in your school lab. It dates back to the 17th century but has lost none of its ap*peal*. To set up for the experiment, start by mounting a 250 to 300 mL rounded flask containing 20 to 25 mL of water onto a ring stand above an electric hot plate. Then insert a 15 to 20 cm length of glass tubing into a one-holed stopper. Attach a short piece of rubber hose to the end of the glass tubing protruding from the top of the stopper. To the lower end of the glass tubing, attach a small bell that can fit through the mouth of the flask. Now you're ready to conduct the experiment.

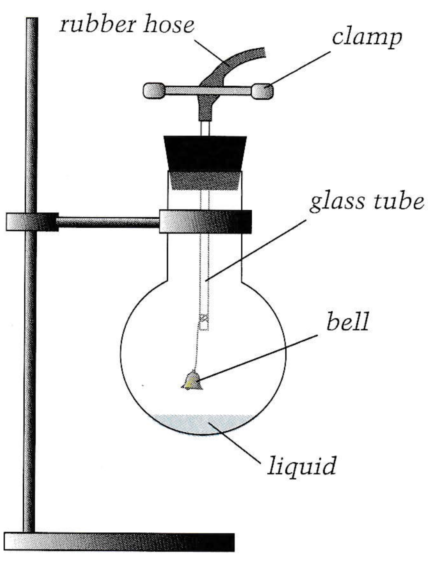


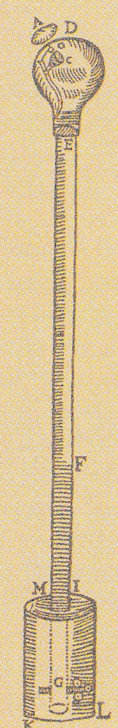
Figure 1

Heat the water in the flask until it boils. Note: You may use an alcohol burner, but be careful—the glass may crack if the wick touches the flask. After the water has been boiling for three minutes, seal the flask with a hose clamp and quickly remove the hot plate. When the flask cools to room temperature, remove it from the holder and shake it carefully. You will hear the bell jingling, but the sound will be rather weak.

The bell is muted because the cooling and condensation of water vapor rarefied the gas in the flask, producing a slight vacuum. Now release the clamp on the rubber hose, wait a few seconds, and then replace it. When you shake the flask again, the bell will sound markedly louder. Why? By removing the clamp, you allowed air to pass into the flask and increase the density of the sound-conducting medium.

To continue the experiment, pour the water out of the flask and replace it with 20 to 25 mL of anhydrous glycerin or ethylene glycol. Repeat the procedure of heating the liquid and then allowing it to cool. At room temperature, the vapor density of these substances is several times less than the vapor density of water. As a result, the vacuum will be stronger than in the previous experiment and the observed effects should be more pronounced. In fact, from over a meter away, you won't be able to hear the bell at all.

An early illustration of the bell-in-a-vacuum experiment, which appeared in A. Kircher's *Musurgia Universalis sive Ars Magna Consoni et*



Dissoni (Rome, 1650). In the setup pictured, a lodestone (A) was used to move the clapper of the bell (C).

ABOUT 200 YEARS SEPARATE the first and last quotations in the box. As you can see, "potential" has been a hard concept to pin down. First it assumed the role of tension, then the guise of an electromotive force, and finally it appeared as a mysterious function. In the future, you may also encounter potential under a different alias, such as contact potential difference, ionization potential, or gravitational potential. You will also learn of the efforts of the glorious crew of scientists—Euler, Laplace, Poisson, Green, Gauss—who have worked to unravel the tangle of terminology surrounding this concept. It should come as no surprise that the concept of potential has drawn admirers from the fields of both math and physics. And why not? The universal character of this notion is connected with a great number of fruitful applications, such as heat conduction, fluid dynamics, and the calculation of gravitational, electric, and magnetic fields.

When solving the following elementary problems, don't forget that the modern theory of potential is a keystone in the foundation for an entire field of study—that of mathematical physics.

Questions and problems

1. The electric potential decreases with the distance from a particular charge. What is the charge's sign?

2. Is there always a potential difference between two conductors, one charged positively and one negatively?

3. A point charge q is located a distance r from the center of an isolated uncharged conducting ball. What is the ball's potential?

4. Does the potential at the center of a charged sphere depend on the charge distribution on its surface?

5. Without touching the surface, a small charged metal ball is placed inside a charged conducting sphere through a small hole in its exterior. The charges of the ball and sphere are opposite and equal in value. How does the sphere's potential change?

6. How does the potential of a spherical capacitor vary with distance r from the center of concentric spheres if the inner sphere of radius R_1 carries a charge $+q$, and the outer sphere of radius R_2 carries a charge $-q$? Graph this dependence.

7. Two conductors are positively charged such that the potential of the first conductor is 100 V and that of the second is 50 V. Would positive charges move from the first conductor to the second if they came into contact? There are no other objects near these conductors.

8. A ball connected to an electroscope is moved along the surface of the charged object shown in figure 1. Would the reading of the electrometer change during this process? Why is a long wire used in this experiment?

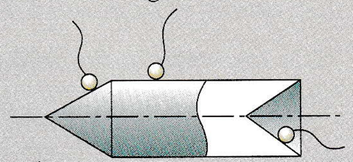


Figure 1

9. A conducting uncharged ball is placed in the homogeneous electric field of a parallel plate capacitor such that its center is midway between the plates. The potentials of the plates are +100 V and –100 V. What form would the surface of zero potential take?

10. An elastic metal ball with a charge q is fixed on an insulated elastic support. A second (identical) ball with the same charge is dropped on it from a height of h. How high will the second ball rebound after the collision?

11. A small object with a charge $-q$ slides along a smooth inclined plane forming a 45° angle with the horizon. Will its velocity near the base of the plane be affected by a fixed charge of $+q$ as shown in figure 2?

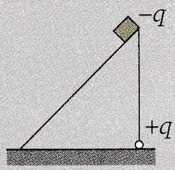


Figure 2

12. The potential difference between points A and B in a circuit containing capacitors is V. If a capacitor with capacitance C is connected to these points, will its charge be equal to CV?

13. An uncharged metal plate is inserted into a charged parallel plate capacitor parallel to its plates, which are not connected to a battery. The thickness of this plate is equal to half the distance between the capacitor's plates. How does the potential difference across the capacitor change?

14. If you had to approach a downed power line, why would you want to take extremely small steps?

15. The potential difference between any two points of a uniform wire ring is zero, but there is a non-

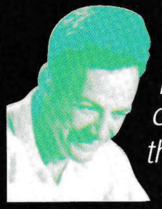
Do you have p

It may depend on how well

"Voltage is an effort performe electrified body trying to get ri give it to other bodies. . .

"The electromotive action manifests itself in two kinds o first electric tension . . ."—An

"Taking into account how desirable it is of such a universal character as electric one particular function instead of scatte forc



"There is a number at each point from one position to another, this placed at one point in space, it w direction in which this number var this number by its usual name—po —Richard Feynman

e potential?

well you know potential!

rformed by each point of an get rid of its electricity and dies..."—Alessandro Volta

on inds of phenomena. I will call the "—André-Marie Ampère

ble it is to be able to calculate the force ectricity, we can focus our attention on scattering it by studying each of these forces individually."—George Green

point in space, and when you go er, this number varies. If an object is e, it will be affected by a force in the er varies most quickly (I'll refer to e—potential)."

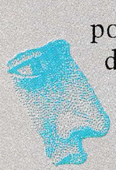
zero current in the ring. How is this possible?

16. Is it possible, while on board an airplane flying in the Earth's magnetic field, to measure the voltage generated between the tips of its wings?

17. A tungsten ball in a vacuum is irradiated with ultraviolet light. How will the ball's potential change over time?

Microexperiment

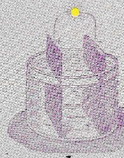
At sea level, the intensity of the potential difference across the distance between your nose and toes is about 200 V.



Can you use this voltage to light an electric lamp? Does this voltage present a danger to your person?

It is interesting that . . .

. . Volta—who discovered the contact potential difference, coined the term *potential*, and was honored by having the unit of potential difference named after him—didn't have the slightest idea how or why his invention, the "voltaic" pile, worked. The French scientist Dominique Arago has described this invention as the most beautiful device ever invented, surpassing even the telescope and the steam engine.



. . . the passage of current through an electrolyte solution results in the generation of an emf that is directed counter to the applied emf. This phenomenon (known as "galvanic polarization") was discovered at the beginning of the 19th century. Later this phenomenon led to the invention of the lead-acid battery.

. . . the problem of charge distribution in a conductor of a given shape was first addressed in the 18th century by the French physicist Charles Augustine de Coulomb. Later, Poisson, attempting to solve just this type of problem, hit upon the notion of using a function that depended on coordinates and a constant value at the conductor's surface.

. . . George Green, the author of *An Attempt to Apply Mathematical Analysis to the Theories of Electricity and Magnetism,* was self-taught. Before entering the University of Cambridge at the age of 40, Green worked as a mechanic and baker while studying science in his free time. It's worth noting that, while he introduced the concept of a potential function, Green didn't relate it to the notion of work, which was a concept that was not yet in use in physics.

. . . electric current can flow not only in a circuit where the voltage across two arbitrary points is zero but also from a point of lower electric potential to a point of higher potential, which is the case inside an electric battery.

. . . there are electric fields in which one can measure voltage but not potential. Fields generated by electromagnetic induction, for example, are "nonpotential" fields, as are those created by transformers and electric motors.

. . . a large electric eel can produce a potential difference up to 600 V and a current about 1 amp. This shocking display of power is the product of a large number of small circuits that contain cells connected in series, each generating 0.15 V. The circuits themselves are connected in parallel, and this natural electric "device" produces a strong electric current that can paralyze or even kill a victim.

. . . by shuffling your feet on a carpet and touching another object, you can generate an electric discharge up to 1 cm in length, which means that your electric potential is in the range of 10,000 to 20,000 V.

. . . the potential difference in a lightning strike between a cloud and the ground reaches 4 billion V, with an average current of 20,000 amps.

. . . the range of voltages used by humans covers twelve orders of magnitude. The maximum attainable potential is limited by the electric strength of the insulators, which is characterized by millions of volts. The minimal voltage used in certain technical applications is a mere fraction of a microvolt.

—Compiled by A. Leonovich

***Quantum* articles about potential**

Arthur Eisenkraft and Larry D. Kirkpatrick. "Electricity in the Air." November/December, 1992, pp. 46–48.

Albert Stasenko. "From the Edge of the Universe to Tartarus." March/April, 1996, pp 4–8.

A. Leonovich. "Of Combs and Coulombs." January/February, 1997, pp. 28–29.

ANSWERS, HINTS & SOLUTIONS ON PAGE 51

PHYSICS CONTEST

Elephant ears

"Sir Isaac Newton was very much smaller than a hippopotamus, but we do not on that account value him less." —Bertrand Russell (1872–1970)

by Arthur Eisenkraft and Larry D. Kirkpatrick

WHY DO ELEPHANTS HAVE such big ears? And why do they have such thick legs? In other words, why do elephants have different shapes than horses? These questions and more can be answered using the laws of scaling that we learn in physics.

Elephant bones are made from the same basic material as human bones. Therefore, the bones must be thicker to support the extra mass of the elephant. But how much thicker? Let's compare an elephant to a horse. A typical horse has a mass around 600 kg and a typical elephant has a mass around 4200 kg, or some 7 times larger. Because all mammals have a density near that of water, the elephant must have 7 times the volume of the horse. If we assume that the two have the same shape (they both have four legs!), the linear dimensions of the elephant must be the $\sqrt[3]{7}$ = 1.9 times the corresponding dimensions of the horse.

Each elephant leg must support 7 times as much weight as a horse leg. Because the compression strength of a beam depends on its cross-sectional area, an elephant leg bone must have 7 times the cross-sectional area of a horse leg bone. In other words, the elephant leg bone must have 2.6 times the diameter of a horse leg bone. Notice that the elephant and the horse cannot have the same shape; the legs must be proportionately larger than the other dimensions. The comparison would be even more dramatic if we compared the elephant to a mouse!

This explains why elephants have such thick legs, but what about the ears? Let's assume for the moment that an elephant eats 7 times as much as a horse because it has 7 times as much mass. As this food is used by the body, it generates heat. Therefore, an elephant must dissipate 7 times as much heat as a horse. We know that the thermal loss is proportional to the difference in the temperature across the skin and to the area of the skin. The surface area of any solid depends on the square of its linear dimensions, so the elephant only has $1.9^2 = 3.6$ times the surface area. This means that the elephant must have a much higher body temperature or some other way of getting rid of the thermal energy. This is one of the roles of the big ears. They increase the surface area and, by moving in the air, keep the air temperature near the skin from climbing very much. The elephant also eats less per unit mass than a horse.

It is interesting to note that although elephants communicate by ultrasound, it is not necessary for them to have big ears for this purpose.

Not all scaling deals with lengths. We can use any factor as a scaling parameter. For instance, the Bohr radius for the hydrogen atom is given by

$$a_0 = \frac{\hbar^2}{mke^2} = 0.0529 \text{ nm},$$

where $\hbar$ is Planck's constant divided by 2π, m is the mass of the electron, k is Coulomb's constant, and e is the electronic charge. What would be the new radius if the electron were replaced by a muon with a mass 207 times as large? (We assume that the mass of the proton is large compared to the mass of the muon.) We do not need to solve for the new radius from scratch; all we need to know is that the radius scales inversely with mass. Therefore, the radius of the muonic hydrogen atom is

$$a_\mu = a_0\left(\frac{m_e}{m_\mu}\right) = \frac{a_0}{207} = 0.256 \text{ pm}.$$

This was one of a series of five problems on scaling that made up one of the three theory questions at the International Physics Olympiad held in Sudbury, Canada, last July. (See Happenings for a report on the Olympiad.) The theory problems were developed under the direction of Chris Waltham, who is a faculty member at the University of British Columbia. Three of the other scal-

Art by Tomas Bunk


CIRCUS
SCIENTIFICUS
TOM
BUNK

ing problems make up this month's Contest Problem.

A. The mean temperature on the Earth is $T = 287$ K. What would the new mean temperature T' be if the mean distance between the Earth and the Sun were reduced by 1%?

B. On a given day, the air is dry and has a density $\rho = 1.2500$ kg/m^3. The next day the humidity has increased and the air contains 2% water vapor by mass. The pressure and temperature are the same as the day before. What is the new air density ρ'? Assume ideal gas behavior. The mean molecular weight of dry air is 28.8 g/mol, and the molecular weight of water is 18 g/mol.

