QUANTUM

JULY/AUGUST 1996

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Girl with a Hoop (1885) by Auguste Renoir

THE REASON FOR THE APPEARANCE OF "GIRL with a Hoop" in Gallery Q is quite straightforward: the hoop. It is a beguilingly simple thing. You can roll it—that's obvious enough. You can "jump rope" with it, which is not so obvious. You can spin it, so that it seems like a semitransparent globe. But the little girl will have a few years to wait before she understands, for instance, that the hoop's center of mass is nowhere to be found along the ring of the hoop itself—it's right in the middle, where there's no hoop at all!

When she has come to the point of studying physics, she will understand the equation for a hoop's "moment of inertia" given on page 6. She will also understand why the hoop rolls back to her when she tosses it forward with backspin. (How many times did she test this out, throwing it further and further, trying to see how far she could send the hoop and still have it obediently return?) Though Renoir's painting offers no evident clue, we must nevertheless conclude that this little girl will someday be able to read *all* of the article that begins on page 4.

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VOLUME 6, NUMBER 6

Cover art by Dmitry Krymov

When we dig a particularly deep hole in North America, we sometimes joke about digging "all the way to China." Well, if you allow yourself to be drawn into our cover, you may find yourself emerging on the other side of the cutting board—on page 10! It won't be an "infinite descent"—a very handy technique in mathematics that may trace its lineage back to the ancient Greeks. But then, you wouldn't want it to be—would you?

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FEATURES

DEPARTMENTS

PUBLISHER'S PAGE

"Can you hear me?"

Some thoughts on the history of human communication

EVERY TIME I SEND A HUGE file in a matter of seconds over a T1 line from my office in Nevada to NSTA headquarters in suburban Washington, D.C., I can't help marveling at the power at my fingertips. Not only has a mind-boggling amount of information—high-resolution pictures, countless words (I don't send audio or video clips, but others do)—been compressed into a few megabytes of digital "space."[1] This information is being transported over thousands of miles of geographic space.

It occurred to me that there are two basic aspects to communication. One is the software—the language, or system of symbols, we use to organize our thoughts. The other is the hardware—our vocal chords, a clay tablet, a telephone line, or a combination of several devices. I came across a marvelous book called *The Timetables of Technology* (Bunch and Hellemans, eds., Touchstone Books, 1993) and have spent many hours extracting from it my own timeline that includes both aspects of communications—the "linguistic" and the "technical." I hope you will find it as fascinating as I do and will excuse the greater length of this Publisher's "Page." Why not insert your own year of birth (as I did!) to get a perspective on what preceded you and what has occurred in your lifetime?

Over the history of *Homo sapiens*, from about 90,000 B.C. to the present, there has been a very slow evolution of human communications, from rudimentary marks on bones and stone to our modern electronic communication systems. A broad view of this history reveals only a few quantum leaps in the technology of communications. For example, it took some 60,000 years for humans to develop the ability to scratch simple counting numbers on bones. It took another 10,000 years or so to extend this ability to include pictures of animals and other objects. Some 7,000 years later, rudimentary writing began to occur in the form of hieroglyphics. It then took almost 2,000 more years, until about 1,000 B.C., for an alphabet to be created. This was one of those quantum leaps in communications.

90,000 B.C.: Fossil evidence of archaic form of *Homo sapiens* is present in Africa. **35,000 B.C.:** *Homo sapiens* is the only member of the genus left. The Neanderthals have disappeared, and technology advances swiftly. **30,000 B.C.:** Paleolithic peoples in central Europe and France used tallies on bone, ivory and stone to record numbers. For example, a wolf bone shows 55 cuts arranged in groups of five. Musical instruments from bird or bear bones were made. **25,000 B.C.:** Ceramic figurines emerge in Moravia. **20,000 B.C.:** Animal engravings are made in north-western Spain, and in France, carved pendants with allegorical scenes are made. **15,000 B.C.:** Figurines are made wearing clothing. A bone map of a region in Ukraine is created. **10,000 B.C.:** Polychrome red and black bison created on ceiling of Altamira Cave in Spain. In France, geometric designs are painted on pebbles. Fired clay tokens are made to express, as a form of written word, the idea of a number. **5,000 B.C.:** Cave painting near the White Sea shows people walking with planks attached to their feet, an early form of skis. **3,300 B.C.:** In Sumeria pictographs are used in horizontal strips on baked clay tablets. Egyptians begin to write using hieroglyphic signs on papyrus. **2,900 B.C.:** Egyptian scribes devise hieratic script, a simplification of hieroglphics. **1,800 B.C.:** An early form of cuneiform writing style emerges in Babylonia. **1,500 B.C.:** The first alphabets are created, one of these by stripping cuneiform characters to 30 signs, each standing for a sound. **1,000 B.C.:** The Phoenician alphabet of 22 signs for consonants is developed and will later be adapted by both the Greeks and the Israelites.

With an alphabet, records of the written word accumulated, so that, 700 years later, at about 300 B.C., some 750,000 "books" (papyrus scrolls) had been placed in the great library of Alexandria. This library was partially destroyed in 50 B.C., and the Christians destroyed a major portion of the library 440 years later, with the final destruction carried out by the Muslims in A.D. 650.

The Chinese invented paper in 150 B.C., and even though it was not at first used for writing, it became, over the centuries, used for money and for pages of books. By A.D. 600

[1]The question of how much "space" a computer file takes up is a curious one: *x* number of floppy disks, or *y* millimeters on a tape drive, or *z* infinitesimal electron "parking spaces" in a RAM chip "garage."

the Chinese had printed whole pages using a block, the precursor to the printing press.

300 B.C.: Some 750,000 "books" on papyrus scrolls, all known books, are placed in the library at Alexandria. **250 B.C.:** Parchment is invented, followed somewhat later by vellum, both sides of which can be written on. **150 B.C.:** The Chinese invent paper, but not for writing on. **100 B.C.:** The *Codex*, leaves of parchment sewn together—the first "book"—appears in Rome. **50 B.C.:** The great library at Alexandria is partially destroyed and many books are either destroyed or lost. Lucretius describes how the illusion of motion can be created by sequential display of frames. **0:** A dictionary of local expressions is made in China. **110:** The oldest piece of paper used for writing is in existence. **390:** Destruction of more of the library at Alexandria by Christians. **600:** The Chinese print whole pages with wood blocks. **650:** Final destruction of library at Alexandria by Muslims.

It took almost 700 more years before block printing was common in Europe, followed rapidly by the invention of the printing press with movable type by Gutenberg and Koster. A brief period of some 300 years produced dramatic improvements in the technology of written communications. Text and type appeared on the same page. Color separations were made to give color to printed materials. But the first book was not printed in the English language until 1474.

870: First printed book, the *Diamond Sutra*. **900:** Printed money is used in China. **1000:** Alhazen describes the camera obscura, precursor to the camera. **1050:** Chinese books are printed with movable type. **1,086:** The *Doomsday Book* is written in England. **1107:** The Chinese invent multicolor printing. **1215:** The Magna Carta is signed. **1290:** Block printing is used in Europe to print pages. **1379:** Cryptography is invented. **1390:** Metal type used for printing in Korea. **1396:** Gutenberg and Koster invent printing with movable type. **1420:** Schoeffer invents "color separation" printing, using blue and red ink. **1460:** Pfister combines woodcuts with movable type to give images and text on one page.

CONTINUED ON PAGE 37

QUANTUM

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A Venusian mystery

"Star light, star bright,
First star I see this morning . . ."
—Fractured nursery rhyme

by Vladimir Surdin

WHEN THE TITLE OF A SCIentific article intended for a general audience contains the word "mystery," one is tempted to turn to the end and look at the answer. In this case, don't bother—this particular astronomical paradox is still waiting to be explained. The purpose of this article is merely to acquaint you with it—maybe you'll be the person who manages to unravel this problem.

The riddle of her rotation

Venus is known to be very similar to our planet in mass and size. However, it's located somewhat nearer to the Sun and makes one revolution around it in 224.7 days (throughout this article "days" will mean "Earth days"). As for the rotation of Venus about its axis, astronomers knew nothing about it for a long time because the details of planet's surface can't be seen through the thick Venusian atmosphere. It was radar that made it possible to break through the planet's cloudy layer and learn that it rotated about its polar axis very slowly and in the direction opposite to its orbital revolution, making a complete turn in 243 days. In this respect Venus is strikingly different from its kindred planets (Earth and Mars), where the daily rotation proceeds in the direction of orbital revolution and the period is far shorter (approximately one day).

Venus offered yet another surprise. As astronomers observed the motion of the clouds in its atmosphere, they saw that the upper layers of the atmosphere moved by themselves, separate from the planet, revolving in the same direction but far more quickly—in just four days. Keep in mind that the atmosphere of Venus is significantly more massive and dense than the Earth's. Naturally some scientists were tempted to explain the paradoxical rotation of Venus by the interaction of the planet with its massive atmosphere.

For example, a manuscript was once sent to a journal with the title: "The Atmosphere of Venus is a Giant Heat Engine." The paper asserted that by absorbing the Sun's heat, the atmosphere of Venus could affect the planet's rotation. To prove it, the following estimate was made. The power of the solar radiation striking the Venusian atmosphere is about 10^{17} W. Thus the planet acquired 10^{34} J of energy over the course of its evolution (billions of years). If Venus rotated like the Earth (that is, with a period of 24 hours), its rotational kinetic energy would be about 10^{29} J. Evidently the atmosphere of Venus has received enough solar energy to stop the planet's rotation and start it rotating in the opposite direction many times. The authors of the manuscript were quite convinced that this estimate proved their hypothesis. Do you agree with them?

The "three pillars of mechanics"

How does one verify a new idea? First off, a physicist thinks about the *conservation laws*. In the problem of the Venusian rotation, the *law of conservation of energy* surely isn't violated. What other quantities must be conserved? There are two—*linear momentum* and *angular momentum*. All of classical mechanics rests on these three "pillars." As we examine the rotation of Venus, we are particularly interested in the angular momentum.

The capacity of an object to main-

Art by Pavel Chernusky

tain a rotation or transfer it to other objects is characterized by its *angular momentum*. There are many features common to both angular momentum and linear momentum.

Students are generally familiar with linear momentum, known in the old days as the "quantity of motion." This is a vector pointing in the direction of the object's motion and equal to the product of its mass and velocity:

$$\mathbf{p} = m\mathbf{v}. \qquad (1)$$

The total linear momentum is conserved when objects interact. One of many examples is that of jumping from a boat: you jump in one direction, the boat moves in the opposite direction. The law can be clearly observed in a skating rink: if you and your friend are standing on the ice in your skates and you push off from one another, you'll move off in opposite directions, and the lighter partner will travel with the greater speed.

However, it's not so easy to observe the implementation of this physical law on the ground. The reason is friction. An automobile starts to move, but the Earth stays put. Strictly speaking, the Earth also acquires a momentum in the opposite direction, but due to its huge mass the corresponding speed is negligible. So when the car's wheels grip the road, we forget about the conservation law: the recoil momentum always goes "into the ground," and the planet's motion doesn't change appreciably—in fact, it remains a convenient reference system. If we know the engine's power and the mass of the automobile, it's easy to calculate the time required to accelerate to a certain speed. To do so, we use only the law of conservation of energy and forget about the conservation of momentum.

In this respect, outer space is more like ice: one must not neglect momentum in space! Here's one example: if the nuclear engine of a spaceship of mass m develops power W, what will the acceleration of the spaceship be? Surprisingly, this problem can't be solved unless we know the mass of the substance ejected from the engine and its speed. The point is that the engine's power is expended not only on pushing the rocket in the forward direction, but also on accelerating the ejected substance (plasma, perhaps) in the opposite direction. In this process both the rocket and ejected substance have momenta that are equal (in magnitude)! In outer space, conservation of momentum is a serious thing.

Of equal importance is the law of conservation of angular momentum **L**, which is also a vector quantity. To illustrate how it works, consider an object that has an axis of symmetry and rotates about it. In this case the direction of the vector **L** coincides with that of the angular velocity vector $\boldsymbol{\omega}$—that is, the vector **L** is directed along the rotation axis such that, if we look toward it, we see the object rotating clockwise (the so-called right-hand rule). Recall that the magnitude of the angular velocity (measured in radians per unit time) is linked to the period of rotation T by the relation

$$\omega = \frac{2\pi}{T}.$$

For a rotating object the angular velocity plays the same role as its linear counterpart does for an object moving in a straight line. In the definition of linear momentum, given by equation (1), its value is linked to the (linear) velocity by the magnitude of the object's inertia—that is, its mass. The angular momentum and angular velocity are related in a similar way, but in this case the magnitude of the (angular) inertia is the "moment of inertia" I:

$$\mathbf{L} = I\boldsymbol{\omega}. \qquad (2)$$

For a physical point with mass m, the moment of inertia is calculated according to the relation

$$I = mR^2,$$

where R is the distance from the point to the axis of rotation. By the way, the angular momentum in this case can be written in another way:

$$L = mR^2\frac{2\pi}{T} = mR\frac{2\pi R}{T} = mRv, \qquad (3)$$

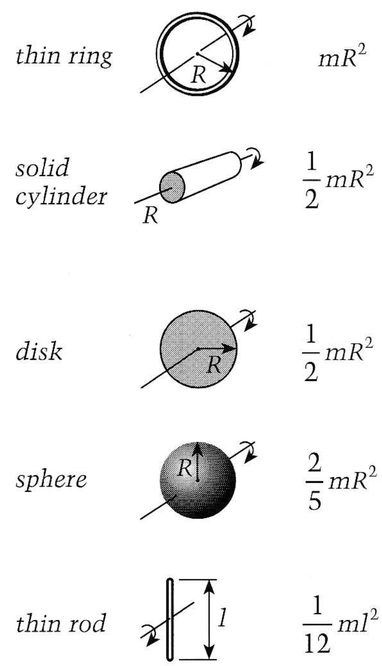
where v is the linear speed of the circular motion. The moment of inertia of a rigid system of bodies is the sum of its constituent parts:

$$I = \sum_{i=1}^{n} m_i R_i^2,$$

which corresponds to an integral over the entire volume of the continuous bodies:

$$I = \int R^2 dm.$$

Shown below are the moments of inertia of some objects that have simple shapes:



Now let's look at how the law of conservation of angular momentum works in a few examples.

Case 1. You sit down on a stool that can rotate about its vertical axis and take a massive wheel that can rotate about the vertical axis (fig. 1). Clearly the angular momentum of the system "person–stool–wheel" is zero. Now try to spin the wheel. In doing so you (together with the stool) begin to rotate in the opposite direction: the angular momentum of the wheel is compensated by the opposite angular momentum of the

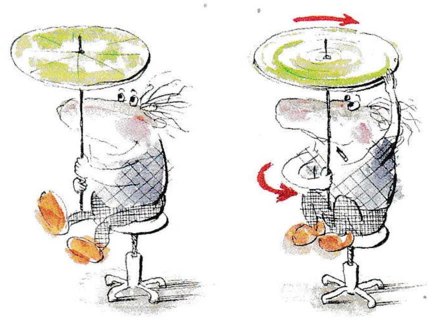


Figure 1

"person–stool" system. The total angular momentum remains zero.

Case 2. A figure skater rotates on the ice with her arms spread wide. Then she presses them quickly to her chest—and begins to rotate much faster (fig. 2). When the skater drew her arms closer to her axis of rotation, she decreased the moment of inertia of her body, which was compensated by an increase in the angular velocity ($\omega = L/I$).

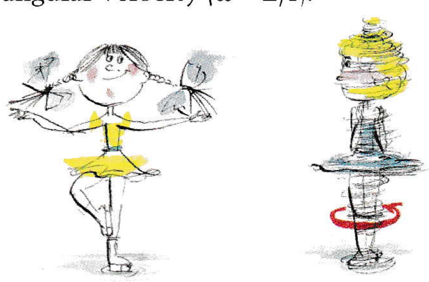


Figure 2

However, common experience tells us that, for bodies on the ground, the law of conservation of angular momentum is just as unimportant as the law of linear momentum. For example, let's modify the experiment with the rotating stool: now you spin the wheel about a horizontal axis—that is, perpendicular to the stool's axis (fig. 3). In this case the stool doesn't rotate! Where is the recoil angular momentum? It went "into the ground"! Naturally our huge planet didn't notice it.

Figure 3

Here's another example: the movement of a swing (fig. 4). Here we use the law of conservation of energy, noting the transition from potential energy to kinetic and back again, but in doing so we pay no attention to the law of conservation of angular momentum. Why? We can see at once that angular momentum is not conserved during the motion—the swing rotates alternately in both directions. Where is the "reservoir" that supplies (and takes away) the angular momentum to (and from) the swing? The Earth again, of course!

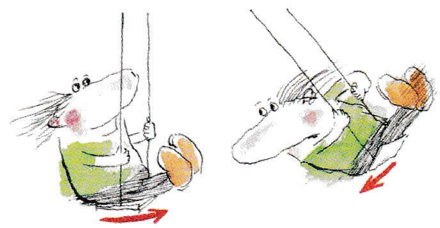


Figure 4

When solving common "earthly" problems, engineers rarely think about conservation of angular momentum. As a rule, all motors are firmly affixed to a massive platform and do not experience recoil rotation when the flywheel begins to move. In some particular cases—say, when one is designing a helicopter—the problem of angular momentum is very important. The helicopter's main rotor continuously imparts an angular momentum to the surrounding air, so a recoil momentum is imparted to the helicopter. To stabilize the aircraft, engineers use either two counterrotating rotors or a tail rotor that rotates in the vertical plane, located as far back from the rotation axis of the main rotor as possible. However, there are few other examples in terrestrial engineering.

On the other hand, when dealing with problems in outer space, one cannot avoid the laws of conservation of angular momentum and linear momentum. If you want to turn a spaceship, you need to fire the steering rockets (in this case the burnt fuel carries off the recoil momentum) or engage the gyrodynes, which are massive flywheels rotating in the direction opposite to that in which the spaceship is to be turned.

To change the momentum of an object, one must apply a force **F**; but to change the angular momentum, one must apply a torque (the moment of the force):

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}, \tag{4}$$

where **r** is the position vector from the point of rotation to the point where the force is applied.[1] Here we use the vector product, which takes into account the different possible directions of the vectors **r** and **F**.

For those who aren't familiar with vector products, we note that the resulting vector for the torque $\boldsymbol{\tau}$ is directed perpendicular to both vectors **r** and **F** and is oriented according to the right-hand rule. This force is equal to

$$\tau = rF \sin \alpha,$$

where α is the angle between **r** and **F** (fig. 5). The value $d = r \sin \alpha$ is

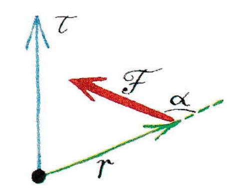


Figure 5

[1]Note that the general definition of the angular momentum of a point mass looks a lot like the definition of the torque in equation (4):

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where **r** is the position vector of the point. In the particular case of a point moving in a circle, we obtain formula (3). The angular momentum of a system of points is equal to the sum of the individual momenta, which results in formula (2) for a rigid object rotating about its symmetry axis. Generally, **L** and $\boldsymbol{\omega}$ are not parallel, but the objects considered here (a planet's atmosphere) have a high degree of symmetry. For objects with spherical symmetry, **L** and $\boldsymbol{\omega}$ are always parallel.—*Ed.*

referred to as the *moment arm*.

When a torque acts on an object for a time interval Δt, the object's angular momentum changes by $\Delta\mathbf{L}$:

$$\Delta\mathbf{L} = \Delta\tau\Delta t.$$

In addition, we mustn't forget that the rotating body accumulates kinetic energy E. Since $v = \omega R$, the value of this energy is

$$E = \frac{1}{2} I\omega^2.$$

It's clear that in this formula for the rotational kinetic energy the moment of inertia plays the role of the mass.

Looking for an answer

What about the enigma we started with? What could change the rotation of Venus so radically (assuming it was born a quite "normal" planet, rotating like Earth or Mars)? Undoubtedly a planet's atmosphere can transform the Sun's energy into mechanical work. Solar energy induces vertical and horizontal air flows that move clouds and sand dunes, push sailboats across the water, and rotate the vanes of windmills. But considering a planet as a system composed of a solid body and an atmosphere, we need to keep in mind the conservation of the total angular momentum: the planet and its atmosphere can exchange angular momentum, but they cannot change the total amount of it.