C. A type of helicopter can hover if the mechanical power output of its engine is P. If another helicopter is produced that is an exact half-scale replica (in all linear dimensions) of the first, what mechanical power P' is required for it to hover?

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington, VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space.

Color creation

In the May/June issue, we asked readers to solve two problems concerned with thin film interference. In the first problem the white light fell on the thin film at an angle of 30°. Readers sending in solutions to this problem chose to make some simplifying assumptions to ease the derivation. Specifically, they assumed that the total distance through the soap film was equal to twice its thickness, ignoring the angle at which the light

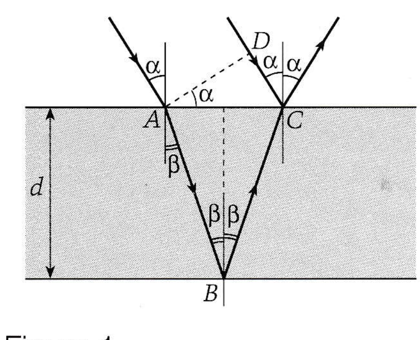


Figure 1

traveled. They also assumed that the ray that reflects from the top surface and the ray that enters the film and reflects from the bottom surface interfere with one another even though they are displaced from one another. We will complete a more thorough analysis of this standard problem.

As shown in figure 1, the beam of light incident at point A refracts into the film, reflects at point B, and then reemerges at point C. A second ray partially reflects at C, and these two rays interfere. The path difference between these two rays determines whether there is constructive or destructive interference. The beam of light arrives along the line AD in phase. We see that the refracted and reflected rays travel additional distances $AB + BC$ and DC, respectively. From the geometry of the problem, we see that

$$AB + BC = \frac{2d}{\cos\beta}.$$

The wavelength in the soap film is smaller than the wavelength in air by a factor of n, the index of refraction. The optical path length is therefore

$$\frac{2d}{\cos\beta} \div \frac{\lambda_0}{n} = \frac{2dn}{\lambda_0 \cos\beta},$$

where β is the incident angle of the ray at the lower surface. The path length DC is given by

$$DC = AC \sin\alpha = 2d \tan\beta \sin\alpha,$$

where α is the angle of incidence at the top surface. The number of wavelengths in this distance is

$$\frac{2d \tan\beta \sin\alpha}{\lambda_0}.$$

Because this wave reflects from a medium with a higher index of refraction, it undergoes a phase shift of 180°. This makes the total number of wavelengths equal to

$$\frac{2d \sin\beta \sin\alpha}{\lambda_0 \cos\beta} + \frac{1}{2}.$$

Constructive interference will occur when the path difference is equal to an integral number of wavelengths k:

$$k = \frac{2dn}{\lambda_0 \cos\beta} - \frac{2d \sin\beta \sin\alpha}{\lambda_0 \cos\beta} - \frac{1}{2}.$$

Because Snell's law applies to these rays of light, we know that $\sin\alpha = n \sin\beta$. Therefore,

$$\begin{aligned} k &= \frac{2dn}{\lambda_0 \cos\beta}\left[1 - \sin^2\beta\right] - \frac{1}{2} \\ &= \frac{2dn}{\lambda_0 \cos\beta}\left[\cos^2\beta\right] - \frac{1}{2} \\ &= \frac{2dn\cos\beta}{\lambda_0} - \frac{1}{2}. \end{aligned}$$

Using Snell's law and the relationship $\sin^2\alpha + \cos^2\alpha = 1$, we can write the equation in terms of α:

$$2k + 1 = \frac{4d}{\lambda_0}\sqrt{n^2 - \sin^2\alpha}.$$

If $k = 0$,

$$d = \frac{\lambda}{4}\frac{1}{\sqrt{n^2 - \sin^2\alpha}} = 0.1\ \mu\text{m}.$$

For a light ray incident on the film along the vertical direction,

$$\lambda_0 = 4d\sqrt{n^2 - \sin^2 0} = 4dn.$$

Because $d = 0.1\ \mu$m, $\lambda_0 = 0.53\ \mu$m. This is a greenish-yellow light.

Part B of the problem asked for the minimum thickness of a film of acetone ($n = 1.25$) placed on glass ($n = 1.50$) such that light coming from the vertical would have destructive interference at 600 nm and constructive interference at 700 nm.

The path difference for the ray reflected off the top surface of the acetone and the ray reflected off the bottom surface of the acetone is simply $2dn$. Because both reflections occur for materials with higher indices of refraction, a phase shift occurs at both surfaces and therefore the 1/2 does not appear in our equation.

For constructive interference, the optical path difference $2dn/\lambda_{0c}$ should equal an integral number of wavelengths:

$$k = \frac{2dn}{\lambda_{0c}}.$$

For destructive interference, the optical path difference $2dn/\lambda_{0d}$ should equal a half integral number of wavelengths:

$$k + \frac{1}{2} = \frac{2dn}{\lambda_{0d}}.$$

Eliminating k and solving for the thicknesses, we find

$$d = \frac{1/2}{2n\left(\frac{1}{\lambda_{0d}} - \frac{1}{\lambda_{0c}}\right)} = 840 \text{ nm}.$$

This is the minimum thickness. There are other possible thicknesses. What are some of them? ◘

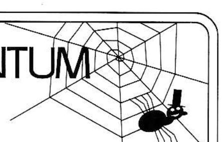
Calling all modem maniacs!

What did you like in this issue of *Quantum*? If you find pen-and-paper communication too old-fashioned, you can send your comments, questions, and suggestions to the managing editor by electronic mail at the following address:

quantum@nsta.org

We look forward to hearing from you.

Visit QUANTUM on the Web!



We've opened up a site on the World Wide Web, so stop by if you can. As with most Web sites, we're still growing. But already you'll find an **index** of *Quantum* articles, a **directory** of personnel and services, **background** information on *Quantum* and its sister magazine *Kvant*, and more. Recent visitors will have found a **preview** of this issue, plus a Brainteaser **contest**.

Point your Web browser to
http://www.nsta.org/quantum

Circular reasoning

Eleven steps to a deeper understanding of inscribed angles

by Mark Saul and Benji Fisher

WELCOME TO *QUANTUM'S* newest column, Gradus ad Parnassum.[1] It will feature problems and theorems that are closely connected with the regular high school mathematics curriculum, but which most regular classes do not get to investigate.

The problems will be grouped by theme. For example, this month's theme is the measure of angles in a circle. If you haven't yet studied these theorems, they are accessible in almost any textbook on geometry.

This month, we have a guest coauthor contributing to the problems. He is Benji Fisher, who got his Ph.D. from Princeton University and has since taught at Columbia University, the Bronx High School of Science, and the Commonwealth School in Boston. The authors would also like to acknowledge the help of the honors geometry class from Bronxville High School.

In what follows, the symbol $\overset{\frown}{AB}$ refers to minor arc AB, unless the major arc is indicated. We will use the following results:

A. An inscribed angle is measured by half its intercepted arc.

B. Inscribed angles intercepting the same arc are equal. In the diagram, if points A, B, X, Y are on the circle, then $\angle XAY = \angle XBY = (^1/_2)\overset{\frown}{XY}$. We say that arc AB, or line segment AB, subtends $\angle AXY$ at point X and $\angle AYB$ at point Y.

C. Conversely, if an angle intercepts an arc equal to twice its measure, then that angle is an inscibed angle (that is, its vertex is on the circle). In figure 1, if points A, X, and Y are on the circle and $\angle XAY = \angle XBY$, then point B is also on the circle.

D. An angle formed by two chords is equal to half the sum of the arcs it intercepts. In figure 2, $\angle APD = {^1/_2}(\overset{\frown}{AD} + \overset{\frown}{BC})$.

E. An angle formed by two tangents, a tangent and a secant, or two secants is equal to half the difference of the arcs it intercepts.

In figure 3a, $\angle APB = {^1/_2}(\overset{\frown}{AXB} - \overset{\frown}{AYB})$ In figure 3b, $\angle APB = {^1/_2}(\overset{\frown}{AB} - \overset{\frown}{AC})$ In figure 3c, $\angle APB = {^1/_2}(\overset{\frown}{AB} - \overset{\frown}{CD})$.

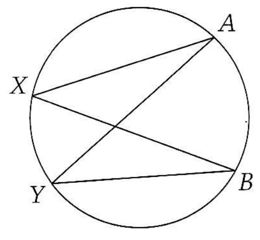


Figure 1

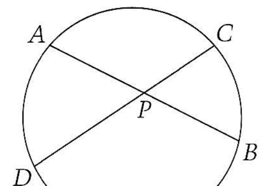


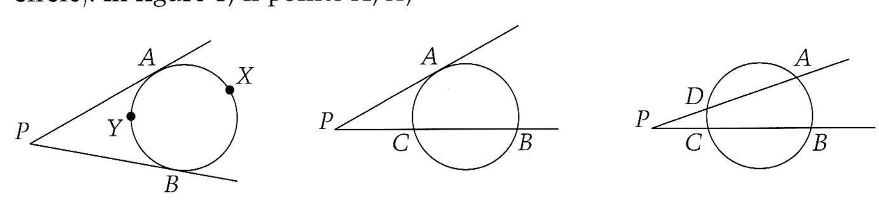
Figure 2

1. Show that if a quadrilateral is inscribed in a circle, then its opposite angles are supplementary. Is the converse of this statement true?

DEFINITION. A quadrilateral that can be inscribed in a circle is called a cyclic quadrilateral.

2. Show that the bisectors of the angles of any quadrilateral form a cyclic quadrilateral.

3. Through the midpoint M of an arc of circle AB we draw any two lines, cutting the circle at points D and E and cutting chord AB in F and G. Show that quadrilateral $DEFG$ is cyclic.



a b c

Figure 3

[1]The phrase *gradus ad Parnassum,* literally "steps to Parnassus," refers to a series of increasingly difficult exercises that help one's skills. (The ancient Romans considered Mt. Parnassus to be the home of the Muses.) The Austrian composer Johann Joseph Fux used the phrase as the title of his treatise on counterpoint, a primer used by Haydn, Mozart, and other 18th-century composers.

4. We know that the three altitudes of a triangle coincide in a point, called the orthocenter. We also know that any triangle can be inscribed in a circle. Show that if we reflect the orthocenter in any side of the triangle, the reflection lands on the circumcircle of the triangle.

5. In triangle ABC, AP and BQ are altitudes. Show that quadrilateral $ABPQ$ is cyclic.

DEFINITION. The three altitudes of a triangle are drawn, and a new triangle is formed by connecting the three feet of the altitudes. This new triangle is called the pedal triangle of the original triangle. (Did you ever notice that the word pedal is an adjective, meaning "pertaining to feet"? Ask a Latin scholar.)

6. Show that the altitudes of the original triangle are the angle bisectors of the pedal triangle.

7. A fixed point is chosen inside a circle, and all possible chords are drawn through that point. Then the midpoint of each chord is chosen. What figure is formed by these midpoints?

In technical terms, this question says: Find the locus of the midpoints of chords of a circle passing through a fixed point. (The word *locus* is just a fancy term for a set of points satisfying a given condition.)

What if the chosen point is the center of the circle?

Can you make up a problem like this but starting with a point outside the circle?

8. If A, B, and C are three points on a circle, we join the midpoints of arcs AB and AC. This line intersects AB at X and AC at Y. Show that $AX = AY$.

9. Given a circle and a fixed chord AB, let CD be a second (variable) chord with a fixed length. (a) Find the locus of points of intersection I of lines AC and BD. (b) Find the locus of points of intersection K of lines AD and BC.

10. (a) Points A and B are fixed points on a given circle, and M is a variable point major arc AB. Point P is a point on segment AM such that $MP = MB$. Find the locus of points P. (b) We extend segment AM to a point Q outside the circle such that $MQ = MB$. Find the locus of points Q. (c) Answer questions (a) and (b) if point M is taken on minor arc AB.

11. Two circles intersect in points A and B. A secant to both circles passes through A, cutting the first circle again at P and the second at Q. A second secant to both circles passes through B, cutting the first circle again at R and the second at S. Show that PR is parallel to QS. ◘

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Van der Waals and his equation

Making an ideal gas real

by B. Yavelov

ON NOVEMBER 23, 1837, IN the town of Leiden in the Netherlands, the first of carpenter Yakobus van der Waals' nine children was born. This boy, Johannes Diderik van der Waals, was destined to become one of the shining stars of the European scientific community and the founder of modern molecular physics.

Because his father's modest income prevented him from pursuing a formal education beyond elementary school, van der Waals had to school himself. While working as a school teacher, he managed to attend lectures on mathematics, astronomy, and physics at the old University of Leiden. It was at this university on June 14, 1873, that van der Waals defended his doctoral thesis "On the continuity of liquid and gaseous states" (the result of his first independent study and his first scientific publication). The defense was a success, but the esteemed members of the Scientific Council showed no particular interest in van der Waals' work. Ten years later, the entire European scientific community began to realize that this work would ensure that his name would have a place in the science pantheon. In 1910 van der Waals was honored with the Nobel prize for physics.

What earned van der Waals such praise? In mathematical language, the answer is simple. Before van der Waals, the gaseous state was described by the equation of an *ideal* gas:

$$PV = RT. \qquad (1)$$

After the publication of his thesis, it was possible to describe the state of a real gas:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT, \qquad (2)$$

where V is the volume of one mole of gas, T the absolute temperature, P the pressure, R the universal gas constant, and a and b are experimentally defined constants characterizing the intermolecular attraction and the volume occupied by the molecules.

Although the ideal gas equation is extremely useful in physics, it can only provide a very rough approximation of the properties of real gases. It can only be used to accurately describe gases with very low densities, where the size and interaction of the molecules that make up the gas are insignificant. The van der Waals equation (or the real gas equation of state) provides a far more realistic model in which the molecules are considered as perfectly rigid balls with small but significant diameters, that are held together by cohesive forces that quickly fade with distance.

Van der Waals described the transition from equation (1) to equation (2) as "applying the corrections." First, his modified equation takes into account that, in addition to the external pressure P, there is an internal pressure a/V^2 resulting from the intermolecular cohesion that tries to bind the molecules together into a single tight cluster in opposition to the chaotic thermal motion. Second, the thermal motion does not take place in the entire volume V occupied by gas, but only in this volume minus that occupied by the molecules themselves. Thanks to van der Waals' corrections, an equation that could only describe nonexistent theoretical objects was transformed into an equation that could describe the behavior of real fluids with densities ranging from rarefied gas to the liquid state. In addition, van der Waals' corrections also permitted scientists to calculate the values of molecular forces and approximate the size of molecules.

Figure 1 shows isotherms calculated at different temperatures with the help of the van der Waals equation. The larger values of V indicate fluids in a gaseous state, and the smaller ones indicate fluids in a liquid state. Isotherms 1, 2, and 3 show that, at temperatures $T > T_c$, each value of P corresponds to only one value of V. In other words, the tran-

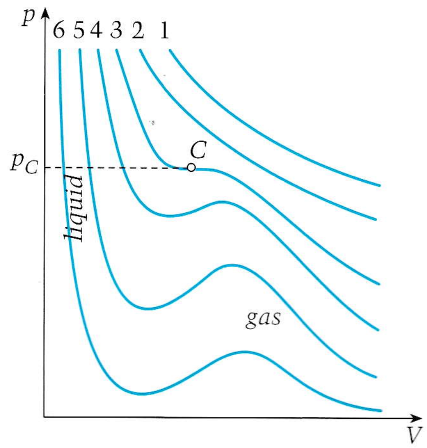


Figure1

sition from a gaseous to a liquid state is continuous, so at such temperatures categorizing matter as either liquid or gas is pointless. Something quite different takes place at $T < T_c$. In this temperature range the isotherms have regions where each value of P corresponds to three different values of V, as indicated by the wavy segments in isotherms 4, 5, and 6.

Let's consider in detail an isotherm for some temperature $T < T_c$ (fig. 2). The wavy segment *ec* (shown by the dash line) corresponds to an unnatural phenomenon: a compression-induced drop in pressure. It is just a mathematical whim of the van der Waals equation (sometimes such whims result in discoveries, however), and in reality the strange segment *ec* indicates that during the gradual change of volume in this region, the matter cannot be homogeneous all the time—some of the time it will be separated into gas and liquid components. Thus, the true isotherm is the broken line *abfg*

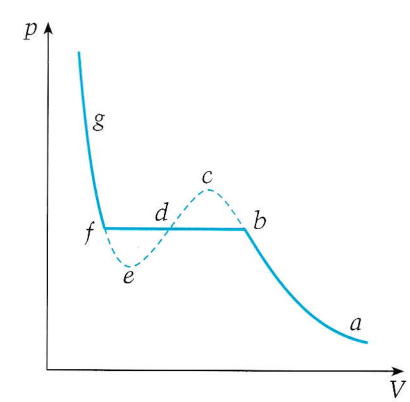


Figure2

with a straight line *bf* in the region of the wavy segment. This straight line connects the branch *ab* corresponding to the gaseous state and the branch *fg* describing the liquid one. Horizontal segment *bf* corresponds to the conversion of gas into liquid (and vice versa) at a given temperature and constant pressure.

At what pressure is the linear segment located? The van der Waals equation cannot answer this question. The position of the horizontal segment is determined by the rule discovered independently by J. Maxwell (1831–1879) and R. Clausius (1822–1888): The height of the segment *bf* should divide equally the areas of the "half-waves" *bcd* and *def*, which lie below and above it.

As the pressure increases, the linear segment becomes shorter until it eventually becomes a single critical point C when $T = T_c$ (fig. 1). According to van der Waals' equation $T_c = 8a/(27bR)$, and the corresponding critical pressure is $P_c = a/(27b^2)$. For example, in the case of water, T_c = 647.3 K and P_c = 22.13 mPa. This results in a value of $b = RT_c/(8P_c) = 30.4 \cdot 10^{-6}$ m^3/mol. For water molecules considered as perfectly rigid balls, the equation yields a diameter $d = 2.9 \cdot 10^{-10}$ m, a value that has a reasonable order of magnitude.

The van der Waals equation is widely used by physicists and engineers for good reason. The equation is easy to work with, and it allows them to describe the properties of matter over a vast range of conditions. Furthermore, it is based on a simplified though realistic model of matter that can be easily interpreted. But just how accurate is van der Waals' equation? At high temperatures and low pressures, it is very accurate, but when the density of a gas approaches the liquid state, the equation can only provide a qualitative description at best.

The van der Waals equation created a problem that the author himself was unable to provide a logical solution to, although he was able to solve the problem intuitively. It seems that for many years after its introduction, no one was able to provide a strict mathematical proof. How could such a practical equation exist without a sound theoretical basis? Some physicists were inclined to believe that a metaphysical answer should be sought.

It wasn't until 1966 that the van der Waals equation was deduced. However, a certain enigma has resisted complete resolution down to the present day. The recent equation was strictly derived for a model in which absolutely rigid balls (molecules) are linked by very weak attractive forces with an infinite range. Van der Waals himself insisted that the forces of attraction described by his equation do not operate at long range but rather over only a few molecular diameters. This situation would elicit little interest if it hadn't turned out that this scientist's physical intuition has yet to be surpassed by the most sophisticated mathematical methods. Indeed, a comparison of experimental data with the results of computer calculations has shown that the van der Waals equation can also be valid for short-range intermolecular attraction, and it does not necessarily have to be weak.

What was the fate of van der Waals after the defense of his famous dissertation? In 1875 he became an academician, and two years later, a professor of physics at the University of Amsterdam. He worked at this position until 1908, when he was forced into mandatory retirement at age 70. In addition to the real gas equation, van der Waals made a number of other important scientific discoveries, some of which weren't appreciated until modern times. He proceeded with his scientific work until 1916 but seemed to tire as the years passed, both physically and creatively. When he died in 1922, his work in molecular physics was being overshadowed by new developments in quantum theory and atomic and nuclear physics. Today, however, his work is once again the talk of the physics community and will certainly lead the way to new important developments in the field.

The ins and outs of circles

What do inscribed and circumscribed circles have in common?

By I. F. Sharygin

INSCRIBED AND CIRCUMSCRIBED circles have a lot in common. Their similarities are the focus of this article, and the two problems that follow will bring out their common properties.

Problem 1. Let the sides of triangle ABC be $BC = a$, $CA = b$, and $AB = c$. Find the lengths of the segments into which the sides are divided by the points where they touch the circle inscribed in triangle ABC.

Problem 2. Let the angles A, B, and C of triangle ABC be known. Find the angles between the sides of the triangle, and the radii of the circumscribed circle drawn to the corresponding vertices.

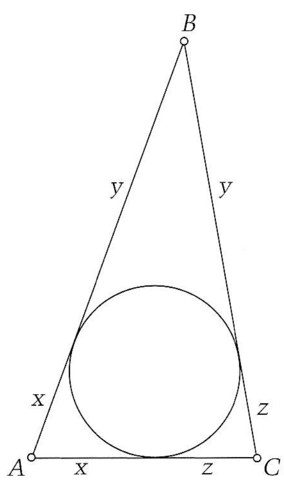


Figure 1

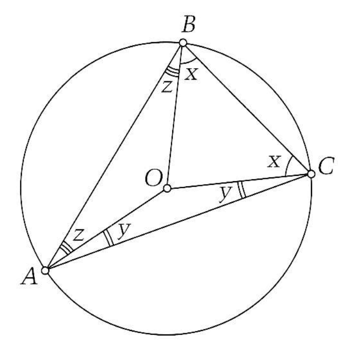


Figure 2

Let's solve the first problem. If two of the segments described in the problem have a common point (a vertex of the triangle), they are equal (fig. 1). These segments will be referred to as x, y, and z. This allows us to obtain the system

$$x + y = c, \; y + z = a, \; z + x = b.$$

Adding all three equtions, we find $2x + 2y + 2z = a + b + c$, so $x + y + z = (1/2)(a + b + c)$. We set this last quantity equal to s (for *semiperimeter*), and subtract each original equation, in turn, from the last we've obtained. We find that

$$x = s - a, \; y = s - b, \text{ and } z = s - c.$$

These formulas belong to a group of "practical" formulas that can come in handy if you have a difficult problem to solve.

One can solve the second problem in a similar way (fig. 2). Angles adjoining one side of the triangle are equal. Denoting them by x, y, and z, we come to the system

$$x + y = C, \; y + z = A, \; x + z = B,$$

from which we compute

$$x = {}^1/_2(B + C - A) = \pi/2 - A,$$
$$y = {}^1/_2(A + C - B) = \pi/2 - B,$$
$$z = {}^1/_2(A + B - C) = \pi/2 - C.$$

Actually, unlike the previous case, it is possible that for the values of some angles to be negative (that is, if we consider oriented angles). Thus, the angles adjacent to the longest side of an obtuse triangle will be negative. You can easily derive the relationships obtained in problem 2 from the main property of central and inscribed angles ($\angle BOC = 2\angle BAC$). The careful reader will have noticed that the algebra we used to solve both problems is exactly the same (especially if we use d to represent the half-sum of the angles of the given triangle instead of $\pi/2$) and consider the profound analogies between the inscribed and circumscribed circles.

In both cases we used only the main property of an isosceles triangle, which holds not only in the usual Euclidean geometry but also in hyperbolic (Lobachevskian) geometry. That's why the relationships we've identified hold true for spherical triangles, for which the "duality" of their angles and sides is fundamental. This duality leads to the duality of their inscribed and circumscribed circles.

Let's proceed to inscribed and circumscribed quadrilaterals. We can examine their properties by working through the following two problems:

Problem 3. Let $ABCD$ be a circumscribed quadrilateral. Then $AB + CD = AD + BC$.

Problem 4. Let $ABCD$ be an inscribed quadrilateral. Then $\angle A + \angle C = \angle B + \angle D$. ($\angle A + \angle C = \angle B + \angle D = 180°$.)

Moreover, these relationships are not only necessary but also sufficient for the given quadrilateral to be circumscribed (or inscribed).

Now let's point out two more relationships, both of which give a necessary and sufficient condition for the existence of a circle inscribed within a given quadrangle, if the latter is not a trapezoid.

Problem 3′. If the opposite sides of quadrangle $ABCD$ are extended, let them intersect as in figure 3. Now, if $ABCD$ is a circumscribed quadrilateral, then $KA + AM = KC + CM$, $KD + BM = MD + KB$. And vice versa, if one of these two equalities is satisfied, then the quadrilateral $ABCD$ will be circumscribed.

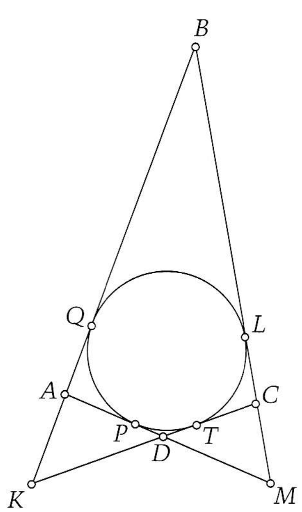


Figure 3

One can also point out the corresponding relationships for an inscribed quadrilateral. We leave this for the reader to explore.

Proof. Let's first prove that the relationships of problem 3′ are necessary. Using the fact that two tangents drawn to a circle from one point are equal, we find

$$\begin{aligned} KA + AM &= KQ - AQ + AP + PM \\ &= KT + ML = KC - CT + CL + MC \\ &= KC + MC. \end{aligned}$$

The sufficiency of the conditions of problems 3, 3′, and 4 are usually proven by contradiction. However, we'll suggest another way that would put greater stress upon the close connection between inscribed and circumscribed circles.

Let's prove the sufficiency of the first condition of problem 3′. Suppose, in quadrilateral $ABCD$ (fig. 4), that $KA + AM = KC + CM$. Mark segments $KE = KA$ on line KC and $MF + MA$ on MB. From the equality

$$KA + AM = KC + CM,$$

it follows that

$$\begin{aligned} CF &= MF - MC = MA - MC \\ &= KC - KA = EC. \end{aligned}$$

Since KA, MA, and CE are equal to KE, MF, and CF respectively, the bisectors of the angles AKD, AMB, and BCD coincide with the perpendicular bisectors of AE, AF, and FE. Thus these bisectors intersect at one

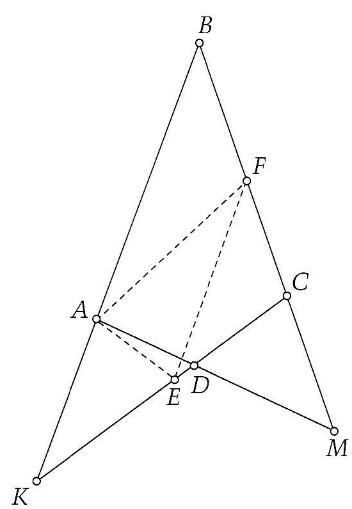


Figure 4

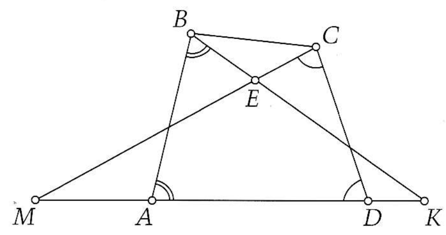


Figure 5

point: the center of the circle circumscribing triangle AEF. This point, being on three angle bisectors, is equidistant from KB and KC, KC and BC, and BC and AM. Therefore, it is equidistant from the sides of quadrilateral $ABCD$. In other words, it is the center of the circle inscribed in it.

To prove the sufficiency of the conditions of problem 4, let's suppose that the sums of the opposite angles of quadrangle $ABCD$ equal each other, and for definiteness, $D < C$, $A < B$. Draw straight lines through the points C and B such that the angles between them and sides DC and AB of the quadrangle are equal to angles D and C respectively (fig. 5). We obtain two isosceles triangles CMD ($CM = MD$) and ABK ($AK = BK$). Triangle BEC is also isosceles ($\angle CBE = \angle ABC - \angle BAD = \angle BCD - \angle CDA = \angle BCE$). The perpendiculars dropped to sides AB, BC, and CD from K, E, and M respectively, coincide with bisectors of the interior angles of the triangle MEK, and therefore meet in one point. This point is equidistant from all four vertices of the quadrilateral. In other words, it is the center of its circumscribed circle as well as the center of the circle inscribed in triangle MEK.

It looks as if inscribed and circumscribed circles courteously bow to each other in these two proofs.

If $ABCD$ is a trapezoid, there are some special features:

Problem 5. There exists a circle inscribed in trapezoid $ABCD$ with bases AD and BC if and only if one of the following equalities holds:

(a) $$TB + BP = DP,$$
$$TC + AP = AD + CP,$$

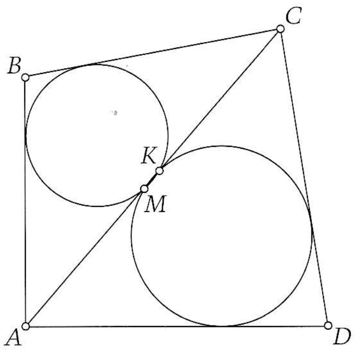


Figure 6

where P is the point where the extensions of the lateral sides of the trapezoid meet, and T is the projection of D on the line BC.