However large the Venusian atmosphere is, its mass is only 1/20,000 that of the entire planet, and the corresponding moment of inertia is 1/10,000 that of the planet. So if Venus rotated initially with a period of 24 hours and then practically stopped and transferred all its angular momentum to its atmosphere, the atmosphere would have to revolve around the planet with a period of 24 hours/10,000 = 8.6 seconds. Such a huge velocity is more than enough to overcome the gravity of Venus and fly off into outer space!

Therefore, we must look for external objects with which Venus could exchange angular momentum. It could be, for example, a satellite of Venus similar to our Moon. (By the way, the tides generated by the Moon slow the Earth's rotation, and in the future it will be very slow, with a period of tens of days. But that's the subject of another article.) A satellite could have done the trick for Venus, and then the planet lost it to the depths of space.

There is still another agent that can affect the rotation of Venus: sunlight! As noted earlier, the power of the solar radiation striking Venus is $W \cong 10^{17}$ W. Thus the force of the light pushing against the disk of the planet is $F = W/c \cong 3 \cdot 10^8$ N. If the reflective properties are uniform over the entire disk, then (due to symmetry) there will be no torque. However, if there is a persistent asymmetry in the Venusian atmosphere between the morning and evening hemispheres, the incoming flow of photons could transfer some angular momentum to the planet.

In the simplest case, if one hemisphere were black and the other white (absorbing and reflecting light, respectively), the light pressure could spin the planet so as to attain a period of a few days. Or, conversely, it could stop the planet's rotation, if it had the same period (a few days). The actual asymmetry of the Venusian atmosphere isn't so drastic, of course, but it exists nevertheless: there is a certain small difference between the morning and evening hemispheres in the altitude of some atmospheric layers and their reflective power. This is seemingly caused by heating of the atmosphere by day and cooling by night.

With the help of the simple formulas described in this article you can work out different possible scenarios for how the rotation of Venus evolved. Don't forget about the particles in the solar wind that bombard the planet—perhaps they, too, can modify its rotation. However, in each of your scenarios the total angular momentum of all the interacting participants must be strictly conserved!

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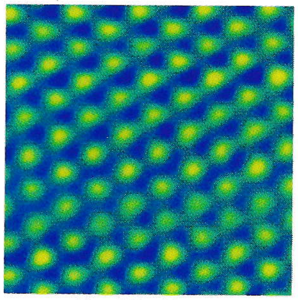
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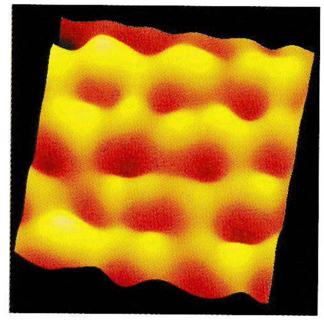
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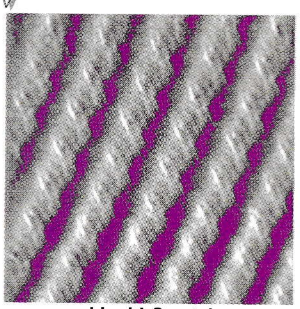
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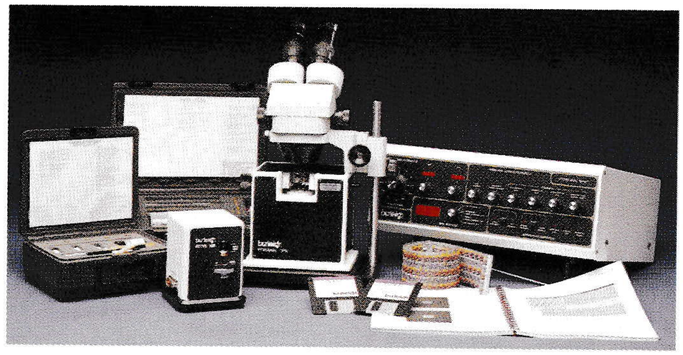

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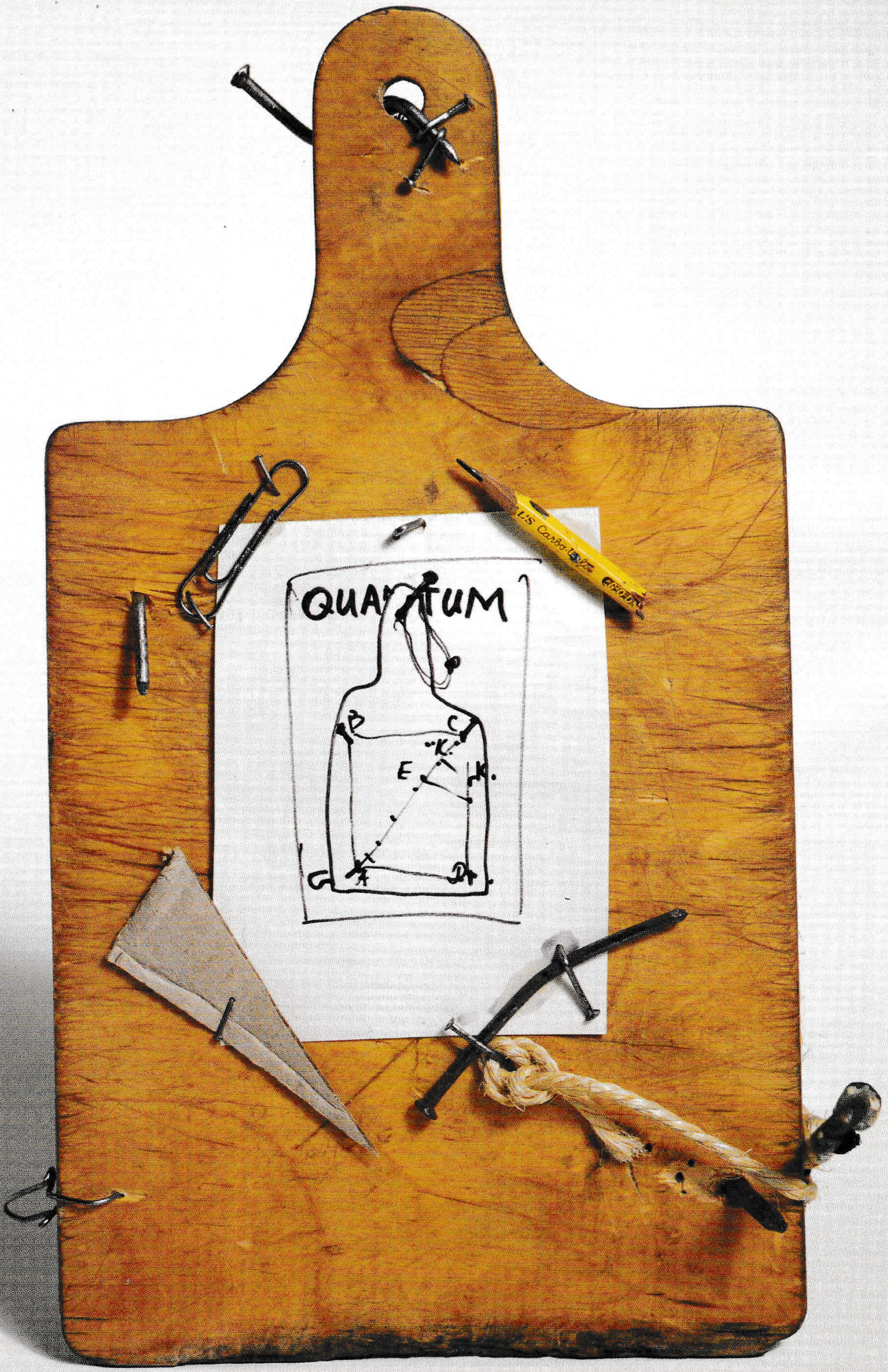
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Infinite descent

A method for getting to the bottom of a wide range of problems

by Lev Kurlyandchik and Grigory Rozenblume

WHICH IRRATIONAL NUMber is the "oldest"? Undoubtedly, $\sqrt{2}$. We don't know exactly who was the first to prove its irrationality, but we're sure the argument went something like this.

The first proof

Suppose $\sqrt{2}$ is a rational number. Geometrically, this means that the diagonal length c of a square is commensurable with its side length a—that is, there exist integers m and n and a segment of length a such that $c = dm$, $a = dn$. Mark $m - 1$ equally spaced points on the diagonal AC and $n - 1$ points on the side DC of a square $ABCD$ (fig. 1). Mark off the segment $AK = AD$ on the diagonal and the segment $DE = KC$ on the side. Their endpoints K and E will fit on points we've marked before. Let's prove that the triangles ACD and KEC are similar to each other. Since they have a common angle C, it suffices to prove that $KC/EC = CD/AC$.

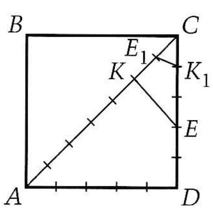

Figure 1

Art by Dmitry Krymov

We note that $KC = c - a$ and $EC = a - (c - a) = 2a - c$. Remembering that $c/a = \sqrt{2}$, we have

$$\frac{KC^2}{EC^2} = \frac{c^2 + a^2 - 2ac}{c^2 + 4a^2 - 4ac} = \frac{3a^2 - 2ac}{6a^2 - 4ac} = \frac{1}{2} = \frac{CD^2}{AC^2}.$$

Thus triangle KEC, like triangle DAC, is an isosceles right triangle, and we can repeat the above construction on its sides CE and CK, marking point K_1 on CE such that $EK_1 = KC$ and point E_1 on CK such that $KE_1 = CK_1$. These points will again be points of division; the triangle K_1CE_1 will again be an isosceles right triangle; and our construction can be reproduced anew to yield another similar triangle K_2CE_2. This process can be continued indefinitely. The triangles K_iCE_i thus obtained are getting smaller and smaller, while the points K_i, E_i always hit our initial division points. But the number of these points is finite! And the number of triangles K_iCE_i is infinite. This contradiction proves the irrationality of $\sqrt{2}$.

Centuries passed. An algebraic—and perhaps simpler—proof was invented.

The second proof

To prove that $\sqrt{2}$ is irrational means to prove that the equation $x^2 = 2y^2$ has no positive integer solutions. Suppose such solutions exist and $x = m$, $y = n$ is one of them.

It follows from the equation that m is an even number, so we can write $m = 2m_1$ for some integer m_1. Substituting this into the equation gives $n^2 = 2m_1^2$—that is, $x = n$, $y = m_1$ is also a solution. Notice that this second solution is "smaller" than the first: $n < m$, $m_1 < n$. Now we can apply the same reasoning to obtain a third solution $x = m_1$, $y = n_1$ (where $2n_1 = n$), which is even "smaller" than the second: $m_1 < n$, $n_1 < m_1$. Proceeding in the same way we'll obtaining ever decreasing solutions of the equation. But once again there's a contradiction up ahead! All the numbers m, n, m_1, n_1, ... are positive integers and they strictly decrease—$m > n > m_1 > n_1 > \ldots$, and an infinite decreasing sequence of positive integers is impossible. Therefore, our initial conjecture was erroneous and $\sqrt{2}$ is irrational.

Essentially, both these proofs followed the same scheme. Assuming that the problem has a solution, we constructed a certain infinite process; whereas, by its nature, the process must stop at a certain point. This method of proof is called the *method of infinite descent*.[1]

The method of descent is often used in a simpler form. Assuming that we have already reached the natural termination point of the process, we make sure that we "can't stop" at this point.

The third proof

Let $x = m$, $y = n$ be the solution to the equation $x^2 = 2y^2$ with the least possible x. The number m must be even, so we can write $m = 2m_1$, and $x = n$, $y = m_1$ is also a solution. But $m > n$, in contradiction to the choice of the solution (m, n) as the "smallest."

This version of the proof shows that the method of descent is related to the method of mathematical induction. They are both based on the fact that any nonempty set of positive integers has a smallest element.[2] The method of descent is especially useful in proving theorems that assert that some situation is not possible.

Now let's consider a number of examples illustrating the diversity of applications of infinite descent.

Diophantine equations

One of the areas in which the method is most frequently used is in the solution of equations in integers, often called Diophantine equations.

Problem 1. Prove that the equation

$$8x^4 + 4y^4 + 2z^4 = t^4$$

has no solution in positive integers.

Solution. Suppose solutions exist. Let $x = m$, $y = n$, $z = p$, $t = r$ be the solution with the smallest possible x. It follows from the equation that r is an even number, so we let $r = 2r_1$. Substituting this into the equation and dividing by two, we get

$$4m^4 + 2n^4 + p^4 = 8r_1^4.$$

Now we see that p is even, so let $p = 2p_1$. Then

$$2m^4 + n^4 + 8p_1^4 = 4r_1^4.$$

We proceed in the same way: $n = 2n_1$,

$$m^4 + 8n_1^4 + 4p_1^4 = 2r_1^4;$$

then $m = 2m_1$,

$$8m_1^4 + 4n_1^4 + 2p_1^4 = r_1^4;$$

and we have arrived at a solution, $x = m_1$, $y = n_1$, $z = p_1$, $t = r_1$, with a smaller x: $m_1 < m$! This contradicts the choice of the initial solution as the "smallest."

The next problem is a little more difficult.

Problem 2. Prove that the equation

$$x^2 + y^2 + z^2 + u^2 = 2xyzu$$

has no positive integer solutions.

Solution. Let x, y, z, u be a solution to our equation. Since the left side, $x^2 + y^2 + z^2 + u^2$, is an even number, there must be evenly many odd numbers—that is, four, two, or zero—among the numbers x, y, z, u. If all of them are odd, the left side of the equation is divisible by four, whereas the right side is not. Suppose that the set $\{x, y, z, u\}$ contains only two odd numbers. The identity $(2k+1)^2 + (2n+1)^2 = 4(k^2 + n^2 + k + n) + 2$ shows that the sum of any two odd squares leaves a remainder of 2 when divided by 4. On the left side of our equation, we are adding two odd squares and two even squares (which are multiples of 4), so the remainder upon division by 4 is still 2. This means that the left side of our equation is not a multiple of 4, while the right side is, and therefore there cannot be exactly two odd numbers among x, y, z, and u. So all the numbers are even: $x = 2x_1$, $y = 2y_1$, $z = 2z_1$, $u = 2u_1$. Substitute this into the equation and divide by four:

$$x_1^2 + y_1^2 + z_1^2 + u_1^2 = 8x_1y_1z_1u_1.$$

We see again that all the numbers can't be odd (otherwise, the left side could not be divisible by 8). Exactly two of the numbers can't be odd either, because in this case the left side wouldn't even be divisible by 4. So all of them again are even: $x_1 = 2x_2$, $y_1 = 2y_2$, $z_1 = 2z_2$, $u_1 = 2u_2$. Another substitution and simplification yields

$$x_2^2 + y_2^2 + z_2^2 + u_2^2 = 32x_2y_2z_2u_2.$$

Repeating the above argument again, we find that the numbers x_2, y_2, z_2, u_2 are all even, and so on. The kth step of the process yields the equation

$$x_k^2 + y_k^2 + z_k^2 + u_k^2 = 2^{2k+1}x_ky_kz_ku_k,$$

where all the variables are even integers. These integers were obtained from those in the initial solution by k successive divisions by two. Therefore, the numbers $x/2^k$, $y/2^k$, $z/2^k$, $u/2^k$ are integers for all $k \geq 0$. And this is, of course, impossible.

The next equation has infinitely many solutions, but can be investigated by the same method.

Problem 3. Find all positive integer solutions to the equation

$$x^2 - 2y^2 = 1.$$

Solution. One solution of the given equation is not difficult to guess: $x_1 = 3$, $y_1 = 2$. From the identity

$$(3x + 4y)^2 - 2(2x + 3y)^2 = x^2 - 2y^2,$$

it follows that for any solution x, y the pair $3x + 4y$, $2x + 3y$ is also a solution. This gives an infinite series of solutions:

$$x_2 = 3 \cdot 3 + 4 \cdot 2 = 17,$$
$$y_2 = 2 \cdot 3 + 3 \cdot 2 = 12;$$
$$x_3 = 99,\ y_3 = 70;$$

and so on. Let's prove that there are no other numbers satisfying the given equation.

Consider a solution x, y. Then $3x - 4y$, $3y - 2x$ is a solution too (this follows from the identity above, if we reverse the sign of y in it). Notice that $9x^2 - 16y^2 > 9x^2 - 18y^2 = 9$, so in particular $9x^2 - 16y^2 > 0$, which

[1] Apparently this method was invented by the Greeks. There is good reason to believe it was the method Fermat tried to apply to his Great Theorem.

[2] Two other methods, related even more closely to infinite descent, were described and discussed in previous issues of *Quantum*: the principle of taking the extreme (see "Going to Extremes" in the November/December 1990 issue) and the method of monovariants (see "Light at the End of the Tunnel" in the March/April 1994 issue).—*Ed.*

means that $3x > 4y$. Similarly, for $y > 2$ we have $4x^2 - 8y^2 = 4 < y^2$, so $4x^2 < 9y^2$ and $2x < 3y$. This means that our formulas transform the solution x, y with $y > 2$ into another positive integer solution $x^{(1)}$, $y^{(1)}$. Not only that, the obvious inequalities $y^2 < 1 + 2y^2 = x^2 < 4y^2$, or $y < x < 2y$, imply $x^{(1)} < x$, $y^{(1)} < y$ (this is left for the reader to check). Now we can form a third solution $x^{(2)}$, $y^{(2)}$ from $x^{(1)}$, $y^{(1)}$, as long as $y^{(1)} > 2$, and so on. This process can't go on forever—that is, at a certain step we'll arrive at a solution $x^{(n)}$, $y^{(n)}$ with $y^{(n)} \le 2$. Since $y^{(n)} \ne 1$ (because otherwise $(x^{(n)})^2 = 1 + 2(y^{(n)})^2 = 3$), we must have $y^{(n)} = 2$. Then $x^{(n)} = 3$, which means that the pair x, y belongs to the series of solutions we started with.

Combinatorial problems

Problem 4. The mass of each of $2n + 1$ weights is an integer number of grams. Any $2n$ of the weights can be divided into two groups equal in number (n weights in each) and in mass. Prove that all the weights are of equal mass.

Solution. Since the mass of any $2n$ weights is even, all the masses are either even or odd (depending on whether the total mass of all $2n + 1$ weights is even or odd). Subtract the mass of the lightest weight (or weights, if there are several) from each of the masses. Some of the weights now have mass 0, and we can easily check that the new system of weights satisfies the same condition. Since some of the new masses are zero, they are all even. Halving all of them, we again obtain a system of weights satisfying our condition. Zero weights are still present in the new system, so all the new masses remain even and we can halve them again. The process can continue in this way forever, which is possible only if all the masses are zero. But this means that the initial masses were equal to one another.

Now let's consider a problem from combinatorial geometry.

Problem 5. Is it possible to cut a cube into several different cubes?

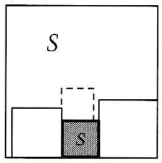

Figure 2

(Cubes are considered different if their edge lengths are different.)

Solution. We start by noticing that if a square S is cut into different squares, then the smallest of them—say, s—cannot border on a side of S. This is because in this case the square bordering on the side of s opposite its "outer" side (being limited by the "neighbors" of s) would have to be smaller still (see figure 2).

Now we can prove that the answer to the question is no. Suppose that we've managed to cut a cube Q into different cubes Q_i. Consider any face of cube Q—say, its bottom base S. The cubes Q_i that stand on the base define a partition of S into different squares. Let S_1 be the smallest of them and Q_1 the corresponding cube. Since S_1 doesn't border on the boundary of S, it is surrounded by larger squares. Their corresponding cubes form a "well" with cube Q_1 at its bottom. Therefore, the cubes bordering on the upper base S_1' of Q_1 define a partition of S_1' into different squares. The smallest of them, S_2, lies strictly inside S_1', so the corresponding cube Q_2 is smaller than Q_1 and lies at the bottom of the next "well." The process can be continued to yield an infinite tower of decreasing cubes, which is impossible.

At first sight, it may seem that a similar argument must work as well for square partitions of a square. But in fact it doesn't. Think why!