(b) $$\frac{AD}{BC} = \cot\frac{A}{2}\cot\frac{B}{2}.$$

Try to prove these statements yourself.

Let's conclude the article with two more problems.

Problem 6. Let $ABCD$ be a circumscribed quadrilateral. Prove that the circles inscribed in triangles ABC and CDA are tangent to each other.

Proof. Let the circles inscribed in triangles ABC and CDA touch AC at points K and M respectively (fig. 6). To prove that points K and M coincide, we need the help of the formula from problem 1:

$$MK = |AM - MK| = \left|\frac{1}{2}(AB + AC - BC) - \frac{1}{2}(AC + AD - CD)\right| = \frac{1}{2}|AB + CD - BC - AD| = 0.$$

Problem 7. Straight lines, parallel to DC and AB respectively, are drawn through the vertices A and C of quadrilateral $ABCD$. The first of them meets the straight line BC at the point B_1, and the second meets the straight line AD at the point D_1. Prove that

(a) if $ABCD$ has an inscribed circle, then AB_1CD_1 has one also.

(b) if $ABCD$ has a circumscribed circle, then AB_1CD_1 has one also.

Proof. We prove only *(b)*, leaving *(a)*, which is easier, for the reader. We confine ourselves to the case when $ABCD$ is not a trapezoid. Denote the points in which straight lines AD and BC and AB and CD meet at K and M respectively. We have to consider two different cases (fig. 7 and fig. 8).

Let P be the point where straight lines AB_1 and CD_1 meet. In this case (fig. 7), $PA = CM$, $PC = AM$. Since $ABCD$ is a circumscribed quadrangle, $KA + AM = KC + CM$ (according to the first equality demonstrated in problem 3′). Replacing AM and CM with PC and PA, we obtain $KA + PC = KC + PA$. Then, according to the second equality stated in problem 3′, quadrilateral AB_1CD_1 can be circumscribed.

In the second case (fig. 8), by applying the second statement of equality from problem 3′ to the quadrilateral $ABCD$, we can conclude that the first condition of the same problem is satisfied for the quadrilateral AB_1CD_1.

Let's finish up with a few more problems.

Problems.

1. Point M is located on base AC of isosceles triangle ABC. If $AM = a$ and $MC = b$, find the distance between the points where the circles inscribed in triangles ABM and CBM touch the side BM.

2. The sides of a circumscribed pentagon have lengths a, b, c, d, and e. Find the lengths of the segments into which side a is divided by the point where it touches the circle.

3. Consider a circumscribed

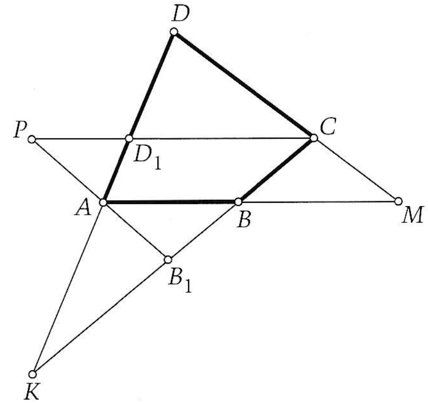


Figure 7

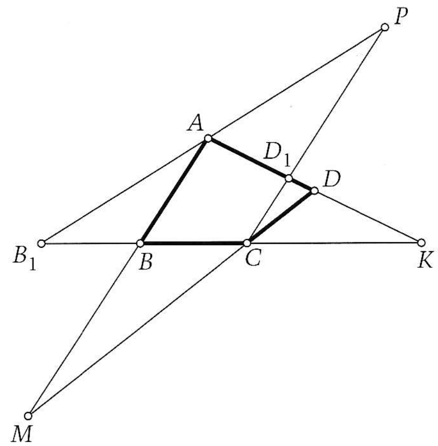


Figure 8

polygon with an even number of sides. Number its sides consecutively and add up the lengths of the odd-numbered sides and the even-numbered sides separately. Prove that these sums must be equal.

4. Straight lines intersecting sides AB and CD of parallelogram $ABCD$ divide it into several trapezoids such that a circle can be inscribed in each of these trapezoids. Prove that the product of the segments into which side AB is divided by the lines is equal to the product of segments into which CD is divided.

5. Points A_1, B_1, and C_1 are found along the sides of triangle ABC such that straight lines AA_1, BB_1, and CC_1 intersect in one point M inside the triangle. Consider the three quadrilaterals AB_1MC_1, BC_1MA_1, and CA_1MB_1. Show that if two of these quadrilaterals can be circumscribed, then the third one can be circumscribed as well.

6. Prove that if the lateral edges of a quadrilateral pyramid are equal, then the sum of the two dihedral angles at the opposite lateral edges of the pyramid is equal to the sum of the other two dihedral angles.

7. Consider two rays, OA and OB. Find the locus of ray OC such that the quantity $\alpha + \beta - \gamma$ (where α, β, and γ are the dihedral angles at the edges OA, OB, and OC respectively) of the corresponding trihedral cone is constant. ◘

AT THE BLACKBOARD II

Number Cells

Can you get there from here?

by Thomas Hagspihl

THIS IS AN INVESTIGATION undertaken by two 14-year-olds, Tim and Hugh. In the following row of cells, you start with 3 and 4:

3	4			

Then you add 3 and 4 to get 7, then 4 and 7 to get 11, ...

3	4	7	11	17

But if you are given the first and last numbers only ...

8				52

what are the missing numbers?

Here are some examples of number cells:

1	2	3	5

3	1	4	5

Tim and Hugh soon saw that the real problem in filling in a number cell is to find the second number—after that, it's easy.

Problem 1. Without reading further, solve this number cell:

2				7

Solution. Trial and error, or a bit of thought, yields

2	1	3	4	7

Tim and Hugh decided to generalize. They started with the four-unit cell (since the three-unit cell is the trivial case). They let the first and last numbers be x and z, and they called the second number y—that is,

x	y		z

The empty block must be $x + y$, and $z = x + 2y$:

x	y	$x + y$	$x + 2y$

Solving for y, we get

$$y = \frac{z - x}{2}.$$

Problem 2. Solve the following number cells:

(a)

4			6

(b)

–3			5

Solution. (a) Taking $x = 4$, $z = 6$, we get $y = (6 - 4)/2 = 1$, which gives the number cell

4	1	5	6

(b) Taking $x = -3$, $z = 5$, we get $y = (5 - (-3))/2 = 4$, which gives the number cell

–3	4	1	5

The general case for a five-unit cell looks like this:

x	y	$x + y$	$x + 2y$	$2x + 3y$

So, $z = 2x + 3y$, or

$$y = \frac{z - 2x}{3}.$$

Problem 3. Solve

4				–7

Solution. Taking $x = 4$, $z = -7$, we get

$$y = \frac{z - 2x}{3} = \frac{-7 - 2(4)}{3} = \frac{-15}{3} = -5.$$

This gives the number cell

4	–5	–1	–6	–7

Will the numbers we find always be integers if the starting numbers are integers? A few examples will soon lead you to the answer no. But how do you know when you'll get integers? In the case of a five-unit cell, it's clear that $z - 2x$ must be divisible by 3 in order to get integers.

Tim and Hugh went on to find

formulas for number cells of length 6 and 7.

Problem 4. Show that

$$y = \frac{z - 3x}{5}$$

for a six-unit cell and

$$y = \frac{z - 5x}{8}$$

for a seven-unit cell.

If we stack the equations, very interesting things start to appear:

$$y = \frac{z - x}{1} \text{ (3 - unit cell)},$$

$$y = \frac{z - x}{2} \text{ (4 - unit cell)},$$

$$y = \frac{z - 2x}{3} \text{ (5 - unit cell)},$$

$$y = \frac{z - 3x}{5} \text{ (6 - unit cell)},$$

$$y = \frac{z - 5x}{8} \text{ (7 - unit cell)}.$$

Look at the denominators of the fractions: 1, 2, 3, 5, 8, ... —*a number cell!*

Also look at the coefficients of x (ignoring the sign): 1, 1, 2, 3, 5, ...—*again, a number cell!*

Tim and Hugh noticed that this cell is special: it's the Fibonacci sequence.

Now Tim and Hugh tried to put these results into a formula to find the second number of any given number cell no matter what the starting numbers or the length are. Let x be the number of cells and let N_x be the last number in the cell. (N_1, N_2, ... will be the first, second, ... numbers.) Also let F_x be the xth Fibonacci number. The formula therefore reads

$$N_2 = \frac{N_x - F_{x-2}(N_1)}{F_{x-1}}.$$

Problem 5. Prove Tim and Hugh's formula.

Problem 6. Show that this formula works for the following number cell:

−3						17

Solution. $x = 7$, $N_x = 17$, $N_1 = -3$, F_{x-2} is the fifth number in the Fibonacci sequence—that is, 5—and F_{x-1} is the sixth number in this sequence—that is, 8. Therefore,

$$N_2 = \frac{17 - 5(-3)}{8} = \frac{32}{8} = 4.$$

The number cell follows:

−3	4	1	5	6	11	17

Problem 7. Solve this cell:

5								2

Solution. $x = 9$, $N_1 = 5$, $N_x = 2$, $F_{x-2} = F_7 = 13$, $F_{x-1} = F_8 = 21$. Therefore,

$$N_2 = \frac{2 - 13(5)}{21} = \frac{-63}{21} = -3.$$

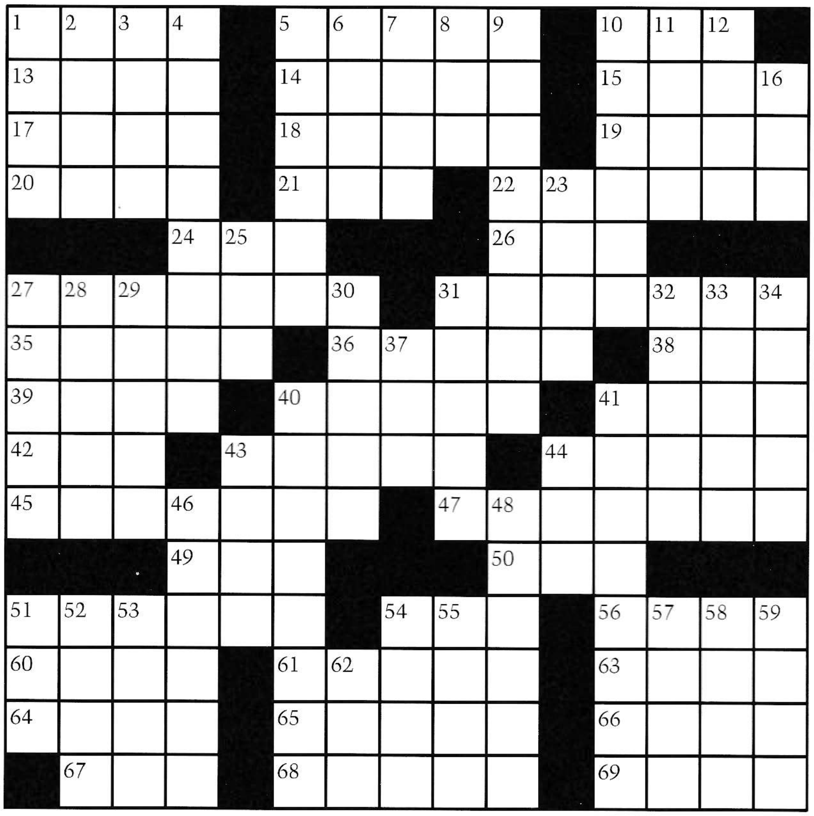
The number cell follows:

5	−3	2	−1	1	0	1	1	2

We hope you will enjoy playing with number cells. ◘

Criss × cross science

by David R. Martin



Across

1 44,462 (in base 16)
5 Rod: comb. form
10 Resinous insect secretion
13 Describe
14 977,834 (in base 16)
15 Toward shelter
17 Monogram part: abbr.
18 Refine metal
19 Archaeologist ___ Leakey
20 Hyperbolic function
21 Moray, e.g.
22 $NaALSi_3O_8$
24 Abscisic acid: abbr.
26 Immoral
27 ___ arsenide
31 ___ birefringence
35 Take in
36 Wrong
38 Anger
39 Lively
40 French school
41 48,858 (in base 16)
42 ___ cycle (Krebs cycle)
43 Separate
44 Fir and cedar
45 Fluorine, e.g.
47 Dissolver
49 ___ X-1 (x-ray pulsar)
50 Mauna ___
51 French mathematician
54 Jump about
56 Passerine bird
60 Decorative case
61 Reddish powder
63 60,906 (in base 16)
64 Forbidden (alt. sp.)
65 Abalone
66 10^{-9}: pref.
67 With: pref.
68 Backs of necks
69 U.S. Treasury agents

Down

1 Landed
2 Radar-jamming transmitter
3 60 C
4 $E + pV$
5 Astronomical grid
6 $C_{34}H_{32}O_4N_4Fe$
7 English chemist Frederick Augustus ___ (1827–1902)
8 Dimercaprol: abbr.
9 Set of related information
10 Wavelength letter
11 Jai ___
12 Qualified: abbr.
16 Sense organ
23 Chemist ___ Onsager
25 Binary digit
27 Enclosed yard
28 711,370 (in base 16)
29 Native
30 City in Ga.
31 Rasps
32 Double-bonded alkene
33 Shakespeare's forest
34 Saccharomyces
37 Sun. follower
40 Fundamental particle
41 Wanting two H atoms
43 Latin consonant sound
44 Middle Eastern group: abbr.
46 Element 76
48 ___' paradox (of night sky)
51 Type of transistor: abbr.
52 Greek letters
53 Red precious stone
54 Jolt
55 Type of molding
57 Swedish botanist ___ Afzelius (1750–1837)
58 ___ Descartes
59 A meson
62 Lake Garda wind

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE SEPTEMBER/OCTOBER PUZZLE

A	B	B	E			F	R	I	T		E	A	B	B
D	A	R	T		T	R	I	C	E		D	A	N	E
E	L	E	C	T	R	O	C	H	E	M	I	C	A	L
A	D	D		R	A	C	E			A	S	C	I	
			R	O	C	K		H	A	L	O			
S	A	L	I	N	E			A	M	I	N	E	S	
U	L	A	M	A		M	A	N	I	C		B	I	G
R	I	B	S		R	E	I	G	N		L	E	V	I
A	B	A		P	E	A	R	S		L	I	D	A	R
	I	N	U	L	I	N			S	E	C	A	N	T
			R	A	D	S		N	A	V	E			
	A	L	A	N			C	A	G	E		I	B	E
E	L	E	C	T	R	O	D	I	A	L	Y	S	I	S
A	L	A	I		A	L	B	A	N		A	L	E	S
R	O	L	L		M	E	A	D			G	E	N	E

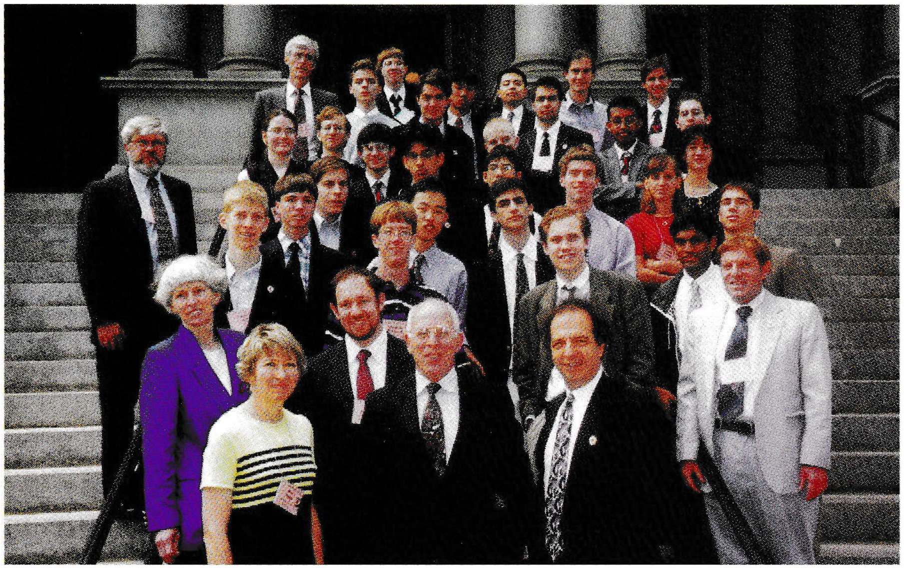
Scores and SNO in Sudbury

The 1997 International Physics Olympiad in Ontario

By Dwight E. Neuenschwander

TWO HUNDRED AND SIXTY-six pre-college students from 56 nations gathered in Sudbury, Ontario, July 13–21, for the 28th International Physics Olympiad (IPhO). Sudbury sits in the heart of the Canadian Shield, a vast region of lakes and forests and two-billion-year-old rock outcrops. Long ago a large meteorite slammed into the Canadian Shield, creating the Sudbury Valley, and producing some of the richest deposits of nickel and copper on Earth. Ontario's geography and livelihood were shaped by the past. With the metals producer Inco Ltd. as principal sponsor of IPhO '97, bounty from the past was invested in the future.

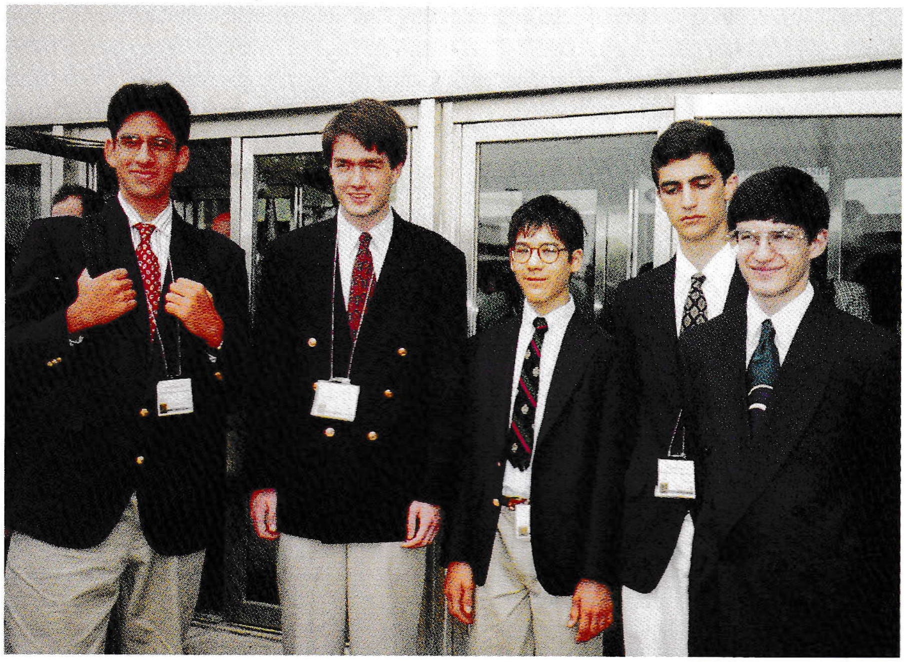
Each member of the U.S. Physics Team won a medal at the 1997 IPhO: one gold (Boris Zbarsky, Rockville, Maryland), one silver (Christopher Hirata, Deerfield, Illinois), and three bronzes (Noah Bray-Ali, Los Angeles, California; Travis Hime, New Canaan, Connecticut; Michael Levin, Chicago, Illinois). In total points accumulated, the U.S. team placed eighth in a very tightly packed point spread at the upper end. Our team members are gracious young people who admirably represented the United States and the American physics community. We have many reasons to be proud of the U.S. Physics Team. This fall, Chris and Travis will begin study at the California Institute of Technology; Michael will attend Harvard University; and Noah and Boris will attend the Massashusetts Institute of Technology.



Members of the 1997 US Physics Team join John Gibbons (front, center), Assistant to the President for Science and Technology, on the steps of the Old Executive Office Building in Washington, D.C.

The opening ceremonies were a celebration of youth. The names of the 56 participating nations were read in English and French as Elgar's "Pomp and Circumstance" was performed by the award-winning jazz band of Lasalle Secondary School. Special music was also provided by 19-year-old Jacinthe Trudeau, champion youth fiddler of Canada. She held the audience spellbound with her vigorous renditions of Irish ballads, waltzes and reels, and the incredible "Orange Blossom Special" that eerily imitates the sound of a railway train! The competition was officially opened by Canadian Space Agency astronaut Julie Payette, who is now in training for a NASA shuttle mission. She captured the IPhO spirit with her words: "We're all in this together. Work together, build partnerships. The future is in our collective hands."

The IPhO competition features experimental and theoretical exams. The object of experimental study was a "bimorph"—two piezoelectric strips fused together such that when



The 1997 US Physics Team medalists. From left to right: Noah Bray-Ali (bronze), Travis Hime (bronze), Christopher Hirata (silver), Michael Levin (bronze), and Boris Zbarsky (gold).

placed in an electric field, the strain differential bends the assembly. A tiny mirror glued to one end of the bimorph, illuminated with a laser pointer, provides an optical lever. The displacement of the reflected laser beam enables one to correlate the bimorph's deflection to the applied voltage. These results were then used to measure the bimorph's capacitance.

Question 1 of the theoretical exam featured a potpourri of scaling problems. For example, if a certain helicopter requires power P to hover, find the power P' required for hovering by a one-half scale copy made of identical materials ($P' = 0.0884\ P$). Question 2 was a study of the stability of certain nuclei against specific channels of radioactive decay, using a phenomenological expression for binding energy as a function of neutron and proton composition. Question 3 examined a theoretical model of aircraft lift in terms of the momentum transferred by the air passing a horizontally moving wing whose control surface is tipped at a small angle above the horizontal.

Some of the excursions for the participants included visiting the Inco ore processing plants and the Big Nickel Mine; spending time with the interactive displays in Science North; a cruise on beautiful Lake Ramsey; an Ojibwa pow-wow; and model rocket launches. Sudbury is home to the Sudbury Neutrino Observatory (SNO), located two kilometers below Earth's surface in Inco's Creighton Mine. Students and coaches were treated to a detailed presentation about SNO by Dr. Art McDonald, the project director.[1]

The beauty of the Olympiad is using physics to bring people together. In a warm sense of community, Sudbury residents became personally involved. Besides Inco's sponsorship and heavy coverage in the local media, some 200 local volunteers came forward to help as guides and workers. Local businesses, government, and utilities went to unusual lengths to welcome the IPhO participants and assist the organizers. At a personal level, the local Rotary Club hosted a luncheon for the coaches and observers on Tuesday; and on Wednesday, all the students had dinner in the homes of local families, and spent the evening with them.

[1]See "The Omnipresent and Omnipotent Neutrino" by Chris Waltham in the July/August 1993 issue of *Quantum*.—*Ed.*

Physics is an international fellowship, cutting through past or present political differences that come between nations. Thus at the closing festivities we found Team USA posing for group photos with other teams that included Iran, Russia, and Vietnam. We found American and Cuban coaches exchanging addresses, looking forward to the day when we can visit one another freely. Thanks to a coach from Kuwait, who had coaches from Sweden to New Zealand attired in the Arabian-style headdress, we learned that Bernie Khoury cuts a dashing figure in Middle Eastern headgear! Friendships were forged through the long days of meetings and examinations, and the sleepless nights of translating and grading.

Again this year, as every year, there was a talent show after the closing banquet, which reveals the strength of that forging. Highlights this year included two students from the Netherlands performing on flutes their own arrangement of Holland's national anthem, then merging it into "O Canada;" and the teams from the People's Republic of China, Taiwan, and Singapore performing together. With arms around one another, they sang in unison a Chinese ballad expressing love for their homeland. The Finns performed Russian folk songs in Finnish; Polish-speaking students from half a dozen countries regaled us in song; we had the huka dance from New Zealand; and a flutist

Top medal winners

	Gold	*Silver*	*Bronze*
Australia	2	1	1
China	3	2	–
Germany	1	2	2
Great Britain	–	2	3
Iran	1	3	1
Poland	–	2	2
Romania	1	2	1
Russia	4	1	–
Singapore	1	1	3
Slovak Rep	1	1	2
South Korea	–	–	5
Taiwan	–	2	2
USA	1	1	3
Vietnam	–	3	2

from Macedonia playing J. S. Bach while a juggler from Bulgaria performed in time to the music. In all, I counted 22 performances, not including the karaoke or the slide show. The finale came with the Canadian students, leaders, and volunteers singing "O Canada." Rising to their feet, everyone joined them in a spontaneous expression of respect and togetherness. We were, indeed, all in this together. On that final night, no one wanted to leave. Finally, at eleven o'clock, when the chartered buses could be kept waiting no longer, in the soft glow of their interior lights the songs in many languages continued as we motored through Sudbury back to the university. Most students were up all night in their dorms, trying to make the moment last.

The next morning, students from around the world clasped hands in farewell. The future is, indeed, in their clasped, collective hands. As the teacher-astronaut Christa McAuliffe observed, "To teach is to touch the future." Through the IPhO we help shape the future by bringing together from many nations the students who will live it.

Dwight E. Neuenschwander *is the academic director of the U.S. Physics Team and the director of the Society of Physics Students at the American Institute of Physics in College Park, Maryland; and a professor in the Department of Physics at Southern Nazarene University in Bethany, Oklahoma.*

Bulletin Board

First Step to Nobel Prize

Five students from the United States earned honorable mentions for their research papers submitted to the international competition "First Step to Nobel Prize in Physics." Katrina A. Bogden was honored for her paper "The Effects of Ionizing Radiation on Erythrocyte Sedimentation Rate," Andrew Hucke for "The Physics of Archery," Craig S. Jones for "The Kirlian Effect and the Identification of Bacteria," Irina Feygina for "An Investigation into the Effects of Extremely Low Temperatures on the Rate of Electron Capture Beta Decay," and Yuki David for "Symmetric Theory of Gravity."

Submissions are now being accepted for the sixth annual competition. The general rules are as follows:

1. All secondary school students regardless of country, type of school, etc., are eligible for the competition. The only conditions are that the school cannot be considered a university college and the age of the. participant must not exceed 20 years on March 13, 1998.
2. There are no restrictions on the subject matter of the papers, their level, methods applied, etc. The papers must, however, have a research character and deal with physics topics or topics directly related to physics.
3. Participants can submit more than one paper, but each paper should have only one author. The total volume (text, figures, captions, tables, references, etc.) of each paper should not exceed 25 normal typed pages (about 25,000 characters).
4. The papers will be refereed by the organizing committee and the best will be given awards. The number of awards is not limited. All awards will be considered equivalent. The authors of the prize-winning papers will be invited to the Institute of Physics for a one-month research stay (scheduled for November 1998). All expenses will be paid by the Institute of Physics during the stay, but participants must pay all travel costs.
5. In addition to the regular awards, the organizing committee may establish a number of honorable mentions. Participants who win honorable mentions receive diplomas, but are not invited to the research stay.
6. Participants should send their papers in duplicate and in English, by March 31,1998, to Maria Ewa Gorzkowska, Secretary of the "First Step," Institute of Physics, Polish Academy of Sciences, al. Lotnikow 32/46, (PL) 02-668 Warszawa, POLAND.
7. Important: Each paper should contain the name, birth date, and home address of the author and the name and address of his/her school.

Additional information on the competition and on the proceedings of past competitions can be obtained from Dr. Waldemar Gorzkowski: phone (022) 435212, fax (022) 430926, e-mail gorzk@gammal.ifpan.edu.pl, or from Dr. Yohanes Surya: phone (062) 21-8211551, e-mail yohaness@centrin.net.id

Current information on the competition and related topics can also be obtained by anonymous FTP at ftp.ifpan.edu.pl in the subdirectory pub/competitions.

Quantum in Kansas

Yan Soibelman, a math researcher and educator in the department of mathematics at Kansas State University, wrote to share his views on *Quantum* and how he uses it to inspire students in his community:

I started to subscribe to *Quantum* a long time ago when I lived in the (now former) Soviet Union, where it is known as *Kvant*. I am pleased to see that *Quantum* preserves the spirit of *Kvant*, and in some ways has surpassed its parent journal.

I helped organize a mathematical olympiad for fifth to eighth grade students in Manhattan (Kansas, not New York). It was sponsored by the Department of Mathematics. The six other deparmental volunteers and myself decided to present participants with four challenging problems that they could work on for several hours, rather than giving them a standard set of 70 problems to solve in 20 minutes. These problems were taken from previous Russian olympiads. Here are a few examples.

1. Prove that the sum $1 + {}^1/_2 + {}^1/_3 + \ldots + 1/p$ (p is prime) cannot be an integer.

2. Is it possible to arrange seven segments on the same plane in such a way that each of them intersects with exactly three others?

3. A triangle is located inside of a rectangle. Prove that the triangle's perimeter is less than the perimeter of the rectangle.