We conclude with a problem that has already appeared in *Quantum* in other situations and with other proofs.

Problem 6. Prove that for $n \ne 4$ a regular n-gon can't be drawn on a square grid such that its vertices coincide with nodes of the grid.

Solution. Consider first the case of a regular (equilateral) triangle

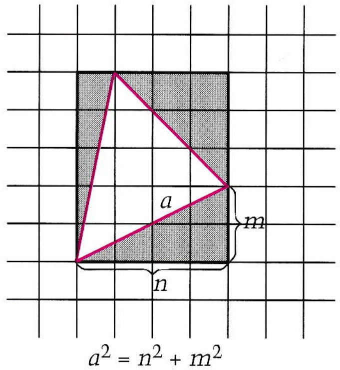

Figure 3

($n = 3$). If its side length is a, then its area is $a^2\sqrt{3}$. The Pythagorean theorem then guarantees that a^2 is an integer (fig. 3), so $a^2\sqrt{3}$ is an irrational number. On the other hand, it's clear that the area of any triangle with vertices at the grid's nodes is rational (see figure 3).[3] This contradiction shows that the required polygon cannot exist for $n = 3$.

The case of $n = 6$ follows from the case of the triangle, because alternate vertices of a regular hexagon form a regular triangle.

Suppose $P_1P_2...P_n$ is a regular n-gon with $n \ne 3, 4, 6$ whose vertices P_i are nodes of the grid. Draw the vectors equal to $\overrightarrow{P_2P_3}$, $\overrightarrow{P_3P_4}$, ..., $\overrightarrow{P_nP_1}$, $\overrightarrow{P_1P_2}$ from the points P_1, P_2, ..., P_n, respectively (fig. 4). Their endpoints will fit the grid's nodes again and form a regular n-gon inside the initial one (why?). We can do the same with the new n-gon and proceed in this way indefinitely. But the square of the side length of any n-gon in this sequence is an integer, and our process decreases it at every step!

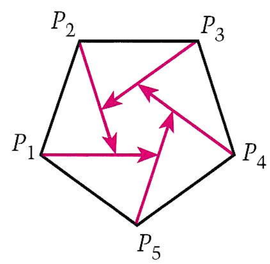

Figure 4

[3]This is true for any polygon on the grid.—*Ed.*

Exercises

1. Prove that unequal sides of an isosceles triangle with a 36° vertex are incommensurable.

2. Prove that the number 7 can't be represented as the sum of the squares of three rational numbers.

3. Solve the following equations in integers: (a) $x^3 - 3y^3 - 9z^3 = 0$; (b) $5x^3 + 11y^3 + 13z^3 = 0$.

4. Prove that the following equations have no solutions in nonzero integers: (a) $x^2 + y^2 + z^2 = 2xyz$; (b) $x^4 + y^4 = z^4$. (Hint for part (b): x^2, y^2, z^2 is a Pythagorean triple. Use the familiar formulas for these triples.[4])

5. Positive integers $a_1, a_2, \ldots, a_n$ ($n > 2$), no two of which are the same, are written around a circle. Then each of the numbers is replaced by the arithmetic mean of its clockwise neighbor and itself. This operation is repeated a number of times. Prove that after sufficiently many iterations some numbers in the set will be fractions (that is, nonintegral rational numbers).

6. A point is given inside a convex polyhedron. Prove that its orthogonal projection on one of the faces lies inside this face. Is this true for nonconvex polyhedrons?

ANSWERS, HINTS & SOLUTIONS ON PAGE 52

[4]These can be found, for instance, in "Arithmetic on Graph Paper" in the March/April 1995 issue of *Quantum*.—*Ed.*

Amended call for manuscripts

Special issue of *Quantum* devoted to *The Limits to Growth*

QUANTUM CONTINUES TO SEEK manuscripts for a special issue commemorating the 25th anniversary of The Club of Rome's groundbreaking study *The Limits to Growth*. However, this issue is now slated to appear in 1997 and will have a broader focus than originally envisioned.

Our first call for manuscripts noted the dramatic advances in technology that have occurred since *The Limits to Growth* first appeared. In 1972 the World3 computer model that underlies this study (and its 1992 sequel, *Beyond the Limits*) was state of the art. Today, however, such models are accessible via the desktop computers available in most American high schools and in many secondary schools around the world. This makes it possible for *Quantum* readers to investigate the concepts from "system dynamics" that may turn out to be the most lasting legacy of the *Limits to Growth* study.

With the aim of informing our readers about the tools and concepts arising in system dynamics, *Quantum* invites articles on the modeling of dynamic systems in a variety of contexts. This issue will also describe various forms of computer software available in support of such modeling efforts. It will include information on freeware that allows one to run and modify an updated version of the World3 model as well as an exposition of the STELLA II software that evolved from the original *Limits to Growth* study.

Prospective authors are invited to send a query to

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Just for the fun of it!

B176

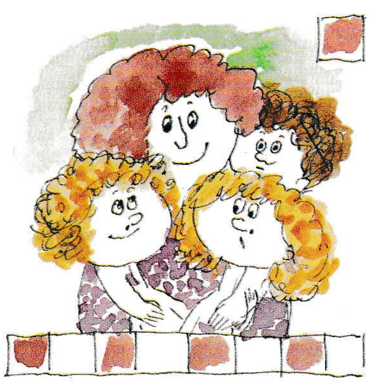
Rearranging coins. Ten coins are arranged in an equilateral triangle pointing down. Move only three coins to make the triangle point up. (A. Khalamaizer)

B177

Drag and read. There was a pile of books on a table. I cautiously pulled a book from the middle of the pile. The books on top of the book I was taking moved along with it, but the books under it stayed in place. Why? (A. Savin)

B178

Rolling coins. Two identical coins touch the side of a rectangle at the same point—one from the inside, the other from the outside. The coins are rolled in the plane along the perimeter of the rectangle until they come back to their initial positions. The height of the rectangle is twice the circumference of the coins and its width is twice its height. How many revolutions will each coin make?



B179

Chess family. Two participants in a family chess tournament were twins. The players were the mother, her brother, her daughter, and her son. It is known that the winner was the same age as the "loser" (the player who ended up in last place) and was the opposite gender as the loser's twin. Who won the tournament? (A. Savin)

B180

Reversing coins. Seven coins are placed heads up around a circle. You're allowed to turn over any five coins in a row. Is it possible to turn all the coins tails up by applying this operation repeatedly?

ANSWERS, HINTS & SOLUTIONS ON PAGE 49

Art by Pavel Chernusky

The power of the Sun and you

Or, Why the gnat is cold blooded

by V. Lange and T. Lange

THE FLOW OF ENERGY FROM the Sun is enormous. Geophysical measurements have shown that even at a distance of 150,000,000 km from the Sun, every square meter of the upper atmosphere oriented perpendicular to the Sun receives 1.4 kJ of solar radiation every second. This value is known as the solar constant $I = 1.4\ \text{kJ/(m}^2 \cdot \text{s)} = 1.4\ \text{kW/m}^2$ and makes it possible to calculate the total power of the solar radiation P_1. To calculate the total power of the Sun's rays, we need only multiply the solar constant by the area of the sphere circumscribed around the Sun with a radius $R = 150{,}000{,}000$ km:

$$P_1 = I \cdot 4\pi R^2 \cong 4 \cdot 10^{26}\ \text{W}.$$

Surely the power level of a human being is much more modest. The average power P_2 generated by a person can be evaluated rather precisely according to the energy content of the food consumed in a day. It's known that a human being who is not involved in hard physical labor should consume about 12 MJ of food per day. Almost all this energy is spent on maintaining one's body temperature and is ultimately dissipated in the surroundings. Only a very small part of this 12 MJ of energy is converted into mechanical work. Dividing 12 MJ by the length of a day (86,400 s) yields

$$P_2 \cong 140\ \text{W}.$$

Therefore, as a generator of energy, the Sun is approximately $3 \cdot 10^{24}$ times more powerful than a human being. All the more unexpected, then, is the result when we compare their specific power—that is, power per unit mass. The Sun's mass M is about $2 \cdot 10^{30}$ kg, and that of a human being m may be taken as 80 kg. From this we obtain

$$P_1/M = 2 \cdot 10^{-4}\ \text{W/kg},$$
$$P_2/m = 1.75\ \text{W/kg}.$$

So the specific power of a human being is almost 10,000 times that of the Sun!

At first glance this result seems way off the mark. Nevertheless it is true.

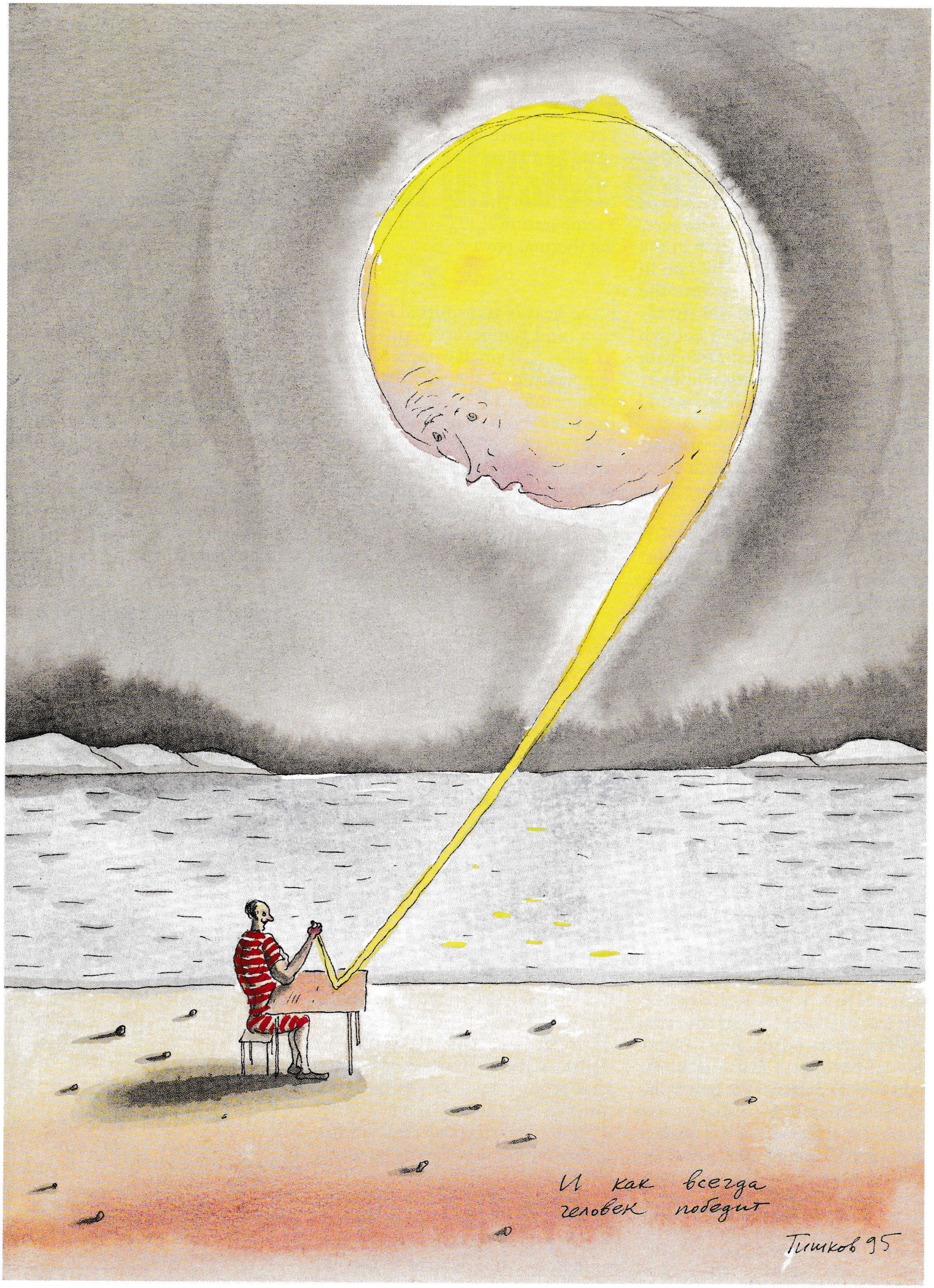
How can this "paradox" be explained? How can the Sun—a giant thermonuclear reactor—lose the specific power "competition" to a mere human being, who acquires energy from chemical reactions that are far "weaker" than the nuclear variety?

It's not hard to find the answer to this question if we assume that the production of thermal energy is more or less uniformly distributed in both the human body and the Sun. As a result, the rate of energy production is directly proportional to the body's volume—in other words, to the third power of the linear size. However, the rate of heat emission is proportional to the surface area—that is, to the square of the linear size. So the larger a body, the weaker the emission rate needed to maintain a certain temperature.

The volume of the Sun is about 10^{27} m^3; its surface area is of the order of 10^{18} m^2. The corresponding human parameters are 10^{-1} m^3 and 1 m^2. Thus the ratio of solar to human volume is about 10^{28}, and their surface ratio is of the order of 10^{18}. In other words, a unit volume of the Sun corresponds to one ten-billionth the surface area of a unit volume of a human being. So it's not surprising that, although the Sun's "metabolism" proceeds at a rate of just

At right: "And, as always, the human will win."

Art by Leonid Tishkov


И как всегда
человек победит
Тишков 95

0.2 mW/kg, the Sun's surface temperature reaches 6,000 degrees.

We'll illustrate the connection between size, energy production, and body temperature with some examples from the life of animals.

The body temperatures of mammals hardly vary at all. In particular, the body temperatures of the giant elephant and the tiny mouse are approximately the same.[1] However, the rate of energy production in the elephant's body is 1/30 the rate in the mouse's. If this rate were the same as in the mouse, the energy produced would not have enough time to leave the elephant's body in order to maintain its normal body temperature. The poor beast would be "baked" in its own hide.

Smaller mammals must produce more energy per unit mass in order to compensate for heat losses and keep their body temperature at the level necessary for normal activity. Thus smaller living creatures must eat more food (again, per unit mass). Tom Thumb, the tiny boy in the fairy tale, would be a terribly voracious little tot—since he is proportional to a normal human being, he would need 20 times more food per kilogram of his mass.

The smallest mammal on Earth, the Etruscan mouse, with a mass of 1.5 g, consumes twice its own mass every day. If this creature is left without food for as little as a few hours, it will die. Or take the hummingbird (with a mass of just 2 g). Practically all of its waking hours are directed at finding and eating food. The only way these birds can endure long nights without food is to drastically reduce their body temperature.

It can easily be shown that very small creatures—say, gnats—cannot be warm blooded. For simplicity, we'll consider the gnat to be a cylinder of diameter d = 0.5 mm and length l = 4 mm. Thus its surface area S and volume V are

$$S = 2\frac{\pi d^2}{4} + \pi d l \cong 10^{-5}\ \text{m}^2,$$

$$V = \frac{\pi d^2}{4} l \cong 10^{-9}\ \text{m}^3.$$

Let's estimate the power "generated" by a gnat.

A body with a temperature T transmits to the surroundings with a temperature T_0 ($T_0 < T$) the following thermal power:

$$P = \alpha S \cdot \Delta T.$$

If the heat is transmitted by radiation, and temperature difference $\Delta T = T - T_0$ is small compared to the temperature T; the coefficient α is proportional to T^3. Depending on the body's reflectivity, α is about 2–5 W/(m^2 · °C) at room temperature. Supposing that the gnat's temperature is 30°C (T = 303 K) and α = 4 W/(m^2 · °C) we find, that when the ambient temperature is 17°C (T_0 = 290 K), the gnat radiates the following thermal power:

$$P \cong 10^{-3}\ \text{W}.$$

Taking the density of a gnat's body to be equal to that of water, we obtain the gnat's mass: $m \cong 10^{-6}$ kg. So the specific power of a gnat must be 10^{-3} W/10^{-6} kg = 10^3 W/kg, or 600 times that of a human being (and 6 million times that of the Sun).

If a human being eats about 1 kg of food per day—that is, about one eightieth of his or her mass—the mass of a gnat's daily intake exceeds its own mass by a factor of 600/80 = 7.5. (Actually, our estimates are a little low, because we didn't take into account the heat lost by convection.) The ambient temperature is often much lower than 17°C; and at 7°C (gnats continue to be rather active under such conditions) the energy losses are almost double, so a gnat would have to consume 15 times its own mass in food! Clearly, it cannot keep its body temperature constant (that is, it cannot be warm blooded).

Looking at the relationships between a body's size and the intensity of heat exchange with the surroundings, we can answer another interesting question: why can a thin wire be melted in the flame of a match, while it's hard to make a thick wire red-hot even in the flame of a gas stove?

The flow of energy that the wire receives from the flame is directly proportional to its surface area $S = 2\pi R l$ (where R is the radius of the wire and l is the length of the portion of the wire being heated). At the same time the rate of heat flow along the wire's axis to its cold (unheated) ends is directly proportional to the cross-sectional area of the wire $S = \pi R^2$. When the radii of the two wires differ by a factor of 10, all other conditions being equal, the thick wire will receive 10 times more heat per unit time than the thin wire, but it will lose 100 times more heat. It's clear that at thermal equilibrium, when the incoming and outgoing heat flows are the same, the temperature of the thick wire will be substantially less. ◘

[1]For a discussion of other physiological constants in mammals, see "From Mouse to Elephant" by Anatoly Mineyev in the March/April 1996 issue of *Quantum*.—*Ed.*

HOW DO YOU FIGURE?

Challenges in physics and math

Math

M176

Polyhedron restored. Andrew cut a cardboard convex polyhedron along all its edges and mailed the set of its faces to Laura. Laura glued them together into a convex polyhedron. Is it possible that the two polyhedrons are not congruent? (N. Vasilyev)

M177

Sequences of squares. (a) Does there exist an infinite sequence of integer squares in which every term starting from the third is the sum of the two preceding terms? (b) Does there exist an infinite sequence of integer squares in which the sum of any two neighboring terms is an integer square? (O. Kryzhanovsky)

M178

Alternating vector sum. An even number of unit vectors are drawn from the same point O on the plane and alternately colored red and blue. Let $\mathbf{r}$ be the sum of the red vectors, $\mathbf{b}$ the sum of the blue vectors. Show that $|\mathbf{r} - \mathbf{b}| \le 2$. (E. Shustin)

M179

Unusual constructions. You have a ruler with two marks on it. With it, you can draw lines as with an ordinary ruler and also mark off segments equal in length to the distance between the two given marks. You are not allowed any other constructions. With this instrument, construct (a) a right angle; (b) a line perpendicular to a given line. (V. Gutenmacher)

M180

Equation for periodicity. A function f satisfies the equation $f(x+1) + f(x-1) = \sqrt{2}\,f(x)$ for all real x. Prove that this function is periodic. (E. Turkevich)

Physics

P176

On the Martian soil. During opposition Mars is located at a distance $l = 5.56 \cdot 10^{10}$ m from the Earth, and its angular diameter is $\alpha = 25''.1$. Find the acceleration due to gravity on the Martian surface if the maximum angular distance between the center of Mars and its moon Phobos $\beta = 34''.5$, while the period of revolution of Phobos around the planet is $T = 2.76 \cdot 10^4$ s.

P177

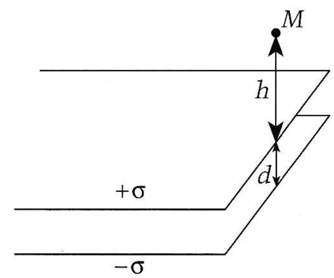
Drop in a cloud. Through observations of how a raindrop falls in a cloud, growing in diameter due to the absorption of tiny drops encountered along way, it was established that drops move with a uniform acceleration. Find this acceleration, assuming the initial size of a drop to be small. Neglect air resistance. (A. Stasenko)

P178

Uranium and hydrogen. An evacuated vessel with a volume $V = 1$ L contains 1 g of uranium hydride UH_3. When heated to a temperature of 400°C, the hydride decomposes completely to yield uranium (atomic mass $A = 238$) and hydrogen. Find the pressure of the hydrogen in the vessel at this temperature.

P179

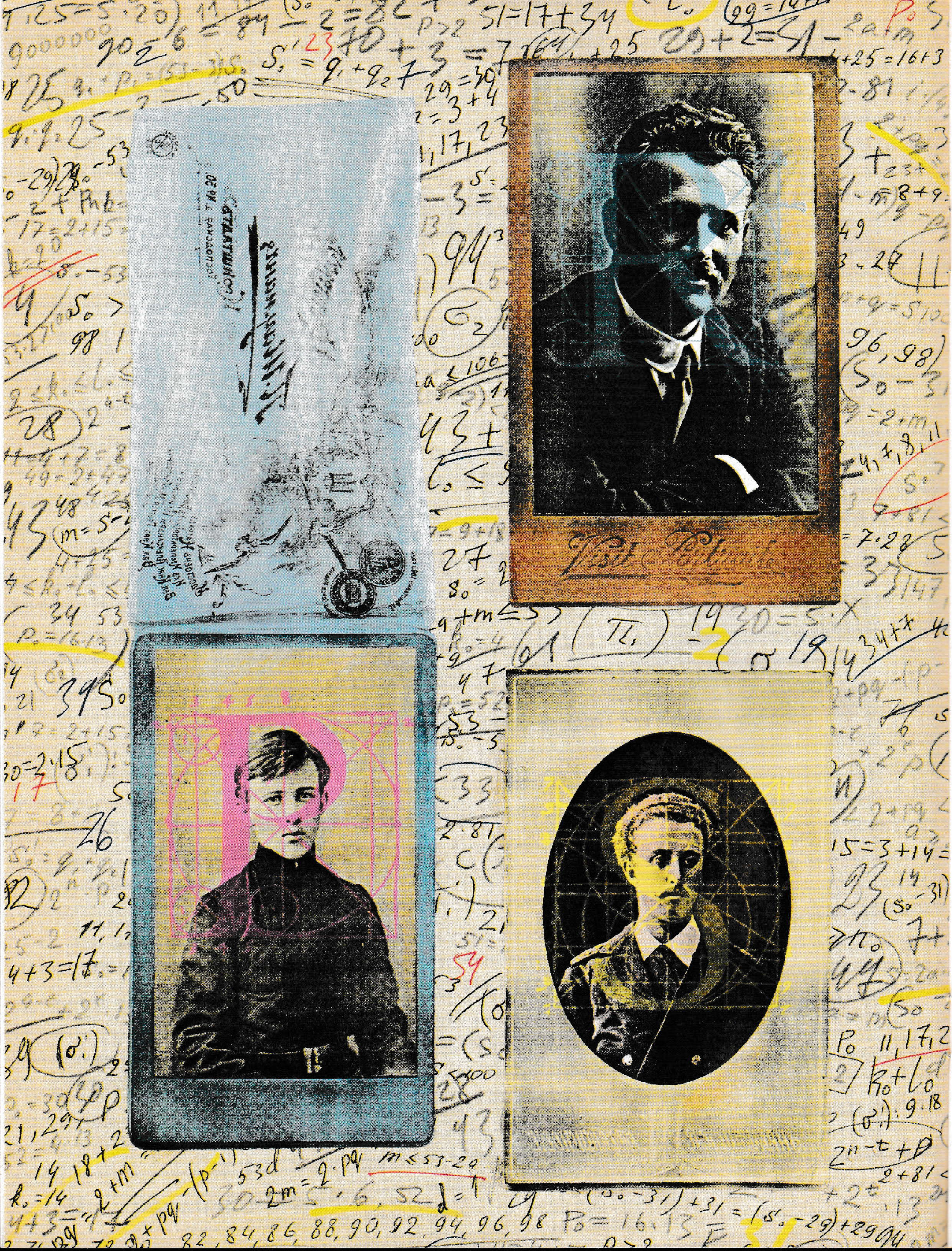
Two charged bars. Two long, wide plates are uniformly charged. The charge densities are $+\sigma$ (the upper plate) and $-\sigma$ (the lower plate). Find the value and direction of the electric field strength at a point M located at a height h above the edge of the upper plate and on the axis lying in the plane of symmetry (see the figure below). The distance d between the plates is small compared to h. (A. Semyonov)



P180

Light beam in an aquarium. A thin-walled aquarium in the shape of a cube with a volume $V = 8$ L is filled halfway with water. A salt is poured into it, so that the refractive index of the salt solution at the bottom is $n_0 = 1.35$. The refractive index decreases with height h according to a quadratic law $n = n_0 - ah^2$, where $a = 1\ \text{m}^{-2}$. A parallel beam of light falls on a side wall of the tank, perpendicular to the surface. At what distance from the aquarium must a screen be placed to obtain the brightest possible strip of light? (A. Olkhovetz)

ANSWERS, HINTS & SOLUTIONS ON PAGE 47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Spinning gold from straw

Or, How two secrets can add up to one certainty

by S. Artyomov, Y. Gimatov, and V. Fyodorov

WE'D LIKE TO TELL YOU about a problem that requires highly sophisticated logic to solve.

A mathematician R said to mathematicians P and S, "I've thought of two natural numbers. Each of them is greater than one, while their sum is less than a hundred. Now I'm going to tell P, in secret from S, the product of these numbers. And I'm going to tell S, in secret from P, their sum." And so he did. R then asked his colleagues to guess the numbers. P and S had the following conversation (P's statements are denoted by the letter π and S's by the letter σ):

"It looks like I can't say what the numbers are." (π_1)

"I knew in advance that you wouldn't." (σ_1)

"Well, then I do know them!" (π_2)

"Then I know them, too." (σ_2)

Now *you* try to guess the numbers!

1. Is it really possible?

At first glance the problem seems insoluble: how can one guess the numbers when no information about them was given?

Let's try an example. Suppose R thought of the numbers 7 and 42. Then the numbers he told P and S were 294 and 49. So what then? P was unable to guess the numbers. No wonder—he only knew their product. Well, not exactly. He also knew that they are natural, they are greater than one, and their sum is less than a hundred. But what good is that?

Denote the unknown numbers by k_0 and l_0 and assume, for definiteness, that $k_0 \le l_0$. Denote the product $k_0 \cdot l_0$ by p_0 and the sum $k_0 + l_0$ by s_0.

Now we can say that P was told that $p_0 = 294$. Then the possible values of k_0 are 2, 3, 6, 7, and 14; then l_0 will be either 147, 98, 49, 42, or 21, respectively. The first two values of k_0 don't suit us, because they make too large a sum: $s_0 > 100$. We are still left with three possibilities, so P indeed cannot guess the numbers.

Let's go on. Mathematician S says that she knew in advance that P would be unable to guess the numbers. How could S know? She must have checked all possible representations of her number s_0 as the sum of two admissible numbers:

$$49 = 2 + 47 = 3 + 46 = \ldots = 24 + 25.$$

R could have thought of any of these number pairs. He told P one of the products $n \cdot (49 - n)$, and S says that P can't guess the numbers from any of them.

But what if for a certain n both numbers n and $49 - n$ are prime? For instance, if R thought of 2 and 47, then he would have given the number 94 to P, and P would easily guess the secret numbers.

So if R thought of 7 and 42, then S, having been given the sum $s_0 = 49$, would have no right to make statement (σ_1). This means that R did not have 7 and 42 in mind.

So, it turns out we *can* say something about the unknown numbers.

Now that we've dispelled our initial doubts, let's figure out where to go next. One method of solution is already clear: we can simply step onto the brute-force, trial-and-error road and check all the pairs k_0, l_0 that satisfy the conditions

$$2 \le k_0 \le l_0 \le 97, \qquad (1)$$
$$4 \le k_0 + l_0 \le 90 \qquad (2)$$

to see which of them "survive" the dialogue (π_1)–(σ_2).

Since the number of possibilities is finite in all cases, we could actually proceed in this artless way and find the answer sooner or later. But that would be boring, wouldn't it? So let's try to restrict the search.

First of all, we'll search through the values of s_0 rather than k_0 and l_0, since there are more than 2,000 possible pairs (k_0, l_0), but fewer than a hundred possibilities for s_0. However, even in this case the brute-force search is long and tedious.

Art by Vera Khlebnikova

2. The Goldbach–Euler conjecture

What information can be derived from (π_1) and (σ_1)? What do these statements mean? The first statement, (π_1), obviously tells us that

> *p_0 is not uniquely factored into the product of two numbers satisfying inequalities (1) and (2).* (π_1')

Statement (σ_1) means that

> *for any decomposition of the number s_0 into the sum of two terms satisfying inequality (1), their product obeys property (π_1').* (σ_1')

The first condition rules out some products; the second rules out certain sums.

In particular, it follows from (σ_1') that s_0 cannot be represented as the sum of two primes. (Otherwise the product of these primes would have a unique factorization into two factors satisfying inequalities (1) and (2) and so would not comply with (π_1').)

But any even number satisfying inequality (2) is representable as the sum of two primes (this is shown by direct verification for the numbers 4, 6, 8, ..., 98). Therefore, s_0 is an *odd* number. In addition, $s_0 - 2$ is a composite number; otherwise $s_0 = 2 + (s_0 - 2)$ would be a decomposition of s_0 into the sum of two primes. After discarding the numbers that don't satisfy these two conditions, we are left with only 24 possibilities for s_0.

The fact (used in the preceding paragraph) that all even numbers from 4 to 98 are representable as the sum of two primes is related to an intriguing mathematical problem. In 1742 a member of the St. Petersburg Academy of Sciences, Christian Goldbach (a German in the service of the Russian state), wrote a letter to Leonhard Euler in which he conjectured that any odd number greater than five is representable as the sum of three primes. In his reply, Euler proposed the hypothesis that any even number greater than two is the sum of two primes. (It's not hard to derive Goldbach's conjecture from Euler's—try it yourself!)

For almost two hundred years both conjectures seemed beyond proof, although they were checked by direct search for numbers up to 9,000,000.

In 1930 the outstanding Russian mathematician L. G. Shnirelman proved the existence of a number k such that any integer $n > 1$ can be decomposed into the sum of no more than k primes. The number k in Shnirelman's proof was rather large; it was later proved that the theorem is true for $k = 20$.

In 1934 another famous Russian mathematician, I. M. Vinogradov, showed that for a certain n_0 any number n, $n > n_0$, is representable as the sum of three primes. It would seem that in our computer era we could rely on the machine to verify all the remaining numbers (from 7 to n_0). However, Vinogradov's constant n_0 is so large ($n_0 > 2^{2^{16}}$) that verification is still beyond the capacity of modern computers.[1]

As for Euler's conjecture, so far there has been no significant progress toward a proof.

3. "Primaries"

We can further reduce the number of candidates for s_0: we can derive from (σ_1') that

$$s_0 < 55. \tag{3}$$

To see why, suppose that, on the contrary, $s_0 \geq 55$. Then s_0 does not satisfy (σ_1'): we can decompose it into the sum of two terms satisfying inequality (1) whose product fails to meet condition (π_1'). This decomposition is $s_0 = 53 + (s_0 - 53)$. Indeed, the product $53 \cdot (s_0 - 53)$ has only one factorization into two factors whose sum doesn't exceed 100, because one of the factors must necessarily be of the form $53d$, since 53 is a prime, and would be greater than 100 whenever $d > 1$. So $d = 1$, and the factorization is unique. But this contradicts property (σ_1') for s_0!

With inequality (3) proved, the number of possibilities for s_0 shrinks to eleven:

$$\begin{gathered}11, 17, 23, 27, 29, 35, 37,\\ 41, 47, 51, 53.\end{gathered} \tag{4}$$

Let's try to establish which of these numbers comply with condition (σ_1') without a direct search. Let s be any of the numbers (4). Since s is odd, if two numbers add up to s, then one is odd and the other is even, and we can write $s = 2a + m$. If s does not satisfy (σ_1'), then for a certain a the product $2am$ is factored uniquely.

This a cannot be equal to one, because the product $2m$ has at least two factorizations. Indeed, suppose $a = 1$. Then the number m is composite, so we can write $m = uv$, where $u > 2$ and $v > 2$, and both factorizations

$$2m = 2u \cdot v = 2 \cdot uv$$

are good for us:

$$2 + uv = 2 + m = s \leq 100$$

and

$$\begin{gathered}2u + v = 2 + uv - (u-1)(v-2)\\ < 2 + uv \leq 100.\end{gathered}$$

It follows that $a \geq 2$.

Now, either $a = m$ or $a \neq m$, and we can investigate each case separately. If $a \neq m$, then $p = 2a \cdot m$ and $p = 2m \cdot a$ are two different factorizations. Since $2a + m = s < 100$ and the factorization of p is unique, we must have $2m + a \geq 100$. At the same time, from $s = 2a + m \leq 53$, it follows that $m \leq 53 - 2a$, and so $2m + a \leq 106 - 3a$. Thus, $100 \leq 2m + a \leq 106 - 3a$, implying $a \leq 2$. So in this case we have $a = 2$, $2m + a = 100$, and $m = 49$, which leads to the only "suspect" value $s = 53$ and its decomposition $53 = 4 + 49$.

In the case $a = m$ the number $s = 3a$ is divisible by 3. Only two of the numbers (4) are multiples of three: 27 and 51. The "suspect" decompositions are $27 = 18 + 9$ and $51 = 34 + 17$.

The number 51 does not satisfy (σ_1')—indeed, the product $17 \cdot 34$ has only one factorization into two factors whose sum is less than 100, so we can exclude it from the list of "candidates for s_0."

The numbers 27 and 53 remain in the list: they satisfy (σ_1'), because for 27 we have $9 \cdot 18 = 2 \cdot 81$ and

[1] This estimate may be out of date now (this article originally appeared in *Kvant* in 1977).—*Ed.*

$2 + 81 < 100$, and for 53 we have $4 \cdot 49 = 7 \cdot 28$ and $7 + 28 < 100$.

So now we have ten "candidates" left: 11, 17, 23, 27, 29, 35, 37, 41, 47, and 53; and all of them satisfy (σ_1').

4. "Then I know them, too"

Finally, let's use statements (π_2) and (σ_2).

We could interpret them the same way we did with the first two above. But there's a shortcut.

From (σ_2) and inequality (3), we'll derive that

$$s_0 < 33. \qquad (5)$$

Suppose this is not true. Then $s_0 \geq 33$, and our mathematician S, decomposing s_0 into the sum of two terms in every way possible, would have found these two variants: $s_0 = (s_0 - 31) + 31 = (s_0 - 29) + 29$. Then her train of thought would have been as follows: if P had been given the product $(s_0 - 31) \cdot 31$, then, using estimate (3) and the fact that 31 is a prime, P would have understood that $(s_0 - 31) \cdot 31$ has only one factorization such that the sum of its two factors satisfies inequality (3). Then P would have guessed the unknown numbers. But the same argument also applies to the product $(s_0 - 29) \cdot 29$.

Therefore, in the case $s_0 \geq 33$, mathematician S would still be unable to identify k_0 and l_0 exactly even after P's statement (π_2), contrary to what happened in the story.

So inequality (5) is indeed true, which leaves only five numbers: 11, 17, 23, 27, 29.

Further, if $p_0 = 2^n \cdot p$, where p is an odd prime and $n > 1$, then P can unambiguously identify the secret numbers, because there is only one odd sum of the form $2^{n-t} + 2^t p$—namely $2^n + p$. So if s_0 has two representations of the form $2^n + p$, then S is unable to find the answer and make statement (σ_2). This observation weeds out three more numbers: $11 = 4 + 7 = 8 + 3$, 23, and 27, leaving only two candidates on the list: 17 and 29.

5. Then we know them, too!

For the number $s_0 = 29$ the last argument fails, because this number has only one representation of the form $2^n + p$ with p odd and prime $(29 = 18 + 13)$. However, slightly modified, it works for the decomposition $29 = 4 + 25$. In the case $p_0 = 4 \cdot 25$ we have only one possible odd sum, $25 = 20 + 5$, other than 29 $(4 \cdot 25 = 5 \cdot 20)$, but then $25 - 2$ is a prime, whereas $s_0 - 2$ must be composite. So in this case S is again unable to guess the numbers, which reduces the list of candidates to one number, 17—that is, either $s_0 = 17$ or the problem has no solution.

So what product p_0 could have been given to P if the sum $s_0 = 17$? Let's search through all decompositions of 17 into the sum of two terms:

$$17 = 2 + 15 = 3 + 14 = \ldots = 8 + 9.$$

For any of the corresponding products except $4 \cdot 13$, mathematician P would be unable to guess the answer and make his statement (π_2). For instance, in the case of the decomposition $17 = 2 + 15$, we have $p_0 = 2 \cdot 15 = 30 = 5 \cdot 6$, but both 17 and $11 = 5 + 6$ satisfy property (σ_1').

So the only possible value for p_0 is $4 \cdot 13 = 52$. With this value mathematician P was able to guess the numbers, because among all factorizations of 52 into two factors only one, $52 = 4 \cdot 13$, yields an odd sum of factors.

Thus, we have $s_0 = 17$, $p_0 = 52$, and the numbers that mathematician R had in mind are 4 and 13.

Problems

1. Is it possible to represent any odd number greater than 3 as the sum $2^n + p$, where p is a prime? If not, give the smallest possible counterexample.

2. Suppose the story told at the beginning of the article is modified as follows. Up to statement (σ_1) it's the same, but then these statements were made:

"And I knew in advance that you would know that in advance." (π_2)

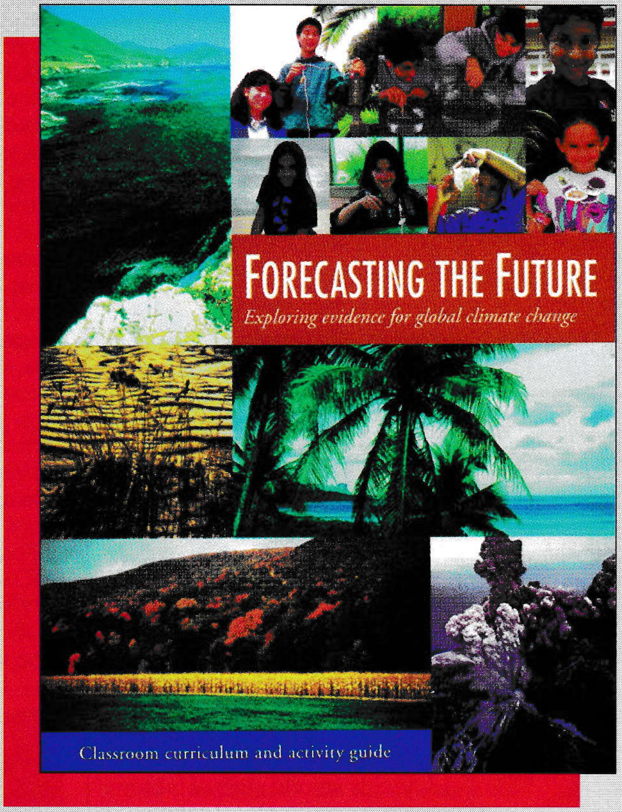
"I don't know what the numbers are." (σ_2)

"Then I know them." (π_3)

Find the numbers in question. (B. Kukushkin)

ANSWERS, HINTS & SOLUTIONS ON PAGE 53

What does the future hold for Earth's climate?



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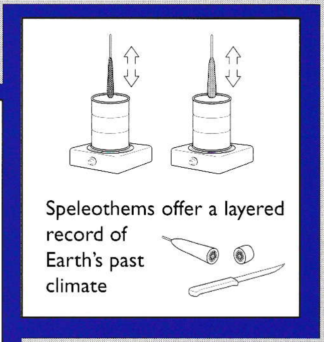

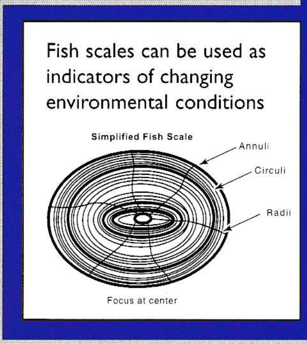

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AT THE
BLACKBOARD

While the water evaporates . . .