The event was advertised in the local paper and drew 24 students, not bad for a small city. The paper also ran a story after the competition that mentioned the winners of the Olympiad (three from each grade). As prizes, winners received subscriptions to *Quantum* and the book *Quantum Quandaries.*

Sincerely,
Yan Soibelman

Dicey CyberTeaser

Brainteaser B216 in this issue, which did double duty as the November/December CyberTeaser on our Web site, gave our contestants a chance to do some quick figuring in their heads—assuming they knew that the pips on opposite faces of a die always add up to seven. If they weren't aware of this essential piece of gaming information, they needed to get their hands on a pair of dice fast, because the answers came flying in this time.

The first ten correct answers were submitted by the following presumed gamesters:

Theo Koupelis (Wausau, Wisconsin)
Tracy Hansen (Galesburg, Illinois)
Pasquale Nardone (Brussels, Belgium)
Bruno Konder (Rio de Janeiro, Brazil)
Rick Armstrong (St. Louis, Missouri)
Oleg Shpyrko (Somerville, Massachusetts)
Jaak Sarv (Tallinn, Estonia)
Keith Watkins (Succasunna, New Jersey)
Hana Bizek (Argonne, Illinois)
Joe Snider (Chicago, Illinois)

Congratulations to our winners, who will receive a *Quantum* button and a copy of the November/December issue. As always, everyone who submitted a correct answer in time was eligible for a roll of the cyberdice to win a copy of our collection of brainteasers, *Quantum Quandaries.*

Our current CyberTeaser awaits you at www.nsta.org/quantum. Just take a deep breath and click on the Contest button.

ANSWERS, HINTS & SOLUTIONS

Math

M216

Clearly $x > 0$. We can estimate the left-hand side by using the Arithmetic-Geometric Mean (AM-GM) Inequality. This theorem says that if a and b are positive real numbers, then $a+b \ge 2\sqrt{ab}$, with equality holding only when $a = b$. We first let $a = x2^{(1/x)}$, $b = (1/x)2^x$, to find that

$$x2^{\frac{1}{x}}+\left(\frac{1}{x}\right)2^x = 2\sqrt{2^{\left(\frac{x+1}{x}\right)}}.$$

But how large can this last expression be? Letting $a = x$, $b = 1/x$ in the AM-GM inequality, we find that

$$x+\frac{1}{x} \ge 2\sqrt{x\left(\frac{1}{x}\right)} = 1,$$

with equality holding only when $x = 1/x = 1$. So $x + 1/x$ is at most 2, and

$$x2^{\left(\frac{1}{x}\right)}+\left(\frac{1}{x}\right)2^x \ge 2\sqrt{2^{\left(x+\frac{1}{x}\right)}} \ge 2\sqrt{2^2} = 4.$$

It is now easy to check that equality holds only when $x = 1$.

M217

Note that each friend must cover 30 miles, and clearly each bicycle must cover the same distance. The total distance they must walk is 30 miles, too. So the total time the trip takes is equal to the sum of the times they walk and ride both bicycles—that is, 1 + 1.5 + 5 = 7.5 hours. Therefore, the trip must take at least 2.5 hours and will take this time if and only if the friends arrive at B simultaneously. So it's sufficient to show that this is possible. Let the first friend walk for x miles and ride the racing bike for the remaining $(30 - x)$ miles; let the second friend ride the mountain bike for y miles and then walk for $(30 - y)$ miles; and let the third friend ride the racing bike for x miles, walk for $(y - x)$ miles, and ride the mountain bike for the remaining $(30 - y)$ miles. Since we already know that each of them must spend 2.5 hours en route from A to B (it's sufficient that this be the case for the first and second friends), we can write the equations to determine x and y. This gives us $x = (45/4)$ miles, $y = (150/7)$ miles. Now we just have to check that for these x and y every biker arrives at the point where the bicycle must be left before the appropriate walker arrives, and we have shown that the minimal time is 2.5 hours.

M218

Let P_1 and Q_1 denote the points where the first and second circle intersect line AD and BC, respectively. Let's prove that points M, C, D, P_1, and Q_1 lie on a common circle. This would mean that points P_1 and Q_1 coincide with P and Q, respectively. In the case shown in figure 1, we have $\angle MP_1A = \angle MAB = \angle MCD$ and $\angle MQ_1B = \angle MBA = \angle MDC$. So $\angle MP_1A$ is supplementary to $\angle MCD$. Similarly, $\angle MQ_1C$ supplements $\angle MDC$. This means that quadrilateral MP_1DC can be inscribed in a circle, which must be the circle through M, C, and D. Similarly, we can show that this cirle passes through Q_1. That is, the circle circumscribed about triangle MCD passes through P and Q. We consider the cases when P and Q lie on the extensions of AD and BC, respectively, in a similar way.

It's enough to show that P_1Q_1 is tangent to the circles shown in figure 1. Since we now know that quadrilateral P_1Q_1CD is inscribed in a circle, we see that $\angle DP_1Q_1$ supplements $\angle DCQ$. Thus $\angle AP_1Q_1 = \angle DCQ_1 = \angle P_1AB$. Therefore $\angle AP_1Q_1 = (^1/_2)\widehat{AMP_1}$, and P_1Q_1 must be tangent to the circle through A and M. Similarly, we can show that P_1Q_1 is also tangent to the circle through B and M. Other cases are considered in the same way.

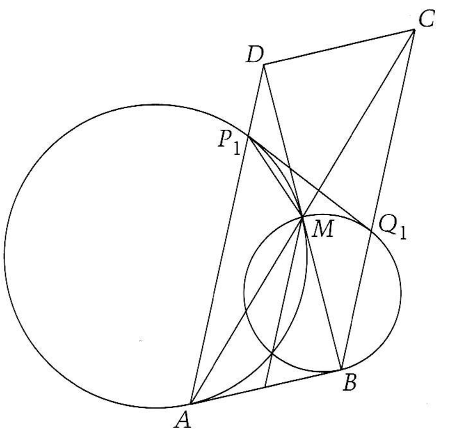


Figure 1

M219

The surface of the resulting polyhedron consists of two equilateral triangles and three equal isosceles trapezoids whose base angles are equal to $90° - \alpha/2$. Consider the development without the larger triangle in figure 2. Since this results in multiple "copies" of the same vertex on the planar development, we have used primes to distinguish

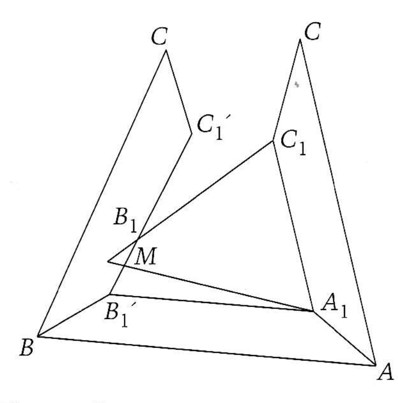


Figure 2

these on the diagram. The development will overlap itself if and only if the segments $B_1'C_1'$ and A_1B_1 meet. If they do overlap, we label their point of intersection as M. Then the inequality $A_1M < A_1B_1$ must hold in triangle A_1MB_1. Calculating the angles of this triangle, we find $\angle A_1B_1'C_1' = 360° - 2(90° + \alpha/2) = 180° - \alpha$, $\angle B_1'AM = 360° - [60° + 2(90° + \alpha/2)] = 120° - \alpha$, and $\angle A_1MB_1' = 180° - (180° - \alpha) - (120° - \alpha) = 2\alpha - 120°$, and rewriting the inequality in the terms of angles, we obtain $180° - \alpha < 2\alpha - 120°$. Thus $\alpha > 100°$. And clearly $\alpha < 120°$—otherwise point D would be on the same plane as A, B, and C.

M220

The conditions of the problem are equivalent to the following statement: Any straight line parallel either to the bisector of the first and third quadrants or to the bisector of the two other quadrants, depending on the direction of the rotation, intersects the graph of the (new) cubic function in no more than one point.

In the first case its derivative is either always greater than or equal to –1, or always less than or equal to 1. In fact, suppose that the derivative is greater than 1 at the point x_1 and is less than 1 at the point x_2. Then if we draw through the corresponding points of the graph straight lines parallel to the bisector of the first and third quadrants, we'll see that at least one of them intersects the graph two (or more) times. Thus the derivative is either always less than or equal to 1 (which is impossible) or always greater than or equal to 1; or (in the second case) it is always less than or equal to –1 (which is also impossible), or always greater than or equal to –1.

Finally we come to the problem: find all a such that for all x either

$$3x^2 + 2ax + 19 \geq 1, \text{ or}$$
$$3x^2 + 2ax + 19 \geq -1.$$

It is sufficient to consider only the second inequality. This leads to $-2\sqrt{15} \leq a \leq 2\sqrt{15}$.

Physics

P216

The force of friction that slows the bead is determined by the normal force. In our case, this force is equal to the centripetal force (directed at the center of the ring) that the ring exerts on the bead:

$$N = \frac{mv^2}{R},$$

$$F_{fr} = \mu N = \frac{\mu m v^2}{R}.$$

In a short period Δt the bead's velocity decreases by

$$\Delta v = a\Delta t = \frac{F_{fr}}{m}\Delta t$$
$$= \frac{\mu v^2 \Delta t}{R} = \frac{\mu v \Delta s}{R}.$$

It's clear that in a given short distance the bead's velocity will decrease by a corresponding fraction. For instance, if after traveling 1 cm the bead's velocity has dropped to 0.99 of its initial value, then after traveling 5 cm its velocity will decrease to $(0.99)^5$ of its initial value.

We can use this property to calculate the bead's final velocity after traveling any distance along the ring. First, let's determine the small segment Δs_0 where the velocity drops to, say, 1 – 0.001, or 0.999, of its initial velocity:

$$\Delta s_0 = \frac{0.001R}{\mu}$$

The entire path $s = 2\pi R n$ contains $N = s/\Delta s_0 = 1{,}000n \times \pi\mu$ such segments, so the bead's velocity at the end will be

$$v = v_0\left(1 - \frac{1}{1000}\right)^N$$
$$= v_0\left(1 - \frac{1}{1000}\right)^{1000N/1000}$$

A simple transformation of the index reveals (to those who have not yet guessed) the base of the well-known natural logarithm, $e = 2.71828...$. Although we started with a rather large number (1,000), we can substitute any number in the formula to confirm our reasoning. Thus we can rewrite our equation as follows:

$$v = v_0\left(1 - \frac{1}{1000}\right)^{1000N/1000}$$
$$= v_0\left(\frac{1}{e}\right)^{n2\pi\mu} = 0.152v_0.$$

Therefore, after three revolutions the bead's velocity will decrease by a factor of $1/0.152 \cong 6.58$.

P217

The temperature of a gas is defined by the mean kinetic energy of its molecules:

$$\frac{1}{2}mv^2 = \frac{3}{2}kT,$$

where k is Boltzmann's constant. This means that the higher the temperature of a gas, the higher the average velocity and average momentum of its molecules.

If the temperature of the vessel's walls is equal to that of the gas, then after coming in contact with the wall, a molecule of the gas will change the sign of its momentum $\mathbf{p}_0$ to $-\mathbf{p}_0$, but not its value. Therefore, the change in momentum is $2\mathbf{p}_0$. If $T > T_1$, the gas molecules will heat up when they come into contact with the wall and acquire additional velocity (fig. 3a). In this case, the change in momentum will be greater than $2\mathbf{p}_0$.

If the wall of the vessel is cooler than the gas, the molecules will lose energy and velocity after coming in contact with the wall. (fig. 3b). In

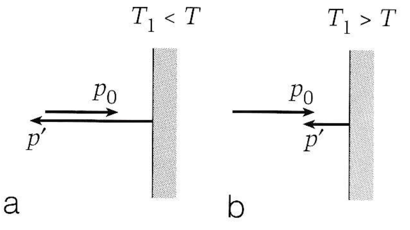


Figure 3

this case, the change in momentum will be less than $2\mathbf{p}_0$. Newton's second law says that a change in momentum is proportional to the mean force acting on the molecules from the walls. According to Newton's third law, the mechanical force acting on the molecules equals the force acting on the walls. Thus at $T_1 < T$, the pressure of the gas on the walls is greater than it is when $T_1 > T$.

P218

The total charge of the probe is zero, so the electric charges induced on its surface can be combined mathematically into pairs of equal yet opposite charges—that is, into dipoles. Consider such a dipole with a distance between charges $a_i << L$, where L is the distance between the probe and the ball. The charges of this dipole are $+q_i$ and $-q_i$. The dipole reduces the potential at the center of the ball by

$$\Delta\phi = \frac{kq_i}{L} - \frac{kq_i}{L+a_i} = \frac{kq_i a_i}{L^2}.$$

The center of the ball is a convenient point for the calculation because the potential generated at this point by the ball's charges does not depend on the charge distribution along the ball's surface.

A dipole is affected by the Coulomb force of the ball:

$$F_i \cong \frac{kQq_i}{L^2} - \frac{kQq_i}{(L+a_i)^2} = \frac{2kQq_i a_i}{L^3} = \frac{2\phi\Delta\phi_i}{k},$$

where Q is the electric charge of the ball and $\phi = kQ/L$ is its potential at a distance L from the ball's center. By summing the forces acting on the dipoles, we get the total force affecting the probe. The same force acts on the ball—it is attracted to the probe inserted into the ball's electric field. Summing small additions to the ball's potential results in the total change in this potential, which is given in the statement of the problem—that is, $\Delta\phi = 1$ V. Thus the force we seek is

$$F_i = \frac{2\phi\Delta\phi}{k} \cong 2.2 \cdot 10^{-6}\, N.$$

P219

It's clear that the force described is directed perpendicular to the plane of the ring. One possible way to find the points corresponding to the maximal force is to take two arbitrary points, position the circuit at some angle to the magnetic field vector, and obtain the formula for the total magnetic force. Then it would be possible to determine the maximal conditions for this formula. This is a rather long and tedious way to solve this type of problem. Instead, let's try to guess the answer!

Clearly we have two choices—either to connect the source at diametrically opposite points and to orient this diameter perpendicular to the field, or to choose two neighboring points (to obtain the maximum current between them) and again turn this segment perpendicular to the magnetic field. In either case, we'll then need to calculate the net magnetic force and compare the results.

With the first approach (connecting at diametrically opposite points), the currents through each half of the ring is

$$I_1 = \frac{V_0}{R/2} = \frac{2V_0}{R}.$$

The force acting on an arbitrary segment is determined by the projection of its length on the diameter, so the total force is

$$F_1 = 2I_1 Bd = \frac{4V_0 Bd}{R}.$$

With the second approach (connecting the neighboring points), the force is entirely determined by the action of the field on the small segment of the ring because the forces affecting the other parts of the ring are practically counterbalanced. The current in the tiny segment of the ring is

$$I_2 = \frac{V_0}{R\alpha/(2\pi)},$$

where α is the small angular size of the segment as viewed from the center of the ring. The force we seek is

$$F_2 = I_2 B\alpha\frac{d}{2} = \frac{\pi V_0 Bd}{R} < F_1.$$

It turns out that the first approach yields the maximal force acting on the ring.

P220

After refraction at the air–glass boundary, all beams that travel from the source to the screen are confined inside the cylinder, and after multiple reflections from its side (at the glass–air boundary) they eventually pass through the opening in the screen.

Indeed, the limiting ray that hits the cylinder's end from the air at an incident angle $\pi/2$ will, after refraction, form the critical angle α with the cylinder's axis. This angle can be found using the equation $\sin\alpha = 1/n$. This beam hits the side of the cylinder at an angle $\phi = \pi/2 - \alpha$ (fig. 4). Since

$$\sin\alpha = \frac{1}{n} = \frac{2}{3} < \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4},$$

then $\alpha < \pi/4$, and $\phi > \pi/4 > \alpha$. Therefore, this beam is totally internally reflected when it strikes the side of the cylinder. Subsequently this beam will continue to be reflected inside the cylinder until it finally reaches the other end of this "light conductor."

Any beam that strikes the surface of the cylinder at an angle less than $\pi/2$ will, after refraction, form an angle $\alpha' < \alpha$ with the cylinder's axis. Therefore, it will strike the side surface at an angle $\phi' > \phi > \alpha$ and will necessarily be totally internally reflected.

Thus the transparent cylinder will direct through the opening all

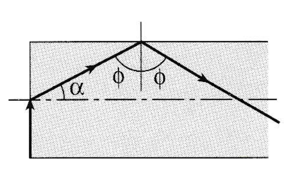


Figure 4

beams that were radiated into a solid angle of 2π steradians. In the absence of such a cylinder, only a small portion of the light will be directed through the opening. Its intensity will be limited by the solid angle $\pi d^2/4l^2$ steradians. Therefore, the cylinder increases the light through the opening by a factor of

$$n = \frac{2\pi}{\pi d^2/4l^2} = \frac{8l^2}{d^2} = 8 \cdot 10^4.$$

Brainteasers

B216

The largest sum of three sides of a die is 4 + 5 + 6 = 15. So we are looking for two numbers, each less than 15, that add up to 27. The only possibilities are 14 and 13, or 15 and 12. But in fact the sum of three visible faces of a die cannot be 13, so the total number of dots on one die is 15, with 12 dots on the other die.

To see that the sum of three visible faces cannot be 13, we can argue case by case. Since $4 \times 3 = 12 < 13$, there must be a 5 or a 6 in a sum of 13. If the largest is 5, the only possibilities are 5 + 5 + 3 or 5 + 4 + 4, but there is only one of each number on a die.

B217

We can think of the perimeter of the quadrilateral as a string with its endpoints attached to points A and B, and with two beads on it. The first

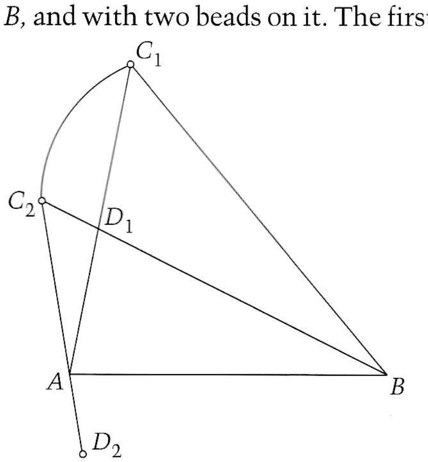


Figure 5

bead, for the position of point C, is 6 units from the endpoint attached to B. The second, for the position of D, is 1 unit from the endpoint attached to A. Beads C and D are 4 units apart. To construct a quadrilateral as specified, we must hold the string taut, and it must not intersect itself or segment AB.

It is not difficult now to see that C varies along an arc of a circle with radius 6 centered at B, and that D varies along an arc with center A and radius 1. But what are the endpoints of these arcs? To find these, we place the string at "extreme" positions. One extreme position is where C, D, and A are collinear (labeled C_1, D_1, and A in figure 5) so that the quadrilateral is actually a triangle. The other extreme position is when C, D, and A are again collinear, but with A between C and D. This position is marked C_2, D_2 in the figure. If D moved along the circle further to the left, the string segment from D to C would intersect AB.

A detailed geometric expression is a bit more complicated to express.

B218

When the ground is very cold (–10°F or lower), the ice crystals that make up the snow will not melt under your weight. They'll snap instead, which produces the crunching noise.

B219

This quantity is equal to 1:

$$\left(\sqrt[3]{2}+1\right)\sqrt[3]{\frac{1}{3}\left(\sqrt[3]{2}-1\right)}$$
$$= \sqrt[3]{\left(\sqrt[3]{2}+1\right)^3 \frac{1}{3}\left(\sqrt[3]{2}-1\right)}$$
$$= \sqrt[3]{\left(2+3\sqrt[3]{4}+3\sqrt[3]{2}+1\right)\frac{1}{3}\left(\sqrt[3]{2}-1\right)}$$
$$= \sqrt[3]{\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)\left(\sqrt[3]{2}-1\right)}$$
$$= \sqrt[3]{\left(\sqrt[3]{2}\right)^3 - 1} = 1.$$

B220

A bit of experimentation will make it clear that it is the center cell that must be removed (see figure 6).

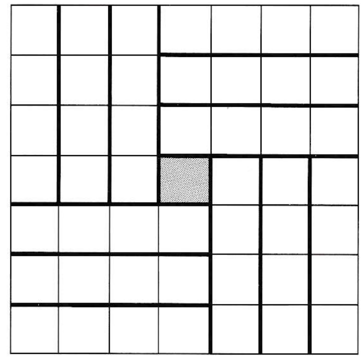


Figure 6

Kaleidoscope

1. Positive

2. No, not in every case. There can be no potential difference if the conductors are placed in the field generated by other charged bodies (see, for example, figure 7, where

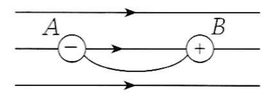


Figure 7

balls A and B (connected by a conducting wire) are charged by induction in an external homogenous electric field.

3. The ball has the same potential throughout its volume. It's equal to the potential at the ball's center created by the point charge: $V = q/4\pi\varepsilon_0 r$. (The potential generated at the ball's center by the charges induced on its surface is zero.)

4. No, it does not.

5. The sphere's potential becomes zero.

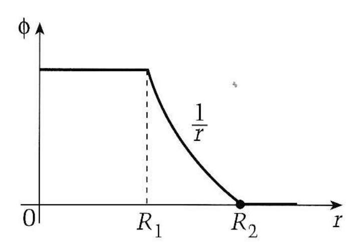


Figure 8

6. See figure 8.

7. Not necessarily. For example, in the case shown in figure 9, the entire charge of the conductor with

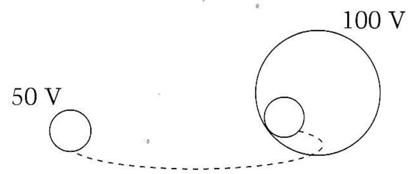


Figure 9

a potential of 50 V will flow to the conductor charged initially with a potential of 100 V.

8. No, it will not, because the potential is the same over the entire surface of the object. A long wire is necessary to prevent the potential of the charged body from affecting the readings of the electroscope.

9. The surface consists of the plane midway between the plates and the ball's surface (fig. 10).

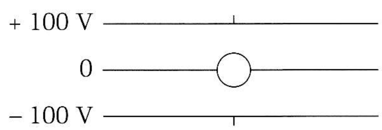


Figure 10

10. To a height of h.

11. No, it will not: the initial and terminal points of the object's trajectory belong to the same equipotential surface.

12. Not necessarily, because the connected capacitor can change the potential difference between points A and B (as seen in the circuit shown in figure 11).

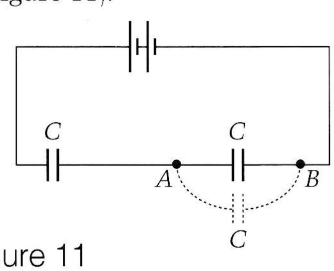


Figure 11

13. It will decrease by a factor of two.

14. Electric current "flows away" symmetrically from a fallen power line (figure 12). The current is greater the closer one is to the wire. The closer one is to the wire, the greater the potential difference between two points on the ground, and thus, the greater the risk of electric shock. The same is true for a tree struck by lightning.

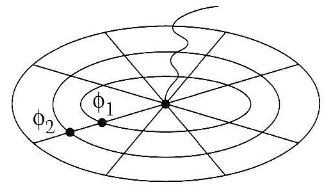


Figure 12

15. This occurs in a ring threaded by magnetic flux that varies uniformly with time, for instance.

16. It's impossible without a wire that is at rest relative to the Earth.

17. The violet light will kick the electrons from the surface of the ball via the photoelectric effect. Therefore, the ball's potential will grow constantly. When the potenial energy of the electrons in the ball's field equals the kinetic energy of the electrons kicked from the ball's surface, a state of dynamic equilibrium is established between the electrons entering and leaving the ball. The ball's charge will not change thereafter.

Microexperiment

The lamp can't be lit because connecting these points with a conductor will immediately equalize their potential. There is no danger to you because when you stand on the ground, you and the ground form an equipotential surface, so the potential difference between your head and your heels is zero.

Gradus

1. Suppose quadrilateral $ABCD$ is inscribed in a circle. Then angle A is measured by $(^1/_2)\overset{\frown}{BCD}$ (fig. 13), and angle C is measured by $(^1/_2)\overset{\frown}{BAD}$. But $\overset{\frown}{BAD} + \overset{\frown}{BCD} = 360°$, so $\angle A + \angle C = 180°$. Since the sum of the angles of the quadrilateral is 360°, $\angle B + \angle D = 180°$ as well.

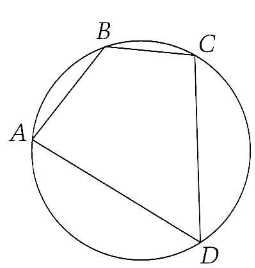


Figure 13

The converse of this statement is also true. Indeed, we can always draw a circle through three of a quadrilateral's four vertices—say, A, B, and D. Let the degree-measure of angle A be a. Then, in this circle, $\overset{\frown}{BD}$ (the arc that does not contain point A) measures $2a$, and $\overset{\frown}{BCD}$ is $360 - 2a$. If $\angle A + \angle C = 180°$, then $\angle C = 180 - a$, which is half of $\overset{\frown}{BCD}$. Therefore, point C must be on the circle that goes through A, B, and D.

2. In figure 14, AQ, BP, CS, and DR are the bisectors of the angles of quadrilateral $ABCD$. We must show that quadrilateral $PQRS$ is cyclic, or,

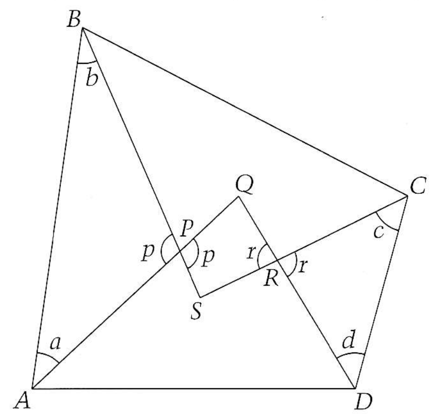


Figure 14

equivalently, that angles QPS, QRS are supplementary. Let p denote the degree-measure of $\angle QPS$, and let r denote that of $\angle QRS$. Then $\angle BPA = p$ and $\angle CRD = r$. From triangles ABP, CRD (see figure 14), $p + a + b + r + c + d = 180 + 180 = 360$. But the sum $a + b + c + d$ contains a copy of half of each angle of the quadrilateral, so this sum equals $360/2 = 180$. It follows that $p + r = 180$, and quadrilateral $PQRS$ is cyclic.

What does $PQRS$ look like if $ABCD$ is a parallelogram? A rectangle? A rhombus? A square? A kite?

3. In figure 15, $\angle E = (^1/_2)\overset{\frown}{MD}$. But

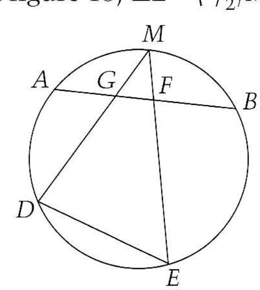


Figure 15

$\angle AGD = \frac{1}{2}(\widehat{BM} + \widehat{AD}) = \frac{1}{2}(\widehat{MA} + \widehat{AD}) = \frac{1}{2}\widehat{MD}$. So $\angle E = \angle AGM$, which means that $\angle E$ is supplementary to angle DGF. Therefore, quadrilateral $DEFG$ is cyclic.

What happens if either of the lines MD or ME cuts line AB outside of segment AB?

4. In figure 16, H' is the reflection of the orthocenter H of triangle ABC in side AB. We must show that H' lies on the circumcircle of the triangle. This is true if $ACBH'$ is a cyclic quadrilateral, or equivalently if angle $AH'B$ is supplementary to angle ACB. But $\angle AH'B = \angle AHB$ (by reflection), which in turn equals $\angle PHQ$, and angle PHQ is certainly supplementary to $\angle PCQ$, because quadrilateral $PCQH$ is cyclic (its other two angles are right angles, and thus supplementary).

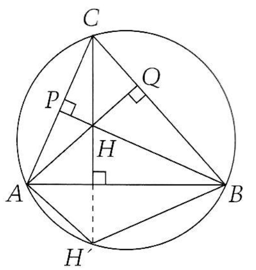


Figure 16

5. In figure 17, AB subtends a right angle at P and also at Q. This means that P and Q are on the circle with diameter AB.

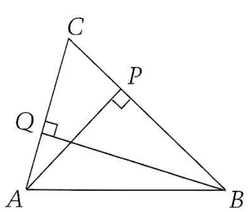


Figure 17

6. By the result of problem 5, quadrilateral $ABPQ$ is cyclic. This means that $\angle APQ = \angle ABQ$ (they both intercept arc AQ on the circle through A, B, P, and Q). For the same reason, (using cyclic quadrilateral $ACPR$), $\angle APR = \angle ACR$ (fig. 18).