Let's think about how, and why, and how fast . . .

by Mikhail Anfimov and Alexey Chernoutsan

HAVE YOU EVER NOTICED how water evaporates from a container? Pour some water into a small pan or glass and observe how the water level drops during the course of a day. Since this is a rather slow process, we have plenty of time to think and calculate. In particular, let's try to evaluate the rate of evaporation and then compare it with our observations.[1]

What is the mechanism of evaporation? We recall that in order to convert some amount of water into vapor at a constant temperature, we need to transfer to the water a certain amount of thermal energy, which is called the latent heat of vaporization. For example, at room temperature T = 290 K, the latent heat of vaporization is 2.46 kJ per gram of evaporated water. Since there are $^1/_{18}N_A$ molecules in 1 g of water (N_A = 6.02 · 10^{23} mole^{-1} is Avogadro's number), we needs to expend energy E_1 = 7.35 · 10^{-20} J to remove one molecule from the liquid phase. In atomic calculations the energy is usually expressed in electron volts (eV). One electron volt is equal to 1.6 · 10^{-19} J, so E_1 = 0.46 eV.

So what is this energy expended on? The answer is pretty obvious: to overcome the attractive force exerted by the liquid on the molecule that would escape. Each molecule interacts with the surrounding molecules. The molecular interaction takes the form of repulsion at small distances ($r < r_0 \sim 10^{-8}$ cm) and attraction at larger distances ($r > r_0$). Inside a liquid every molecule is surrounded on all sides by other similar molecules, so the resulting (average) force is zero. However, this is not true for a molecule that tries to leave the water's surface and escape into the air. This molecule is attracted by a number of surface molecules, and there is no counterbalancing force. Thus, to overcome the attractive forces and eventually leave the water, the molecule must have a rather high kinetic energy. Just for comparison: the average kinetic energy of the translational motion of a water molecule is $^3/_2kT$, where k = 1.38 · 10^{-23} J/K is Boltzmann's constant—that is, 0.038 eV at T = 290 K, which is one order of magnitude less than the energy E_1 needed to pull a molecule out of the liquid phase. Therefore, only a few water molecules are able to escape the surface of the water. These are molecules that happened to be near the surface and had acquired energy an order of magnitude greater than the average due to random collisions.

Now we look at the latent heat of vaporization from another point of view. Clearly the consumed heat is not directly transferred to the molecules that escape from the water's surface. These molecules get the extra energy stochastically from the neighboring molecules. However, since only the most "energetic" molecules have a chance to escape, in the liquid phase less energy is left for each remaining molecule on average. If the liquid does not compensate the energy loss by taking a certain amount of heat from the surroundings, its temperature drops. In order to keep the temperature constant, the liquid must acquire an amount of heat equal to the heat of vaporization.

Here many students fall into the same trap: "Since the escaping molecules have anomalously high energies, the vapor must be warmer than the liquid." This isn't true, of course. Only at the very beginning of its "free" flight does a molecule have any extra energy. In overcoming the attractive forces, the molecule loses much of its energy, so the average energy of the "newcomer" molecules is equal to that of the vapor at the same temperature.

"This is all very fine," you may be thinking, "but we still haven't taken the first steps toward evaluating the rate of evaporation of water. Not only that, it's clear now that we need a much clearer, quantitative understanding of the 'structure' of the liquid and the way its molecules move if we're to come up with any

[1]See also physics challenge P168 in the March/April issue.—*Ed.*

sort of rational estimate." Nevertheless, we can do all the necessary calculations. Here's how.

It turns out that all we need to do is mentally place our glass of water in a closed container. In a little while the container gets filled with saturated water vapor, and the process of evaporation stops. Strictly speaking (and this is the point!), evaporation continues just as before, but the number of escaping molecules is equal to the number of molecules that enter the liquid. This situation is referred to as "dynamic equilibrium" between the liquid and gas. Now we can estimate the rate of evaporation indirectly, focusing not on the liquid but on its saturated vapor. This trick turns on the fact that vapor is a much simpler thing than liquid. In any case, we can obtain a rather good estimate if we consider the saturated vapor an ideal gas.

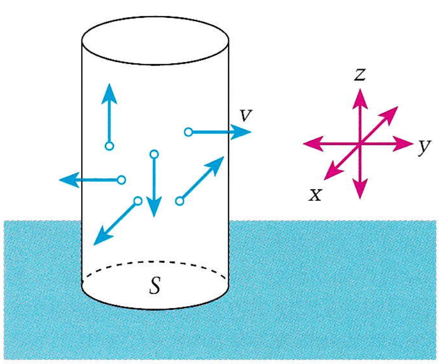
For simplicity we'll make some additional assumptions. In particular, let's consider all the molecules to have the same velocity v and to move only in six permissible directions—that is, parallel to the coordinate axes (one axis is vertical). Every second 1/6 of the molecules in a cylinder of height v (see the figure below) strike an area S of the liquid's surface:

$$\frac{\Delta N}{\Delta t} = \frac{1}{6} nvS,$$

where n is the concentration of molecules (a strict calculation yields the factor 1/4 instead of 1/6). To estimate the velocity v we use the formula for the root mean square speed $v_m = \sqrt{3kT/m}$ (where m is the mass of one molecule). The concentration of saturated vapor n can be expressed in terms of its pressure P_s by means of the ideal gas equation $P_s = nkT$. Then for the mass of water "falling out of" the saturated vapor and onto the surface of area S we get

$$\frac{\Delta m}{\Delta t} = \frac{1}{6} mnv_m S = \frac{1}{2} P_s S \left(\frac{M}{3RT}\right)^{1/2},$$

where M is the molar mass of water and R is the gas constant. Substituting the numerical data into this equation (P_s at 17°C is $0.02 \cdot 10^5$ Pa), we find that 0.16 g of water leaves the saturated vapor and lands on a 1-cm^2 surface of water per second. If this were the rate of evaporation, the water level would drop 1 cm every 6 seconds!



Clearly, this result is far from reality. We have suffered an instructive defeat, and now we must find out why.

The first thing we need to look at is the experiment itself. Evaporation as it actually occurs is slowed notably because the air in the room is not dry but humid to some extent. Humidity at a level of about 60–80% would of itself cause the evaporation to slow by a factor of only 3–4. However, if there were no air flow (for instance, that produced by a fan) over the surface of the liquid, the humidity right at the surface would be close to 100%, which would slow the evaporation drastically. Our estimate of the rate of evaporation, on the other hand, relies on the ideal case, where a return of molecules to the surface is totally absent.

It turns out, however, that our estimate doesn't even correspond to the ideal case. We're off by a factor of 30, but now the reason lies in the calculation itself. The point is, only 3–4% of the molecules striking the surface of the liquid are caught and then absorbed into its depths. Most of the molecules bounce off the surface.

Now it's time to sum up. If an absolutely dry flow of air picked up all the molecules that escaped from the water, the water level in the glass would drop 1 cm not in 6 seconds but in 3 minutes. It's still faster than you would have expected, isn't it? ◘

Finding the family resemblance

An attempt to categorize integer representation problems

by George Berzsenyi

JUST AS I WAS CONTEMplating potential topics for this column, I received an e-mail message from my friend Dr. Harold Reiter, asking for some problems that might fit the framework for an article he plans to publish in *Mathematics and Informatics Quarterly*, an excellent international publication available from the present author. Harold is chairing the committee in charge of the American High School Mathematics Examination (AHSME). Prior to publishing his paper, he plans to present it at A Participating Conference in Mathematical Problem Solving, to be held from the 13th through the 17th of August, 1996, at the Centre for Education in Mathematics and Computing of the University of Waterloo. The timing of this conference should allow for my readers to share their thoughts with Harold, who can be reached by phone (704 547-4561), by fax (704 510-6415), or via the Internet (hreiter@email.uncc.edu).

In the proposed article Harold wishes to categorize problems that arise from considering a fixed set G of generators and a process P for producing integers from the members of G. Each choice of G and P gives rise to a set R of results, and many competition problems concern various aspects of R, like the smallest positive integer contained (or not contained) in R, the largest member of R, the nth member of R for some positive integer n, or the number of elements of R. In yet some other problems one may be interested in how many different ways the elements of R can be generated. One may also categorize the problems by the methodology most applicable for their solutions, by the cardinalities of G and R, as well as by the level of difficulty of the resulting problems. In what follows, we will state some problems that fit into Harold's proposed scheme, asking readers to communicate similar problems (with references and solutions) to Harold.

How many numbers can be expressed as the sum of four distinct members of the set {17, 21, 25, 29, 33, 37, 41}?

Find the 100th positive integer that can be expressed as the sum of distinct powers of 3.

Use each of the nine digits 1, 2, 3, ..., 9 exactly twice to form distinct prime numbers whose sum is as small as possible.

If pairs of distinct elements of S are added, the following ten numbers are obtained: 1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991. What are the elements of S?

Problem posing is an activity complementary to problem solving, and I strongly recommend it to my readers.

Feedback

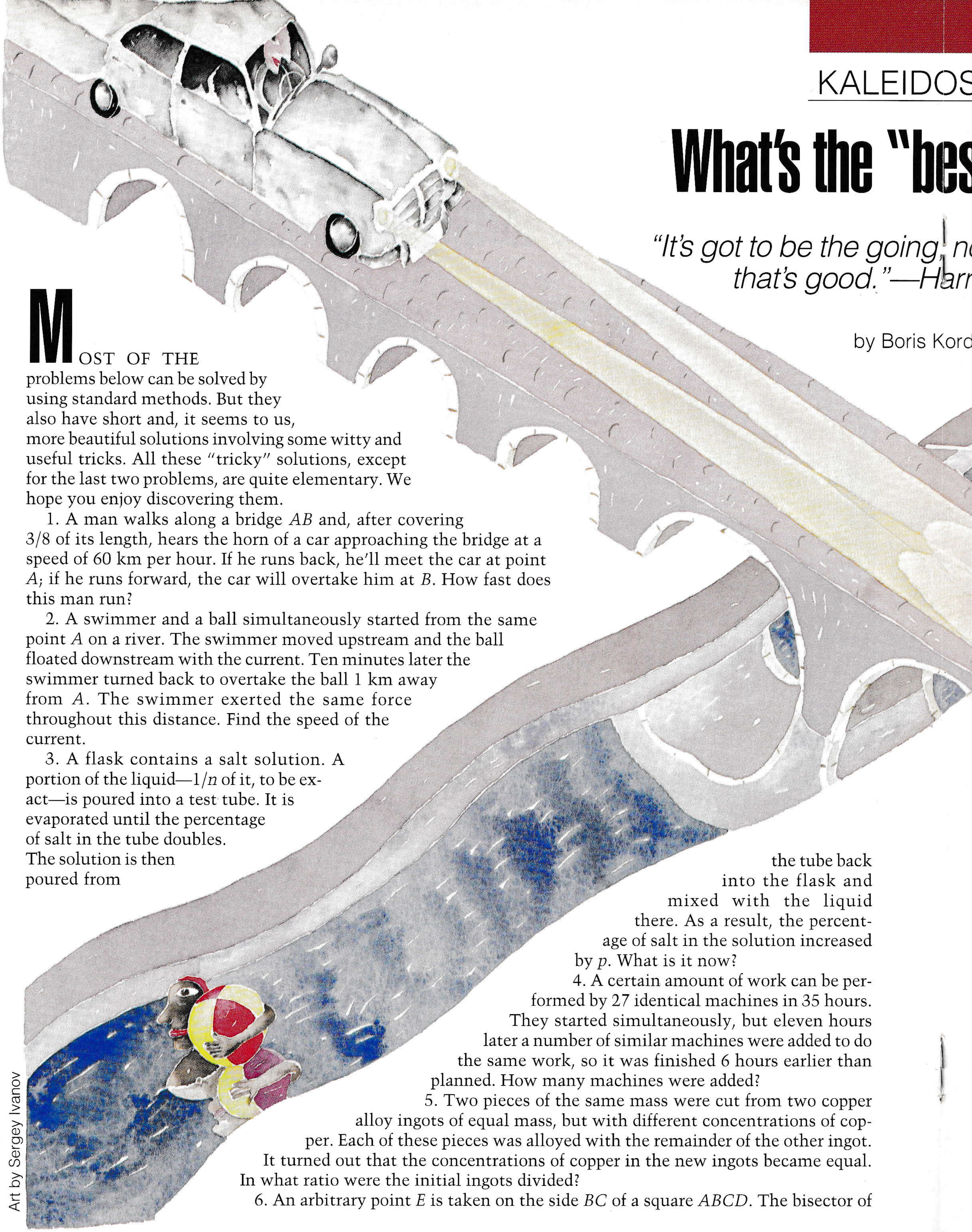
Readers who are still interested in the P_t-sets described in my column in the March/April 1991 issue of *Quantum* might enjoy reading two recent articles on the topic. One of these was written by my young protégé Vamsi K. Mootha ("On the set of numbers {14, 22, 30, 42, 90}," *Acta Arithmetica*, 71.3 (1995), pp. 259–63), while the other was authored by Andrej Dujella ("Generalized Fibonacci numbers and the problem of Diophantus," *Fibonacci Quarterly*, 34.2 (1996), pp. 164–75).

In an earlier column I also promised to provide further references on the triangle construction problems discussed in my July/August and September/October columns in *Quantum*. First of all, the article by Roy Meyers appeared posthumously ("Update on William Wernick's 'Triangle constructions with three located points,'" *Mathematics Magazine*, 69.1 (1996), pp. 46-49). Secondly, the book he called to my attention prior to his death was *Die Konstruktion von Dreicken*. It was written by Kurt Herterich and published by Ernst Klett Verlag of Germany. Roy also wrote to me about two related articles published in Germany. I will report on them as soon as I obtain copies.

What's the "bes

"It's got to be the going, n
that's good."—Harr

by Boris Kord



Art by Sergey Ivanov

MOST OF THE problems below can be solved by using standard methods. But they also have short and, it seems to us, more beautiful solutions involving some witty and useful tricks. All these "tricky" solutions, except for the last two problems, are quite elementary. We hope you enjoy discovering them.

1. A man walks along a bridge AB and, after covering 3/8 of its length, hears the horn of a car approaching the bridge at a speed of 60 km per hour. If he runs back, he'll meet the car at point A; if he runs forward, the car will overtake him at B. How fast does this man run?

2. A swimmer and a ball simultaneously started from the same point A on a river. The swimmer moved upstream and the ball floated downstream with the current. Ten minutes later the swimmer turned back to overtake the ball 1 km away from A. The swimmer exerted the same force throughout this distance. Find the speed of the current.

3. A flask contains a salt solution. A portion of the liquid—$1/n$ of it, to be exact—is poured into a test tube. It is evaporated until the percentage of salt in the tube doubles. The solution is then poured from the tube back into the flask and mixed with the liquid there. As a result, the percentage of salt in the solution increased by p. What is it now?

4. A certain amount of work can be performed by 27 identical machines in 35 hours. They started simultaneously, but eleven hours later a number of similar machines were added to do the same work, so it was finished 6 hours earlier than planned. How many machines were added?

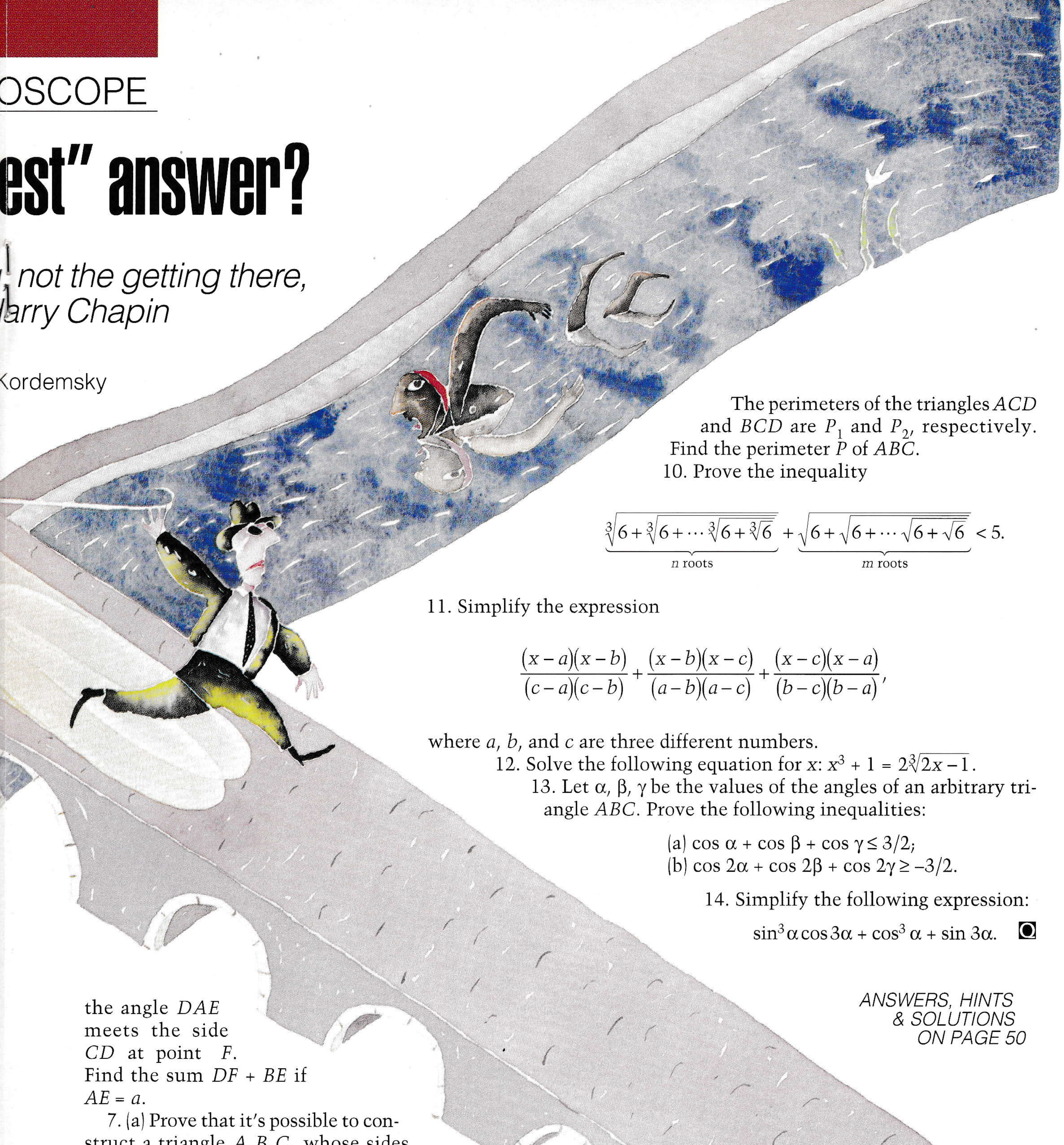
5. Two pieces of the same mass were cut from two copper alloy ingots of equal mass, but with different concentrations of copper. Each of these pieces was alloyed with the remainder of the other ingot. It turned out that the concentrations of copper in the new ingots became equal. In what ratio were the initial ingots divided?

6. An arbitrary point E is taken on the side BC of a square $ABCD$. The bisector of

est" answer?

not the getting there,
larry Chapin

Kordemsky



the angle DAE meets the side CD at point F. Find the sum $DF + BE$ if $AE = a$.

7. (a) Prove that it's possible to construct a triangle $A_1B_1C_1$ whose sides are equal in length to the medians of a given triangle ABC. (b) Prove that the triangle $A_2B_2C_2$ constructed from the medians of the triangle $A_1B_1C_1$ described in part (a) is similar to triangle ABC. (c) Find the ratio of the areas of triangles $A_2B_2C_2$ and ABC.

8. Let K be the midpoint of the median AM in triangle ABC, and let the line BK meet AC at L. Find the area of the quadrilateral $LKMC$ if the area of ABC is 1.

9. The altitude CD is dropped on the hypotenuse AB of a right triangle ABC. The perimeters of the triangles ACD and BCD are P_1 and P_2, respectively. Find the perimeter P of ABC.

10. Prove the inequality

$$\underbrace{\sqrt[3]{6+\sqrt[3]{6+\cdots\sqrt[3]{6+\sqrt[3]{6}}}}}_{n\text{ roots}} + \underbrace{\sqrt{6+\sqrt{6+\cdots\sqrt{6+\sqrt{6}}}}}_{m\text{ roots}} < 5.$$

11. Simplify the expression

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)},$$

where a, b, and c are three different numbers.

12. Solve the following equation for x: $x^3 + 1 = 2\sqrt[3]{2x-1}$.

13. Let α, β, γ be the values of the angles of an arbitrary triangle ABC. Prove the following inequalities:

(a) $\cos\alpha + \cos\beta + \cos\gamma \le 3/2$;
(b) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma \ge -3/2$.