But angle CAB is complementary both to $\angle ACR$ (in triangle ARC) and $\angle ABQ$ (in triangle AQB). Therefore, $\angle ACR = \angle ABQ$, so $\angle APR = \angle ACR$. Therefore, $\angle ACR = \angle ABQ = \angle APQ$, and $\angle APQ = \angle APR$.

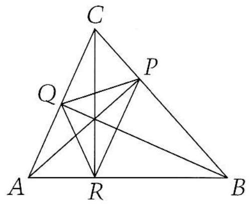


Figure 18

This shows that AP bisects angle QPR. In a similar manner, we can show that the other two altitudes of triangle ABC are angle bisectors of triangle PQR.

7. Let the center of the circle be O, and let the fixed point be labeled P. We can experiment by drawing the diameter through P (fig. 19).

The "perpendicular" to this diameter from O is just the point O itself, which must be a point of the locus. If we draw the chord through P perpendicular to this diameter, we find that point P lies on the locus as well. Furthermore, it is not difficult to see that diameter OP is a line of symmetry for the locus. This leads us to suspect that the locus is a circle with diameter OP.

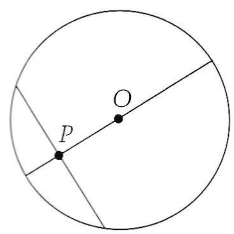


Figure 19

To confirm our suspicions, we take any chord AB through P inside O, together with its midpoint M (fig. 20). Line OM is then perpendicular to AB. This means that the segment OP subtends a right angle at the midpoint of any of our chords, and the locus of midpoints is a circle with diameter OP.

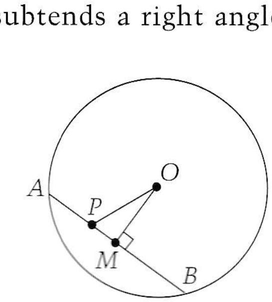


Figure 20

If P is on the circle, the locus is a circle that is tangent internally to the given circle. If P is outside the circle, the locus is that portion of the circle with diameter OP that lies inside the given circle.

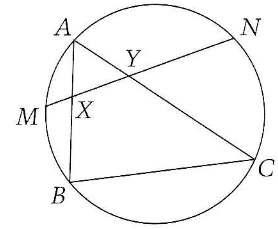


Figure 21

8. In figure 21, M and N are midpoints of $\widehat{AB}$ and $\widehat{AC}$, respectively. We have $\angle AXN = \angle MXB = \frac{1}{2}(\widehat{MB} + \widehat{AN}) = \frac{1}{2}(\widehat{AM} + \widehat{NC}) = \angle AYM$. Thus triangle AXY is isosceles, and $AX = AY$.

9. Figure 22 shows situation (a), with two possible positions (CD and $C'D'$) of the chord CD. Since $CD = C'D'$, we know $\widehat{CD} = \widehat{C'D'}$. So $\angle I = \frac{1}{2}(\widehat{AB} - \widehat{CD}) = \frac{1}{2}(\widehat{AB} - \widehat{C'D'}) = \angle I'$.

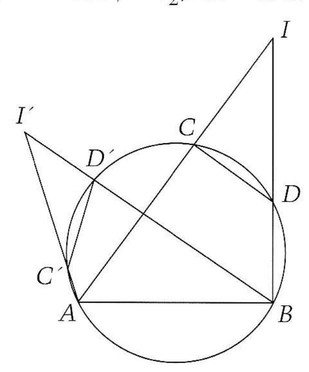


Figure 22

Thus both I and I' are on an arc of a circle passing through A and B.

Figure 23 shows situation (b). In this case, angle AKB is half the sum

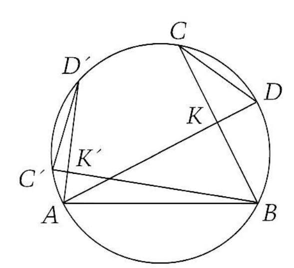


Figure 23

of $\widehat{CD}$ and $\widehat{AB}$, so this angle again does not depend on the position of chord CD. The locus is an arc of another circle.

What can you say about the center and radii of these two circles?

10. (a) Let a be the measure of $\angle AMB$ and note that a does not depend on the position of point M on the circle. Then triangle PMB is isosceles, and $\angle APB$ measures $180 - (90 - a/2) = 90 + a/2$ (fig. 24). Since this also does not depend on the position of M, point P varies along an arc of a circle through A and B.

(b) In the same way (from isosceles triangle BMQ) we find that $\angle AQB = a/2$, so Q varies along another arc through A and B.

(c) An analogous proof will show that these two new loci are again arcs of circles. The four arcs form two circles. Surprisingly, the arc of

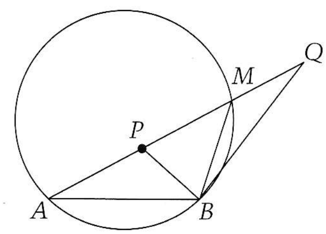


Figure 24

P when M is on major arc AB completes a circle with the arc of Q when M is on the minor arc, and vice versa.

One can guess at the locus by taking the special cases where M coincides with A (so line MA is tangent to the circle), where M coincides with B (so that P and Q also coincide), and where M is the midpoint of either arc (so that P coincides with A). In all cases, note that $\angle PBQ$ is a right angle.

11. In figure 25, quadrilateral

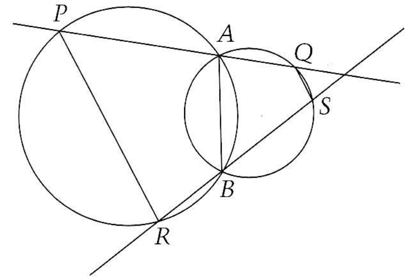


Figure 25

$PABR$ is inscribed in the larger circle. Thus $\angle PRB$ is supplementary to $\angle PAB$. Clearly $\angle BAQ$ is supplementary to $\angle PAB$, so $\angle PRB = \angle QAB$.

Similarly, quadrilateral $ABSQ$ is inscribed in the smaller circle, so $\angle QSB$ is supplementary to $\angle QAB$ and thus is supplementary to $\angle PRB$.

But this says that $PR \parallel QS$, since a pair of consecutive angles (along transversal RS) are supplementary.

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Senior Vice President
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Hindsight

Know when to hold 'em and when to fold 'em

by Dr. Mu

WELCOME BACK TO COWCULATIONS, THE column devoted to problems best solved with a computer algorithm.

The other day, I was out in the pasture relaxing when Bessie, a Holstein friend of mine, happened by and struck up a conversation. She had been working on a cowculation over the weekend, was very proud of her solution, and just had to tell someone. So I listened. Seems that she had been out in the barnyard recently where Farmer Paul showed her some interesting farm facts. His figures revealed there were 100 animals (cows plus chickens) on the farm, with a total of 324 legs. What he was trying to cowculate was the number of cows and the number of chickens. But Farmer Paul had been away from algebra for many years and didn't like equations anyway, so he said, "Bessie, can you cowculate this for me without using any equations?"

Bessie, who is proud of her ability to find the unusual solution, ruminated on the problem and soon announced she had an insight. "Suppose," she said, "all the cows stand on their hind legs?" Then, clearly, there must be a total of 200 legs on the ground. That leaves 324 less 200, or 124, legs up in the air. But they all belong to the cows, so there are 62 cows, leaving 38 chickens.

Art by Mark Brenneman

I was very impressed with her insightful approach but couldn't resist commenting, "Bessie, that wasn't insight, that was hindsight!"

Hindsight is what I needed last month when I went on an extended gambling spree. Each night after milking, I took off for the Half Moo Inn in Cheddarville, Wisconsin, to relax in a friendly game of poker. To cut any big losses, I promised I would lose no more than \$50 a night. I kept a record of my winnings and losings for the month.

```
winnings = {29, -7, 14, 21, 30, -47, 1, 7,
   -39, 23, -20, -36, -41, 27, -34, 7, 48, 35,
   -46, -16, 32, 18, 5, -33, 27, 28, -22, 1,
   -20, -42};
```

I started out with a win of \$29 the first day, but finished out with a \$42 loss on the last day. My total winnings for the month were:

```
Apply[Plus, winnings]
-50
```

I guess just getting away from the farm for a few hours was worth \$50 in entertainment value. But looking back with 20/20 hindsight, I wonder how much money I would have been ahead if I had known when to quit—when to hold 'em and when to fold 'em.

Turns out that if I had started playing on the 16th day and stopped after the 26th, I would have walked away with \$105. That's a \$155 shift in my fortunes, which would go a long way in cow comforts—even today.

This raises an interesting programming problem, which is your "Challenge Outta Wisconsin."

COW 7. Given a sequence $\{x_1, x_2, \ldots, x_n\}$ of integers of length n, write an efficient algorithm (of order n) to find a subsequence $\{x_L, x_{L+1}, \ldots, x_H\}$ of consecutive terms with the largest sum. Return the beginning and ending indices L, H, and the maximum sum $\sum_{i=L}^{H} x_i$. If there is more than one such subsequence, return any one.

A bar chart

Here is a graphical view of my winnings:

Solution one—brute force

Why not just try all possibilities? For each lower index L from 1 to n and each higher index H from L to n, sum the subsequence $\{x_L, x_{L+1}, \ldots, x_H\}$ and see if it's the best so far. If it is, save it along with the indices L and H. Here is this algorithm in Mathematica. A version of this problem and solution appears in *Programming Pearls,* by Jon Bentley (Addison-Wesley, 1986).

```
bruteForceOne[X_] := Module[
   {MaxSoFar = 0, L = H = 0, Low = High = 0,
     SubSub = 0,
   n = Length[X], j},
   For[L = 1, L <= n, L++,
     For[H = L, H <= n, H++, SubSum =
         Sum[X[[j]], {j, L, H}];
       If[SubSum >= MaxSoFar, Low = L; High =
         H];
       MaxSoFar = Max[MaxSoFar, SubSum]
       ]
   ];
   {MaxSoFar, {Low, High}}
]
```

```
bruteForceOne[winnings] // Timing

{0.44 Second, {105, {16, 26}}}
```

Not bad for a first approach, but not very efficient. For large values of n it slows down dramatically since the number of computer operations (thus the time) is proportional to n^3. One n for each loop and one n for the sum computation. To get an estimate for the time it would take to find an answer for a sequence of length 1,000, consider the following cowculations:

```
Clear[timeInSeconds, timeInHours]
solution = Solve
   [timeInSeconds/1000^3 == .44 seconds/30^3, timeInSeconds]
{{timeInSeconds -> 16296.3 seconds}}
timeInSeconds = timeInSeconds /. First[solution]
16296.3 seconds
timeInHours = timeInSeconds/(3600 seconds/hour)
4.52675 hours
```

I'd like to decrease this time a bit. Can we do better?

Solution two—better brute force

We can cut it down by one power of n by cowculating the sum as we move along. Here is the algorithm given by Bentley, which runs in time proportional to n^2:

```
bruteForceTwo[X_] := Module[
   {MaxSoFar = 0, L = H = 0, Low = High = 0,
     SubSub = 0,
   n = Length[X], j},
   For[L = 1, L <= n, L++,
```

```
      SubSum = 0;
      For[H = L, H <= n, H++, SubSum = SubSum +
         X[[H]];
         If[SubSum >= MaxSoFar, Low = L; High =
            H];
         MaxSoFar = Max[MaxSoFar, SubSum]
         ]
   ];
   {MaxSoFar, {Low, High}}
]
bruteForceTwo[winnings] // Timing

{0.11 Seconds, {105, {16, 26}}}
```

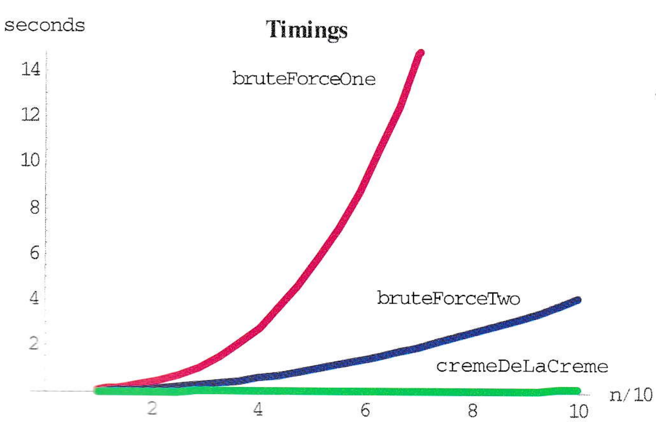
That's a bit better. To get an estimate for the time it would take to find an answer for a sequence of length 1,000 consider the following cowculation:

```
Clear[timeInSeconds]
Solve
[ timeInSeconds/1000^2 == .11 seconds/30^2 , timeInSeconds ];
timeInSeconds/(60 seconds/minutes) /. First[%]

2.03704 minutes
```

Speed is everything

The following graph shows how timings grow with n for all algorithms from the least efficient to the most efficient. Your challenge is to find the most efficient. I want the crème de la crème solution that will run in less than one second for n equal to 1,000.



Time is all important here.
Brute force is out, this is clear.
To do it right,
You'll need insight,
For the crème de la crème to appear.
—Dr. Mu

Computer art

Mark Brenneman's picture of me playing poker is an early attempt to branch out into 3D. Mark started out using the software Ray Dream Studio to build the walls in 3D blocks. Next he added the props—bar, chairs, tables, and so on. Then we used the software Poser to pose the players and export them to Ray Dream Studio. Next he created my beautiful body in Ray Dream Studio by joining over 40 separately drawn pieces. The objects were linked by joints which allowed me to move into just the right pose. PhotoShop and Detailer were employed to create objects that are attached to the surfaces to give texture to the art. Finally, the scene was assembled and 13 light sources identified. Ray Tracer was used to calculate what each pixel should look like considering all light sources and mirrors. It took Mark's 133 MHz Pentium 10 hours to render the final high-resolution drawing. Faster algorithms could really help in this field.

And finally...

The cowculations sent in on COW 6 will appear in the next issue. From now on solutions to COW_{n-2} will appear in COW_n. This gives all cowhands another two months to ruminate on possible solutions before they email them to me at drmu@cs.uwp.edu. Past solutions are available at http://usaco.uwp.edu/cowculations.

If competitive computer programming is your goal, then stop by the USA Computing Olympiad web site at http://usaco.uwp.edu. The 1997 USA team is leaving November 29 for the International Computing Olympiad in Cape Town, South Africa. This is an all-expense-paid trip for the crème de la crème high school computer programmers. Check it out and get on the mailing list for the 1998 season.

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