14. Simplify the following expression:

$$\sin^3\alpha\cos 3\alpha + \cos^3\alpha + \sin 3\alpha.$$

ANSWERS, HINTS & SOLUTIONS ON PAGE 50

PHYSICS CONTEST

Boing, boing, boing . . .

"Nature is an endless combination and repetition of a very few laws. She hums the old well-known air through innumerable variations."—Ralph Waldo Emerson

by Arthur Eisenkraft and Larry D. Kirkpatrick

WHILE WATCHING THE Olympic Games in Atlanta this summer, you may be reminded once again of the versatility of physics. The equations for projectile motion can be used to analyze many different track and field events at the Olympic Games—shotput, discus, hammer throw, javelin, high jump, long jump, triple jump, and pole vault. Now, the athletes are not required to understand all of this physics, but we know that the information is important. Coaches study the physics principles behind the events, and sports physicists analyze the events to improve the athlete's performance.

When we begin our study of projectile motion, we simplify the mathematics by assuming that there is no air resistance, which is definitely not true for the javelin and the discus. Under this assumption, we are fortunate that the motions in the vertical and horizontal directions can be analyzed separately. The horizontal motion is one with a constant velocity because there is no horizontal force acting—that is,

$$x = x_0 + v_{0x}t.$$

The vertical motion is one with a constant acceleration due to the force of gravity:

$$y = y_0 + v_{0y}t - {}^1\!/_2 gt^2.$$

We then proceed to analyze motion on a flat plane, where we are given a launch speed v_0 and a launch angle θ. If we choose the origin of our coordinate system at the launch site so that $x_0 = y_0 = 0$, our equations reduce to

$$x = v_0 \cos\theta\, t,$$
$$0 = v_0 \sin\theta - {}^1\!/_2\, gt.$$

We can then solve these simultaneous equations for the range x and the time of flight t:

$$t = \frac{2v_0 \sin\theta}{g},$$
$$x = \frac{2v_0^2 \cos\theta \sin\theta}{g}.$$

These equations can be used as a first analysis of the long jump. Because the maximum range occurs for $\theta = 45°$, the jumper should leave the ground at 45° (provided v_0 is the same independent of angle).

The analysis of the shotput is more difficult because the shot is launched at a different height than it lands. Our second equation is then quadratic.[1] We often analyze the simpler case in which the projectile is launched horizontally.

One of the problems on the preliminary exam used to select this year's US Physics Team was an interesting example of this last type of projectile motion problem. A ball is dropped vertically, falls a distance h, and strikes a ramp inclined at 45° to the horizontal. The ball undergoes a perfectly elastic collision. This means that the velocity component parallel to the surface of the ramp remains the same, while the perpendicular component reverses directions. How far down the ramp does the ball land after the first bounce?

For the free-fall portion of the motion, our second equation becomes

$$-y = -{}^1\!/_2 gt^2,$$

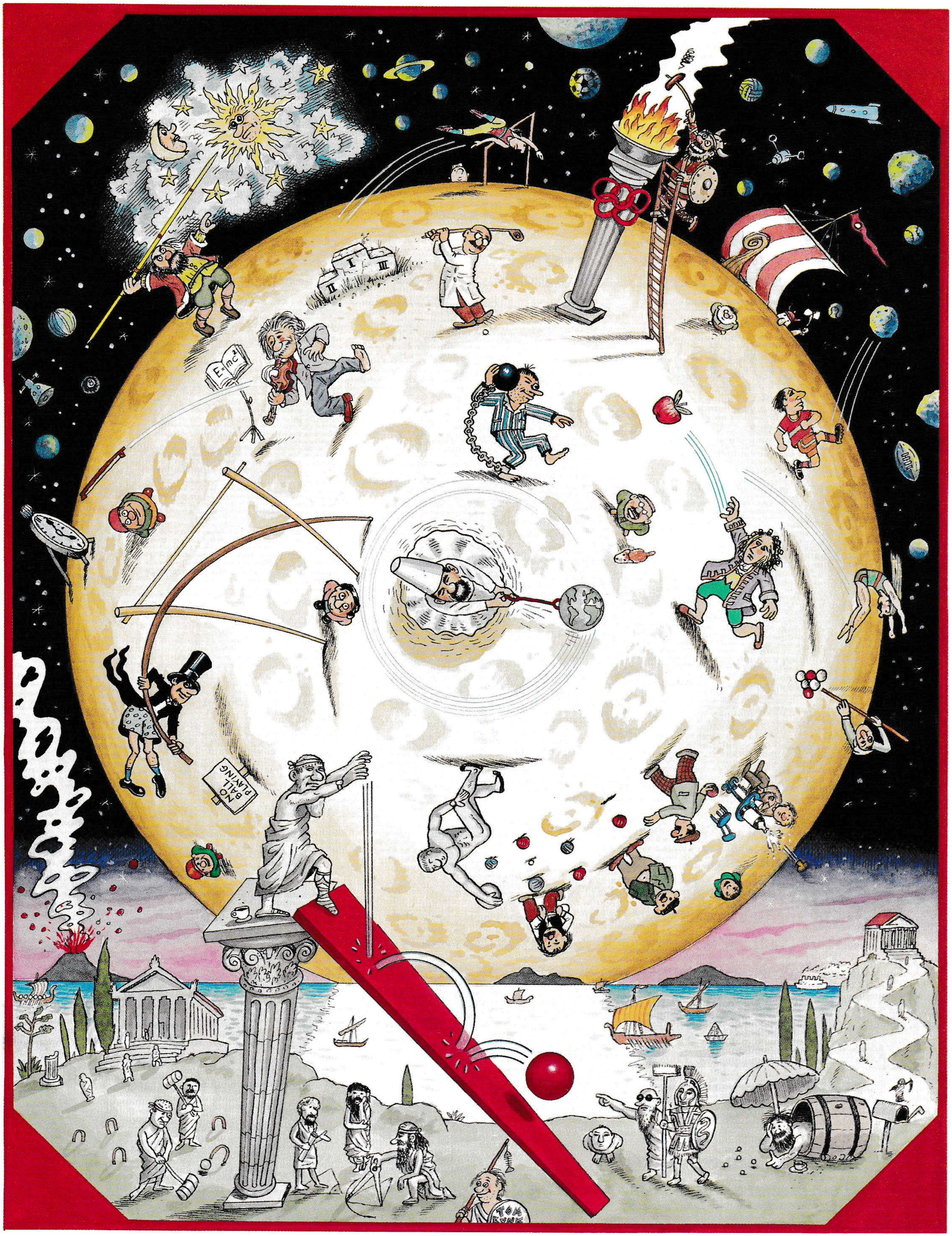
and therefore the time t_0 to reach the ramp is

$$t_0 = \sqrt{\frac{2h}{g}}.$$

We can use the conservation of energy or the kinematic equation $v = -gt$

[1]See "How the Ball Bounces" in *Quantum* (March/April and November/December 1991).

Art by Tomas Bunk



E=mc²
NO BALL PLAYING
TOM

to obtain the speed v_0 at impact:

$$v_0 = \sqrt{2gh}.$$

The choice of 45° for the ramp angle makes the problem easy, because the ball leaves the first bounce in the horizontal direction. This choice (and setting the new origin at the location of the first bounce) also leads to simple coordinates for the location of the second bounce, $(L_1/\sqrt{2}, -L_1/\sqrt{2})$, where L_1 is the distance measured down the ramp. Therefore, our equations for the projectile motion become

$$\frac{L_1}{\sqrt{2}} = v_0 t_1,$$

$$\frac{-L_1}{\sqrt{2}} = -g t_1^2.$$

Solving for L_1, we obtain our answer:

$$L_1 = 4\sqrt{2}h.$$

In trying to find a suitably challenging problem for our readers, we played around with changing the angle of the ramp, but things get messy and nothing very interesting emerged. However, when we looked at the second bounce, something very interesting emerged that prompted us to look at the third bounce, and the fourth . . .

A. Complete the analysis of the first bounce by showing that the time t_1 in the air equals $2t_0$, the speed of impact is $\sqrt{5}v_0$, and the angle θ_1 that the ball makes with the vertical is given by $\tan \theta = 1/2$.

B. For the second and third bounces, find the distance the ball travels down the ramp (in terms of L_1), the time in the air (in terms of t_1), the speed at impact (in terms of v_0), and the tangent of the angle with the vertical at impact.

C. Generalize your answers to the nth bounce and give physical reasons for the existence of the observed patterns.

Please send your solutions to *Quantum*, 1840 Wilson Boulevard, Arlington VA 22201-3000 within a month of receipt of this issue. The best solutions will be noted in this space and their authors will receive special certificates from *Quantum*.

Focusing fields

The contest problem in the January/February issue asked you to determine the boundaries of the magnetic field that would focus charged particles diverging from point P to a point R a distance $2a$ away.

The best solutions came from the wide range of readers that *Quantum* reaches. The best solution by a high school student came from Christopher Rybak, a senior at The Prairie School in Racine, Wisconsin. The best solution by a college student was written by Joseph Hermann, an undergraduate at the University of Missouri. Arthur Hovey, a physics teacher at Amity Regional High School in Woodbridge, Connecticut, and Daniel Dempsey of Canisius College in Buffalo, New York, also contributed excellent solutions. Prof. Dempsey sent along an exciting research paper he had published in *The Review of Scientific Instru-*

ments (Vol. 26, No. 12, 1141–45, Dec. 1955). In this paper, Prof. Dempsey provides solutions to the contest problem where the magnetic field boundaries are limited to straight lines and circles.

We know that the particles will travel in straight lines when they are not in the magnetic field and will travel in circular paths when they are in the magnetic field. These circular paths are defined by the relation

$$qvB = \frac{mv^2}{R}.$$

Since all particles in this problem have the same velocity, mass, and charge, they will all travel along circular paths of a specific radius:

$$r = \frac{mv}{qB}.$$

Upon leaving the magnetic field, the particles will travel along the tangent at the point where they leave the magnetic field. The solution to the problem consists of finding a boundary line where all circles of radius r have tangent lines that meet at point R. The centers of the circles of radius r lie on the y-axis.

Figure 1 shows the geometry of the solution. The equation of the circle is

$$x^2 + (y - b)^2 = r^2.$$

The similarity of the triangles yields

$$\frac{y-b}{x} = \frac{a-x}{y}.$$

Combining these equations by eliminating $(y - b)$, we find that the locus of points is defined by the relation

$$y = \frac{x(a-x)}{\sqrt{r^2 - x^2}}.$$

This function can most easily be viewed by using a graphing calculator or a spreadsheet and graphics program. It is symmetrical with respect to the y-axis.

The solution depends on the relationship between the radius of the circle and a. If $r < a$, the boundaries of the field extend to infinity, and all ions entering the field can be focused. See figure 2 for the case $a = {}^5\!/_3 r$. If $r = a$, the boundaries have finite positions, and once again all ions entering the field can be focused (fig. 3). If $r > a$, the boundaries begin to "flatten" and therefore restrict the angles at which ions can leave point P and arrive at R. See figure 4 for the case $a = {}^5\!/_8 r$. ◘

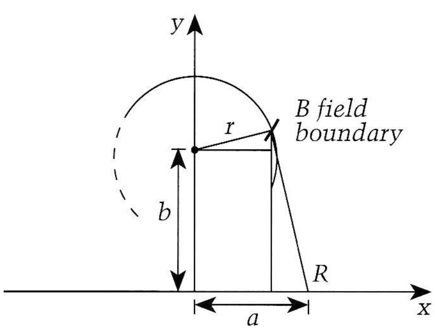


Figure 1

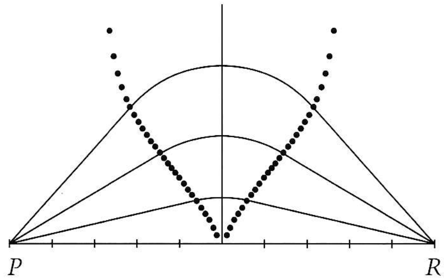


Figure 2

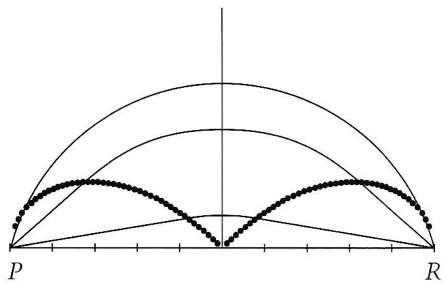


Figure 3

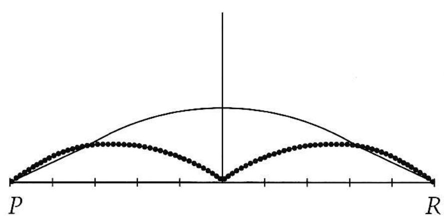


Figure 4

Dragon the Omnipresent

Or at least, the Persistent!

by Vladimir Dubrovsky

IN THIS ARTICLE WE'LL PROVE what is perhaps the most remarkable property of the dragon curves. This property was mentioned in the article "Dragon Curves" by Vasilyev and Gutenmacher in the September/October 1995 issue of *Quantum*, which was devoted to these beautiful creatures.

Although I'll repeat all necessary definitions, the reader would do well to read that article, as well as "Nesting Puzzles—Part 2" in the March/April issue, for a better and deeper understanding of the subject.

One of the definitions of dragon curves describes them as follows.

Draw a line segment OD_1 (fig. 1). Rotate it by 90° about its endpoint D_1. Then rotate the two-segment polygon OD_1D_2 by 90° about its endpoint D_2 (the image of O under the first rotation); rotate the four-segment polygon $OD_1D_2D_3$ thus obtained about D_3; and so on. After n rotations we get a 2^n-segment polygonal line $O...D_{n+1}$ with right angles at all its inner vertices. This polygon is called a *dragon design* (of the nth rank).

Figure 1 shows the particular case of a dragon design generated by rotations in the same direction (clockwise). It's called the *main dragon design*. Since a dragon design of a higher rank continues the corresponding designs of lower ranks, we can imagine them extended indefinitely and consider *infinite dragon designs*. Sometimes it's convenient to round off the corners of a design (the blue curve in the figure). This smoothing procedure yields what's called a *dragon curve*.

Now imagine we draw four perpendicular segments from the point O (fig. 2) and use each of them as the beginning of the corresponding infinite main dragon design. All their segments and vertices fit on the edges and nodes of an infinite grid of unit squares. The theorem we are going to prove holds that *the four main dragon designs just described trace all the edges of the grid, each edge once.*

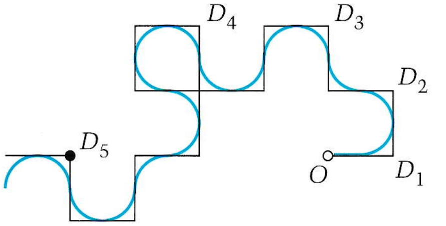


Figure 1

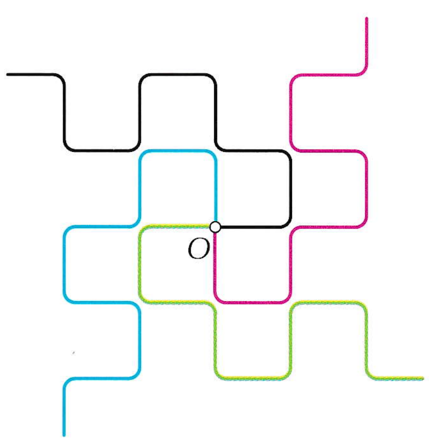


Figure 2

This theorem was discovered by Chandler Davis and Donald Knuth (the exact reference can be found in "Dragon Curves"), and the reasoning below follows in the footsteps of their original proof with certain modifications.

. . . each edge once

First, let's prove that no dragon design, whether it is a "main" design or not, traces the same edge of the grid twice (although it can make a loop and return to the same *node*).

This can be done using simple geometric considerations.

It's not hard to show, by comparing the numbers of steps in all four possible directions, that the length of any loop on a dragon design is a multiple of four and, consequently, that the first and the last segments in such a loop are always perpendicular to each other (see the solution to exercise 7 in "Dragon Curves"). It follows that a dragon design cannot trace the same segment AB in two opposite directions, because a loop of the form $AB...BA$ begins and ends with coincident rather than perpendicular segments.

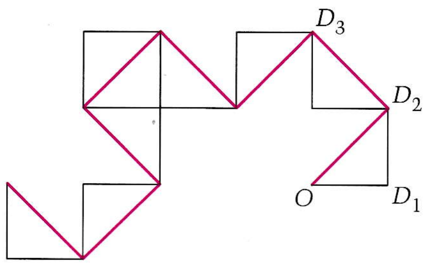


Figure 3

Another consequence of this property is that if a dragon design hits the same point A twice, then the directions in which it leaves it both times are either the same or opposite (because in the path $AB\ldots CAD$ the segment CA must be perpendicular to both "exit directions" AB and AD). Now we can prove that the same segment cannot be traced twice in the *same* direction.

Suppose that, on the contrary, this is possible and choose the dragon design of the smallest rank that traces some segment (say, AB) twice —that is, it has a piece of the form $KABL\ldots MABN$. Notice that the path joining, in order, all *even* vertices of any dragon design is also a dragon design (whose rank is one less—see figure 3), because it can be drawn by the same "quarter turn" rule starting with the segment OD_2 instead of OD_1. Since the number of edges in the loop $ABL\ldots MA$ is even, either both segments AL and AN or both segments KB and MB belong to the "shortcut" (red) design. Consider the first case. By our construction, we have two possibilities: $AL = AN$ (fig. 4a) or $AL \perp AN$ (fig. 4b). The first possibility contradicts our choice of the initial design (the shortcut design turns out to have coincident edges and a smaller rank). The second assumes that the shortcut design leaves point A along two *perpendicular* directions AL and AN, which contradicts the remark above. The argument for the case where the shortcut design contains the edges KB and MB is practically the same (both possibilities $KB = MB$ and $KB \perp MB$ lead to a contradiction).

Now we can prove that two main dragon designs issuing from O at right angles cannot have a common segment. Indeed, the rank n pieces of these designs can be viewed as two branches of one rank $n + 1$ dragon design (not necessarily "main"). But we know that it cannot trace the same segment twice.

It remains to consider two main dragon designs that start in opposite directions (say, up and down). If we replace them with the corresponding dragon curves, which pass around the grid nodes by definition, the fact that the designs have a common segment transforms into the fact that the curves intersect, thus forming a loop. Then the two dragon curves corresponding to the other two designs also form a loop (rotated by 90° relative to the first one). Clearly the two loops have at least one common point other than the origin. And this means that two dragon designs issued *at right angles* have a common segment, which was proved to be impossible.

So the four dragon designs do not overlap (and thus we've solved exercises 8 and 9 from "Dragon Curves").

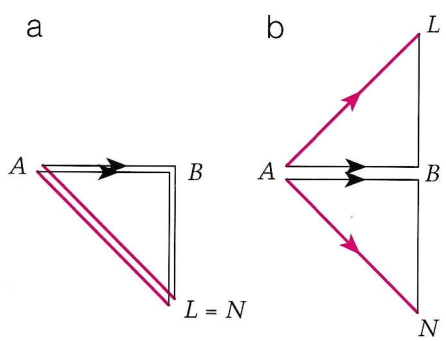


Figure 4

Dragon curves and complex numbers

The idea of the following proof of the "grid-filling" property is quite plain. Roughly speaking, we'll simply calculate the coordinates of a dragon design's nodes and use this formula to show that any point of the grid either belongs to at least two (of our four) dragon designs or occurs twice on the same design. Since a dragon design enters any of its nodes along a grid edge and leaves it along another (perpendicular) edge, and no edge is traced twice by the four designs, then each of the four edges issuing from any node, and therefore, any edge at all, belongs to one of the designs.

In fact, rather than coordinates proper, we'll use complex numbers to represent points of the grid. This is more convenient, because in terms of complex numbers, the 90° rotation (the main tool in constructing dragon designs) is just a multiplication by the imaginary unit i (fig. 5). In particular, the formula for the "eastbound" design (whose first segment joins the origin to the point (1, 0), or the complex number $1 + 0i$) becomes the formula for the "north-," "west-," or "southbound" designs after multiplying by i, -1, or $-i$, respectively.

So, denote by $d(n)$, $n = 0, 1, 2, \ldots$, the nth node of the eastbound design (and the corresponding complex number). To begin with, consider its "turning points" $d(0) = 0$, $d(1) = 1$, $d(2) = 1 + i$, and so on (figure 6 on the next page shows that $d(2^k) = D_{k+1}$). The point $d(2^{k+1})$ is obtained by a 90° clockwise rotation of the origin about $d(2^k)$. It follows that

$$\begin{aligned} d(2^{k+1}) &= d(2^k) + i(d(2^k) - 0) \\ &= (1 + i)d(2^k) \\ &= (1 + i)^2 d(2^{k-1}) \\ &= \ldots = (1 + i)^k d(2) \\ &= (1 + i)^{k+1}. \end{aligned}$$

Now let's take a number n that is not a power of two. Suppose $2^{k_1 - 1} < n < 2^{k_1}$. The point $d(n)$ is $n_1 = 2^{k_1} - n$ edges away from $d(2^{k_1})$, so, by the construction, it's obtained under the

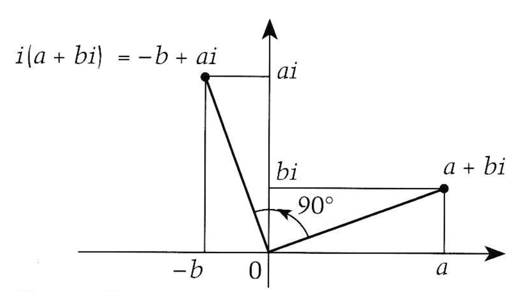


Figure 5
Multiplication by the imaginary unit i, which turns a point (complex number) $a + bi$ by 90° about zero.

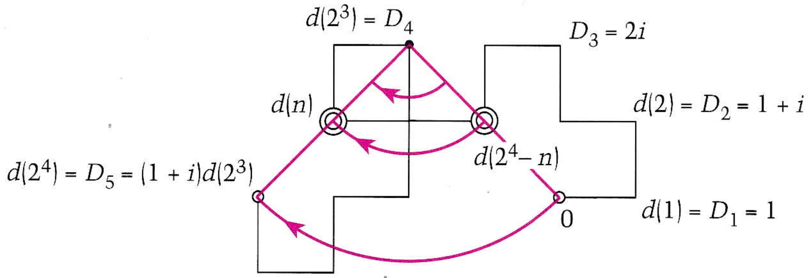


Figure 6

90° clockwise rotation about $d(2^{k_1-1})$ from the point $d(n_1)$, n_1 edges away from the origin. Algebraically, this can be written as

$$\begin{aligned} d(n) &= d(2^{k_1}) - \left(d(2^{k_1}) - d(n)\right) \\ &= d(2^{k_1}) - id(n_1) \\ &= (1+i)^{k_1} - id(n_1). \end{aligned}$$

If n_1 is not a power of two, we can repeat our reasoning for n_1, find the number k_2 such that $2^{k_2-1} < n_1 < 2^{k_2}$, define $n_2 = 2^{k_2} - n_1$, and write

$$\begin{aligned} d(n) &= (1+i)^{k_1} - i\left((1+i)^{k_1} - id(n_2)\right) \\ &= (1+i)^{k_1} - i(1+i)^{k_2} - d(n_2). \end{aligned}$$

Proceeding in the same way until we get some $n_m = 2^{k_m}$, we'll finally arrive at the formula

$$\begin{aligned} d(n) &= (1+i)^{k_1} - i(1+i)^{k_2} \\ &\quad -(1+i)^{k_3} + i(1+i)^{k_4} \\ &\quad + \cdots + (-i)^{m-1}(1+i)^{k_m}, \end{aligned} \tag{1}$$

where $k_1 > k_2 > \ldots k_m \geq 0$. This gives the representation of a complex number as the sum of powers of a fixed number, multiplied by certain coefficients. It resembles the representation of an integer using notation with a given base—but here the "base" is the complex number $1 + i$. Also, here the nonzero coefficients cycle with a period four: *each of them (except the first) is the previous one times $-i$.* Such representations of complex integers are called *revolving representations* (the term was coined by Davis and Knuth).

It's clear from the construction that the highest-order "digit" in the revolving representation of a node on the eastbound dragon design is always 1; for the north-, west-, and southbound designs the corresponding digits are i, -1, and $-i$, respectively.

Let's look again at the numbers $k_1, k_2, \ldots, k_m$ of nonzero "digits" in the revolving representation of $d(n)$. By definition we have

$$\begin{aligned} n &= 2^{k_1} - n_1 = 2^{k_1} - \left(2^{k_2} - n_2\right) \\ &= 2^{k_1} - 2^{k_2} + 2^{k_3} \\ &\quad - \cdots + (-1)^{m-1}2^{k_m}. \end{aligned} \tag{2}$$

This representation may ring a bell for *Quantum* readers: it cropped up in our investigation of the Chinese Rings puzzle (see "Nesting Puzzles—Part 2" in the March/April 1996 issue) and was called there the *folded binary representation* of n (again, after Davis and Knuth). Now the interesting connection mentioned there between this and similar puzzles, on the one hand, and dragon curves, on the other, becomes clear. However, let me refer you to that article for additional details of this connection and folded representations. The really important fact for us now is that any positive integer has exactly two folded binary representations—one with an even number of nonzero digits, the other with an odd number (see problem 3 in "Nesting Puzzles").

More exactly, one of these representations will have m nonzero digits with numbers $k_1 > k_2 > \ldots > k_{m-1} > k_m$, where $k_{m-1} > k_m + 1$. The other representation will have $m + 1$ nonzero digits with numbers $k_1 > k_2 > \ldots > k_{m-1} > k_m + 1 > k_m$. The first $m - 1$ terms in any two such representations are the same. The last term in the first representation is $(-1)^{m-1}2^{k_m}$ and the second ends in $(-1)^{m-1}2^{k_m+1} - (-1)^{m-1}2^{k_m} = (-1)^{m-1}(2^{k_m+1} - 2^{k_m})$, which is numerically the same. For instance, $19 = 32 - 16 + 4 - 1 = 2^5 - 2^4 + 2^2 - 2^0 = 2^5 - 2^4 + 2^2 - 2^1 + 2^0$. We can see that our representation of $d(n)$ is of the first type (with $k_{m-1} > k_m + 1$). However, its "sister representation" yields the same point in the plane, because $(1 + i)^{k+1} - i(1 + i)^k = (1 + i)^k(1 + i - i) = (1 + i)^k$.

So we can calculate the location of the nth node of a given dragon design. Conversely, any complex integer that has a revolving representation is a node of one of our four designs: its highest-order digit shows which of the four design it belongs to, and the numbers $k_1, \ldots, k_m$ yield, by formula (2), its distance from the origin along the dragon path.

Revolving representations

The next step is to show that any complex integer has a revolving representation. Actually, we'll prove a slightly stronger fact: any complex integer $z = a + bi$ has four revolving representations with the lowest-order digits equal to 1, i, -1, and $-i$. For instance,

$$\begin{aligned} 1 &= -i(1 + i)^2 - (1 + i) + i \\ &= -i(1 + i)^2 - 1 = (1 + i) - i. \end{aligned} \tag{3}$$

These four representations will be called 1-, i-, (-1)-, and $(-i)$-representations, respectively.

We'll use induction over $|z|^2 = a^2 + b^2$. For $|z|^2 = 1$, the statement is true by equations (3) (the representations for i, -1, and $-i$ are obtained by multiplying those for 1 by these numbers themselves). Suppose it's true for all u such that $|u|^2 < N$ and take any z with $|z|^2 = N$. If $z = a + bi$ is divisible by $1 + i$—that is, if $z = (1 + i)u$, where u is a complex integer—then $|u|^2 < N$ (the reader can check that in fact $|u|^2 = |z|^2/2$). So the number u has revolving representations of all four kinds. Multiplied by $1 + i$, they yield the representations for z.

Exercise. Prove that a complex integer $a + bi$ is divisible by $1 + i$ if and only if the numbers a and b are of the same parity.

It follows from this exercise that if the number z is not divisible by $1 + i$, then each of the four numbers $z \pm 1$, $z \pm i$ is. To obtain, say, the 1-representation of z, we take the number $z - 1$ and divide it by $1 + i$. This gives us $z - 1 = (1 + i)u$. For the numbers z in question, $|u| < |z|$, so u has an i-representation. Substituting this representation into $z = (1 + i)u + 1$, we come up with the required representation for z. In this case, by choosing the i-representation for u, we secure the correct alternation of nonzero digits in the representation of z. Try to figure out how the proof should be modified to get the other three representations of z.

Filling the grid

The last part of the proof is very short.

Take any complex integer z ($z \neq 0$) and its 1- and (-1)-representations. Each of them shows that point z is hit by the corresponding dragon design at a certain moment.

If the highest-order digits in the two representations are different, then *the two designs are different*.

If the digits are the same (for instance, both of them are ones), then z is visited twice by the same (eastbound) dragon design, but *at two different moments*. Indeed, the number m of nonzero digits in one of these revolving representations must have the same parity as in the other (because the lowest-order digit $(-i)^{m-1}$ in the one case is 1, and in the other it's -1). Therefore, the two corresponding folded binary representations of the moments n such that $d(n) = z$ yield different values of n. (Recall that one of the two possible folded representations of a given integer always has an odd number of nonzero digits, and the other has an even number.)

As was explained above, this means that each edge of the grid belongs to one of the four dragon designs.

PUBLISHER'S PAGE CONTINUED FROM PAGE 3

1474: First book printed in English. **1482:** Euclid's *Elements* produced by Ratdolt, the first printed book with geometric figures. **1485:** First book censorship decrees issued—books perceived as "dangerous." Ratdolt prints *De Sphera*, first book to use more than two differently colored inks on the same page. **1500:** Some 20 million copies of about 35,000 books have been published and printed, 77 percent in Latin, and 45 percent religious.

It's no wonder that the period from about A.D. 100 until the 1400s is regarded as the "dark ages." A substantial expansion of the circle of human beings capable of reading the written word did not take place until the great quantum change that resulted from the invention of the printing press. The printing press, the use of paper for printing, and all of the innovative means of producing and distributing books in hundreds of languages required only the next 200 years.

1559: Publishing of the *Index libvrorum prohibitorum* by Papal authorities—books forbidden to be read by Catholics. **1610:** The *Mercurius gallo-belgicus* newspaper starts publication in London. The *Nieuwe Tijdingen* newspaper is published in Belgium. **1638:** First printing press installed on American continent at Cambridge, Massachusetts. **1642:** Pascal invents adding machine. **1671**: Newton demonstrates that white light is made up of the colors of the rainbow. **1679:** Mechanical computer is invented by Leibniz that can add, subtract, multiply and divide numbers. **1802**: A method of creating images on silver nitrate using a camera obscura is reported by Wedgwood. **1805**: Punched cards are used to control the operation of a loom. **1807**: A machine is invented that makes paper by a continuous process, instead of one sheet at a time. **1811:** The first mechanical press is invented in London. It prints 3,000 sheets per hour.

The next quantum leap in communication technology came about in several areas: our knowledge of science, especially optics and electromagnetics; the observation that the sense of motion of images could be produced by rapidly changing still images; mechanical devices to count, control machines; and do arithmetic; and the recording and replication of images of scenes and people—the development of photography.

1822: The first permanent photograph is made. Babbage develops the precursor to a computer, a difference machine for calculating values of logarithms and trigonometric functions. **1824:** Braille creates method of using raised dots so that the blind can read. **1829:** Burt invents a primitive typewriter. **1832:** Babbage conceives, but does not successfully build, the first computer. **1838:** Daguerre invents process for producing silver images on copper plates, an early and popular form of photography. **1839:** Talbot creates first photographic negatives, reports his invention of photography. **1847:** Hoe invents both the rotary and web presses, which can print 18,000 sheets on both sides per hour. **1856**: A phono-autograph is invented in France that produces a trace produced by sound on a rotating drum, precursor to the phonograph.

In electromagnetics, it was the discovery of the empirical laws of electricity and magnetism, leading to the theory of electromagnetism, as expressed in Maxwell's equations, that produced the entire industry of radio. This was initiated by Hertz, who demonstrated that electromagnetic waves could be produced and that they moved at a speed determined by the constants in the force equations for magnetism and electric charge. This demonstration also showed that light itself was just another kind of electromagnetic radiation. Marconi quickly followed with the invention of a primitive form of radio, and when Lee DeForest invented the vacuum tube with a control grid, amplification was possible—the birth of radio. Not far behind was a peculiar merger of "moving pictures" and optical physics, leading to the invention of television. Photography and radio thus merged into television.

1858: Light is observed that is emitted by electrons hitting a screen in an evacuated tube. **1861:** The first color reproduction is demonstrated, using red, green, and violet filters. **1867**: Sholes develops the typewriter with the "qwerty" keyboard. **1869:** Hauron shows how to create "pixels" of different color that are perceived in ways that produce color images. **1874:** Baudot creates a binary code that uses five bits to represent characters. **1875:** Carey proposes a form of "television." **1876:** Bell patents the telephone. **1877:** Edison invents phonograph. **1880:** 54,000 telephones are in use in the US. **1881:** Marey develops precursor to the movie camera. **1888:** Hertz demonstrates that a varying electric current produces electromagnetic waves that can be detected at a distance. Edison and Dickson invent a motion picture machine with sound synchronization. Eastman introduces the box camera and rolls of film. Smith describes a device that records sound by magnetism. **1890:** Hollerith develops a census system using punched cards. **1893:** Magnetic sound recorder is invented by Poulsen. **1894:** Marconi builds and demonstrates his first radio transmitter and receiver. **1895:** Lumiere makes first motion picture, 35-mm film at 16 frames/second.

Almost concurrent with these great changes in communications, mechanical controls for weaving machines and other equipment, along with mechanical devices that would add, subtract, multiply and divide, led to the first mechanical or vacuum-tube computers. These were powerful and useful devices, but massive in scale and accessible only to a very tiny group of people, mostly scientists.

1896: Hollerith, who used punched cards for the census, founds Tabulating Machine Company, later to be renamed International Business Machines (IBM). **1904:** Fessenden transmits speech by modulating an electromagnetic wave. **1906:** Fessenden transmits music and speech via EM waves to ships at sea. **1908:** Lippman wins Nobel Prize for invention of color photography. **1910:** Seven million telephones are used in the US. **1911:** Swinton describes elements of a modern television system, using CRTs in both the transmitter and receiver. **1912:** DeForest invents vacuum tube that amplifies signals. **1915:** Benedicks discovers that a germanium crystal can convert AC to DC, a precursor to the computer chip. **1919:** RCA is founded.

It was another great advance in physics that led to the next sea change in communications. The invention of the transistor, followed by integrated circuits, allowing electronics to be reduced dramatically in physical size and consumption of electric power. These chips have so evolved that our entire communications and computer industries utilize them almost to the exclusion of the older technologies of vacuum tubes, resistors, capacitors, and inductors.

The next chunk of chronology has quite a few items. Is it merely our proximity to these events that makes them seem so important?

1920: First commercial radio station, KDKA is established. **1920:** DeForest develops improved motion picture machine. **1924**: Zenith produces portable radio. **1925:** Baird produces first television image of a human face. Zworykin files patent for first color television system. **1926:** Talking motion pictures are introduced. **1927:** Electromechanical analog computer invented by Bush at MIT. Neill invents tape with metallic layer for sound recording. **1928:** Television broadcasts begin. **1929:** Bell Labs develops color television system. **1931:** CBS starts television broadcasting. Stereo sound is patented. *Quantum*'s Publisher, Bill Aldridge, is born. **1934:** Turing conceives the Universal Turing Machine, the basis of modern computers. **1935:** Kodachrome introduced by Kodak. **1936:** Olympic Games are televised from Berlin. **1937:** Carlson invents xerography. **1938:** Zuse completes the first working computer to use a binary code. **1939:** Stibitz and Williams build the Bell model 1 computer and introduce the terminal. **1941:** Zuse makes computer with error-detecting code and punched tape for data entry—the first computer controlled by a program. **1942:** Shockley begins work leading to development of transistor. **1944**: The spinning disk drive is invented by Eckert. **1945:** Grace Hopper coins the term "computer bug." **1946:** Mauchly and Eckert Demonstrate ENIAC. **1947:** Gabor develops concept of holography. Land invents instant camera. **1948:** Practical magnetic drum for computer storage is introduced. **1948:** Shockley, Brattain, and Bardeen announce discovery of transistor. The first cable TV systems appear in the US. **1951:** A military supercomputer called Atlas has magnetic drums that store one megabyte. Grace Hopper develops first compiler to translate codes into binary machine code. **1953:** Commercial color TV begins in the US. Townes develops the maser, precursor to the laser. **1954:** FORTRAN programming language created. **1955:** First transistor computer built. Optical fibers introduced. **1957:** Gordon Gould conceives the laser and gets patent only after 1986. **1958:** Integrated circuit on one silicon chip developed. Stereo records are introduced. Schawlow shows how a laser works. **1959**: Feynman describes process of etching that forms basis of modern chip design and construction. Hopper and Phillips invent COBOL language. **1960:** Packet switching to exchange data for computers invented. **1961:** Time-sharing of computers developed. **1963:** Video Disk invented. **1964:** ASCII standard developed. ARPA conceived, precursor to the Internet. IBM develops first CAD software. **1966:** First fiber optics used for transmission of data. **1967:** Computer networking devised. **1968:** Computer mouse invented. **1969:** RS-232 devised to exchange data between computers. Bubble memory invented. **1970:** Floppy disk introduced. RCA introduces MOS chips. **1971:** First microprocessor patented. **1972:** Speech recognition systems introduced. Intel develops first 8-bit microprocessor, 8008, replaced later by the 8080 chip. Landsat I is launched. **1973:** LSI chips introduced. **1974:** Electronic scanners developed. **1975:** Ethernet developed for local area networks. Laser printer produced. **1977:** Apple II introduced. Microsoft is founded. **1979:** Motorola introduces the 68000 chip, used by the Macintosh computer. **1981:** Microsoft's MS-DOS adopted by IBM for its personal computer. **1982:** PostScript system invented. **1983:** Apple introduces mouse associated with cursor on screen with Lisa computer. IBM XT introduced. **1984:** CD-ROM introduced. **1985:** Microsoft develops Windows. **1986–96:** Explosion in creation of more complex microprocessors, memory chips, use of video and audio with computers and communications. Expansion of software, including creation of Windows 95. Internet becomes major means of communications.

Homo sapiens is now about to enter the most dramatic shift in communications technology, since the invention of the printing press. There has been a rapid convergence

CONTINUED ON PAGE 40

Osmosis the Magnificent

A powerful force—but is it an engine for perpetual motion?

by Norayr Paravyan

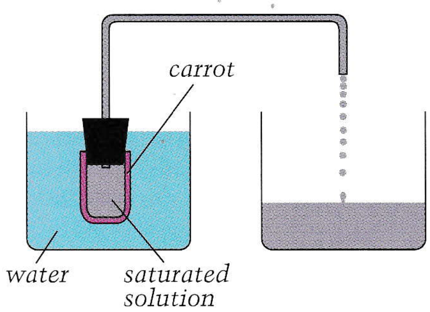
IMAGINE A SOLUTION SEPArated from a pure solvent by a semipermeable partition that lets the small molecules of solvent pass through, but stops the larger molecules of solute. Due to the natural tendency for concentrations to become equal throughout the volume, unilateral molecular diffusion of the solvent will take place. This spontaneous movement of the solvent to the solution (separated from it by the semipermeable membrane) is called osmosis. Usually it's said that the solvent penetrates the solution under the force of osmotic pressure. In other words, osmotic pressure serves as a quantitative characteristic of the phenomenon of osmosis.

Osmotic pressure can be measured—it's equal to the extra pressure needed from the solution side to stop osmosis. It has been established both experimentally and theoretically (the molecular theory of solutions) that the osmotic pressure is proportional to the concentration of a solution and its (absolute) temperature. It would be interesting to verify this relationship—but how? To carry out our experiments, we need a special device—an osmometer, which you probably don't have in your school physics lab.

There's a way around this—we'll make our own osmometer. Take an ordinary carrot, 10–12 cm long and 3–4 cm in diameter. Carefully wash and scrape it and then cut off the tip and the top. Using a special drill for cutting rubber stoppers (ask for it in the chemistry lab), drill out the center of the carrot. The thickness of the walls of the "vessel" produced should be 3 to 8 mm. If you can't lay your hands on a rubber drill, you can get by with a narrow penknife. Also, you can use something else in place of the carrot: a beet, turnip, rutabaga, potato—whatever you have at hand. Now find a rubber stopper to match your vegetable vessel—one that has a hole in it and matching tubing. Our osmometer (see the figure on the next page) is now ready, and we can proceed with our experiments.

Art by Sergey Ivanov

Fill the vessel (up to the brim) with a saturated solution of any water-soluble substance—table salt (sodium chloride), Glauber's salt (sodium sulfate decahydrate), sugar—and then close the vessel tightly with the plug with the tubing. Be careful that there are no air bubbles under the plug—otherwise this air will offer serious "resistance" to the osmosis. Put the filled vessel into a glass filled with ordinary tap water. Rather than hold the vessel in your



hand, attach it to a laboratory stand with a clamp connected to the plug. Make sure the water level in the glass is higher than that in the vegetable vessel. Also, place an empty can or cup at the other end of the tubing.

After a few minutes fluid driven by osmotic pressure will pour out of the tube into the empty can. In about 30–40 minutes the water level in the outer glass will have dropped significantly. How long will this process continue? At first glance, there are no limitations—we can always add more water to the outer glass. But that's only at first glance. In fact, however long this experiment proceeds, it will necessarily stop. This is because the water penetrates the semipermeable partition (the walls of the vegetable vessel) into our osmometer, resulting in a continual decrease in the concentration of salt in the solution. In addition, the concentration is decreased because some of the solution is leaving the osmometer via the tubing. So we have pure water coming into the system, and salt solution coming out. When the concentrations of the solutes inside and outside of the vegetable vessel become equal, the fluid stops dripping out of our device, and the experiment comes to an end.

I should point out that an experiment with this particular setup may last quite a long time (two or three hours)—if, of course, you have the patience to add water to the outer glass of the osmometer. However, the process will necessarily stop. Alas, a *perpetuum mobile* cannot be constructed on the basis of osmotic pressure (or on any other basis, as you well know!).

PUBLISHER'S PAGE CONTINUED FROM PAGE 38

of technologies related to the printed word, images (fixed and dynamic), and computers. This shift is rapidly making the printed word virtually obsolete. Within the next 50 years, the use of paper for newspapers, textbooks, novels, and reference works will cease almost entirely. In their place will be ROM chips containing the information one wants, which are simply placed in a small, inexpensive computing and display device—you will buy a novel, say, as ROM chip and use your book-size device to "read" it on the display screen. Material will also be downloaded from a variety of sources on some new super-Internet system connected by fiber optics to every home. Such a system will combine what is now found on the Internet with what we now get on television. The merger of television and computer will then also be complete.

The remainder of my chronology doesn't come from *The Timetables of Technology*. It constitutes my predictions for the next 30 years.

1997: Cable TV lines used to transmit and receive computer data via the Internet. **1999:** Fiber optics widely used to connect homes so that graphics and data are easily and rapidly exchanged. **2005:** Widespread use of a merged system of TV, computer, and Internet, with homes connected by fiber optics. **2006:** Widespread use of electronic books that use ROM chips to deliver the content of textbooks, novels, and reference materials. Printed books decline rapidly in use. **2008:** Dynamic illustration routinely used in books. **2010:** Symbolic communication now a merger of dynamic graphics and other images, with minimal use of words as symbols to represent these images. **2020:** Communications uses other sensors associated with touch, taste, and smell to communicate, leading to a further reduction in the use of word symbols. **2025:** Few printed books in use anywhere. Everything is electronic and optical.

I wonder where these fundamental changes will leave us, and what will be lost. We already see the use of graphics and moving pictures in television and on the Internet replacing the strings of words that used to elicit those images in the mind of the reader. Great books of the past were great because of the writer's ability to invoke vicarious experiences in the reader by means of words. The printed word opened the world and the universe to billions of people over the past 300 years. A picture may be "worth a thousand words," but the ability to use 1,000 words to paint a picture may in the long term have been a far more valuable skill. But who's to say?

—Bill G. Aldridge

Bulletin Board

Freshman wins Schafer Prize

Ioana Dumitriu, a freshman at New York's Courant Institute of Mathematical Sciences, is the winner of the seventh annual Alice T. Schafer Mathematics Prize. The Schafer Prize is awarded to an undergraduate woman in recognition of excellence in mathematics and is sponsored by the Association for Women in Mathematics (AWM). Dumitriu will receive a cash prize of $1,000.

The Schafer Prize was established in 1990 by the executive committee of AWM and is named for AWM former president and one of its founding members Alice T. Schafer, who has contributed a great deal to women in mathematics throughout her career. The criteria for selection includes, but is not limited to, the quality of the nominees' performance in mathematics courses and special programs, an exhibition of real interest in mathematics, the ability to do independent work, and, if applicable, performance in mathematical competitions.

Two runners-up were also selected: Karen Ball, a senior at Grinnell College, and Wungkum Fong, a senior at the University of California at Berkeley. Each will receive $150. AWM also awarded an honorable mention to Tara S. Holm from Dartmouth College. The prize presentation will take place on the evening of July 22, 1996, at the AWM Banquet (held in conjunction with the AWM Workshop) in Kansas City, Missouri. The AWM Workshop and Schafer Prize Session are held in conjunction with the SIAM Annual Meeting.

"There were many outstanding nominees this year, each with her own style and her own strengths," stated Ruth Charney of Ohio State University, chair of the 1996 Schafer Prize Committee. "It was very difficult to choose a winner. We are pleased to be able to recognize these four exceptional young women."

The Schafer Prize is funded by an endowment with continuing contributions from AWM members and others. Additional contributions will help ensure the long-term viability of the prize. Checks made payable to "ATS Prize Fund" may be sent to AWM, 4114 Computer and Space Sciences Building, University of Maryland, College Park MD 20742-2461.

The Association for Women in Mathematics, founded in 1971, was established to encourage women to study and have active careers in the mathematical sciences. Equal opportunity and the equal treatment of women in the mathematical sciences are promoted. AWM has more than 4,500 members, both women and men, from the United States and around the world, representing all parts of the mathematical community.

Web research service

Information Access Company, a provider of electronic reference information to libraries and schools, announces the debut of Cognito!®, a new student and family research tool on the World Wide Web. Cognito! is a subscription-based service that gathers and indexes more than 995,000 articles and documents from diverse sources—magazines, encyclopedias, reference books, pamphlets, and Internet sites. Editors at Information Access continually add new articles and cross references to Web sites.

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Coin-rolling CyberTeaser

Things got a little sticky at the corners as contestants mentally maneuvered two coins around a rectangle in the July/August CyberTeaser at *Quantum*'s Web site (brainteaser B178 in this issue). Most everyone recognized the problem at the *inner* corners; it was the *outer* corners that tripped some people up. The following e-mailers were the first to submit an answer that satisfied our judge:

John Condon (Houston, Texas)
Leonid Borovskiy (Brooklyn, New York)
11th Grade Math Club: Ilknur Kocak, Murat Tanoren, Sýmay Parmaksiz, Ozsel Beleli (Izmir, Turkey)
Nikolai Yakovenko (College Park, Maryland)
Keith Grizzell (Gainesville, Florida)
Nikolai Kukharkin (Princeton, New Jersey)

Jim Grady (Branchburg, New Jersey)
Lorenzo Maccone (Ivrea, Italy)
Mahesh Madhav (Morton Grove, Illinois)
Peter Onyisi (Arlington, Virginia)

The joint entry by the 11th Grade Math Club presented us with a dilemma: how many *Quantum* buttons to award? In this case, four. But in the future, such combined entries will be eligible for a single button. And only the "corporate name" will be published among the winners. So if it's fame you're after, submit your entry as an individual!

Our congratulations to the winners, who will receive a copy of this issue of *Quantum* and the coveted *Quantum* button. Everyone who submitted a correct answer (up to the time the answer is posted on the Web) is eligible for a drawing to win a copy of *Quantum Quandaries*, the new collection of the first 100 *Quantum* brainteasers. Our thanks to everyone who submitted an answer—right, wrong, close, or ever so close. The new CyberTeaser is waiting for you at http://www.nsta.org/quantum.

What's happening?

Summer study ... competitions ... new books ... ongoing activities ... clubs and associations ... free samples ... contests ... whatever it is, if you think it's of interest to *Quantum* readers, let us know about it! Help us fill Happenings and the Bulletin Board with short news items, firsthand reports, and announcements of upcoming events.

What's on your mind?

Write to us! We want to know what you think of *Quantum*. What do you like the most? What would you like to see more of? And, yes—what *don't* you like about *Quantum*? We want to make it even better, but we need your help.

What's our address?

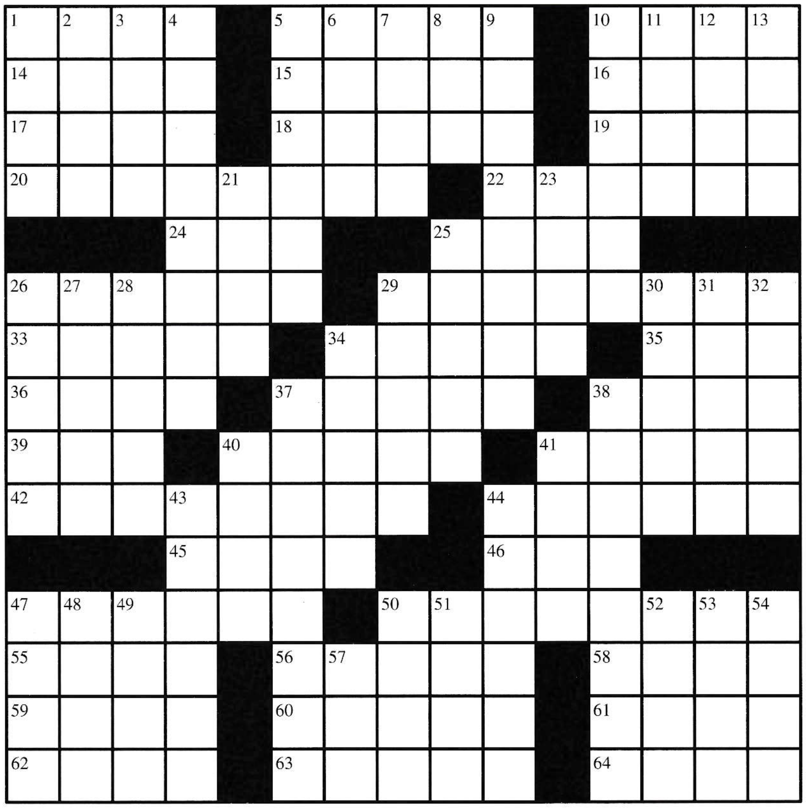
Quantum
National Science Teachers Association
1840 Wilson Boulevard
Arlington VA 22201-3000

Be a factor in the QUANTUM equation!

Volume 6 (1995—96)

Criss × cross science

by David R. Martin



Across

1 Acid's opposite
5 Six-atom ring
10 British beach
14 43,756 (in base 16)
15 Surveyor's instrument
16 Esker
17 English chemist Frederick Augustus ___ (1827–1902)
18 Fireplace item
19 10^{-9}: pref.
20 Summation
22 Cumulus and cirrus
24 Tap
25 Participant: suff.
26 Delicately suggestive
29 ___ *Park*
33 Path
34 Atheist's claim (2 wds.)
35 Computer language
36 60,076 (in base 16)
37 Schwarzenegger role
38 Ohre (in Germany)
39 Geologic unit of time
40 Glass-to-metal seal alloy
41 Smudge
42 German math. ___ of Cusa (1401–1464)
44 Bitterly
45 Conger and Moray, e.g.
46 Annex
47 Polish–US track athlete ___ Walsh (1911–1980)
50 Lamp oil
55 Phosphatase standardization unit
56 ___ box (for handling radioactive materials)
58 ___ *fixe* (obsession)
59 Boyfriend
60 Devoured
61 A Greek national park
62 Swedish botanist ___ Afzelius (1750–1837)
63 Not a soul (2 wds.)
64 Space agcy.

Down

1 47,786 (in base 16)
2 43,709 (in base 16)
3 Plant starter
4 Earth's orbital plane
5 Petrological unit
6 Null
7 ___ Pavlov
8 After deductions
9 Fermi particle
10 Free electron suppliers
11 Botanist Katherine ___
12 Type of gate
13 God of love
21 Lofty
23 Element 82
25 Saccharide
26 Dutch painter (1626–1679)
27 Arrow poison
28 764,588 (in base 16)
29 ___ Salk
30 Geneticist Ruth ___
31 Type of gas
32 Convey
34 Exploding stars
37 Fibrous protein
38 Photosensitive material
40 Cuckoo
41 Alone
43 Element 2
44 Peaceful
47 One of the Leeward Islands
48 ___ off (irate)
49 60,330 (in base 16)
50 Zither's cousin
51 Type of parity
52 Concept
53 Tennis court items
54 61,098 (in base 16)
57 ___-tse (Chin. philosopher)

SOLUTION IN THE NEXT ISSUE

SOLUTION TO THE MAY/JUNE PUZZLE

A	B	A	S		A	A	C	F	E		Q	C	D	
D	E	B	A		U	D	A	L	L		U	B	A	C
A	N	E	W		R	E	D	Y	E		A	D	D	A
M	O	L	T		O	R	E		C	A	R	B	O	N
			O	A	R				T	A	T			
S	C	H	O	L	A	R		F	R	E	S	N	E	L
T	O	O	T	S		A	E	R	I	E		I	C	E
R	A	S	H		C	U	S	E	C		S	T	E	P
A	C	E		F	E	M	T	O		H	O	R	D	E
P	H	A	E	D	R	A		N	U	C	L	E	A	R
			D	D	E				R	H	E			
J	O	V	I	A	N		A	L	A		N	A	S	T
A	L	E	S		K	I	N	I	N		O	B	I	E
N	A	N	O		O	S	A	M	U		I	B	A	R
	V	A	N		V	O	T	E	S		D	E	L	A

ANSWERS, HINTS & SOLUTIONS

Math

M176

Yes, this can happen. An example of the set of faces that can be glued into two differently shaped polyhedrons is given in figure 1. Each polyhedron consists of two congruent irregular quadrilateral pyramids glued together along their bases. The bases are rectangles, and in one case the pyramids are symmetric about their common base (fig. 2a), while in the other they are symmetric about the center of the base (fig. 2b).

Try to think of an example with a smaller or even the smallest possible number of faces. Is it necessary that the volumes of the two polyhedrons be the same?

Notice that if all edges of a polyhedron are of different length or if they are simply marked by different labels on the adjacent faces, so as to fix the order in which the faces are glued together, then Laura will get the same polyhedron as Andrew. This fact is far from obvious. The general case is known as Cauchy's theorem on convex polyhedrons. (N. Vasilyev)

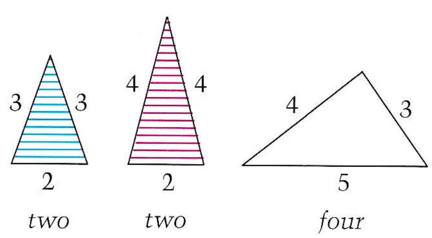


Figure 1

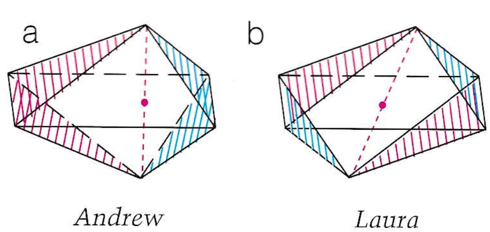


Figure 2

M177

(a) The answer is no. Suppose such a sequence exists. Without loss of generality, we can assume that any two consecutive terms in it are relatively prime. (If their greatest common divisor d is greater then 1, then the next term—their sum—and the previous term—their difference—both will be divisible by d, and we can extend this argument to the entire sequence. So all the terms can be reduced by d. It should also be mentioned that d here is necessarily a perfect square.)

Now, if $a^2 + b^2 = c^2$, where a, b, and c are relatively prime numbers, then c is odd, and one of the numbers a and b is even. (If a and b are both odd, then the sum of their squares takes the form $(2n + 1)^2 + (2m + 1)^2$ and so is even but not divisible by 4, and thus cannot be a square.) On the one hand, all the numbers in the sequence starting with the third must be odd; on the other hand, some of them are necessarily even. This contradiction proves that our answer is no.

(b) Here the answer is yes.

It suffices to find a number a representable in two different ways as the product of numbers of different parity. That is, we need $a = bc = mn$ and that $0 < b^2 - c^2 < 2a < m^2 - n^2 = k(b^2 - c^2)$. Then we verify that the squares of the sequence

$$x = b^2 - c^2,$$
$$y = 2a,\ kx,\ ky,\ k^2x,\ k^2y,\ \ldots$$

satisfy the condition of the problem. For instance, $x^2 + y^2 = (b^2 - c^2)^2 + (2bc)^2 = (b^2 + c^2)^2$, or $(kx)^2 + y^2 = k^2(b^2 - c^2)^2 + 4b^2c^2 = (m^2 - n^2)^2 + 4m^2n^2 = (m^2 + n^2)^2$.

The smallest number with the two required factorizations is six: $6 = 3 \cdot 2 = 6 \cdot 1$. The additional conditions also hold: $3^2 - 2^2 = 5 < 12 < 36 - 1 = 7 \cdot (3^2 - 2^2)$. This number yields the sequence 5, 12, 35, 84, 245, …, whose squares satisfy the given condition.

M178

Let's represent the difference $\mathbf{r} - \mathbf{b}$ as the sum of "black" vectors (fig. 3) each of which is the difference between a red vector and the next blue vector in the clockwise direction. Black vectors subtend disjoint arcs of the unit circle centered at O.

Suppose that $\mathbf{d} = \mathbf{r} - \mathbf{b}$ is a nonzero vector (otherwise there's nothing to prove). Draw the diameter of the circle parallel to $\mathbf{d}$. The projection of $\mathbf{d}$ on this diameter is vector $\mathbf{d}$ itself. Since the projection of the sum of vectors equals the sum of the projections of these vectors, $\mathbf{d}$ equals the sum of the projections of the black vectors.

Consider all black vectors whose projections on the diameter have the same direction as $\mathbf{d}$. Clearly, the sum of the lengths of their projections is no smaller than $|\mathbf{d}|$ (since $\mathbf{d}$ is the sum of these vectors as well as others that point in the opposite direction). On the other hand, their projections (more exactly, the projections of the corresponding chords) are disjoint segments of the diameter, so their total length is no greater than 2. Thus, $|\mathbf{d}| = |\mathbf{r} - \mathbf{b}| \le 2$.

For the set of two opposite unit vectors, $|\mathbf{r} - \mathbf{b}| = 2$, so our estimate is exact.

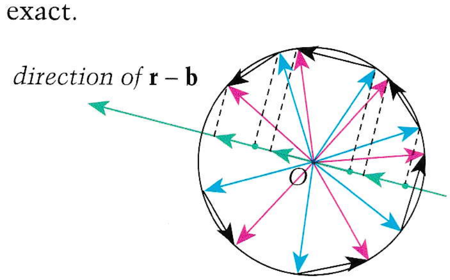


Figure 3

M179

Suppose for definiteness that the given segment is of unit length.

(a) For the first construction, draw a line (fig. 4), mark off three points A, B, C on it such that $AB = BC = 1$, draw another line through B, and mark off segment $BD = 1$ on it. Then $\angle ADC = 90°$ because it is an angle inscribed in the circle ACD and subtended by its diameter AC.

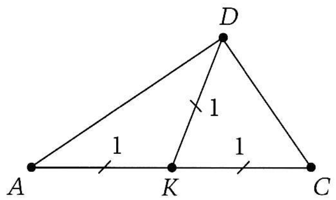


Figure 4

(b) Similarly, we can construct two right angles, AEC and AFC, subtended by the same diameter AC, where points A and C are taken on the given line (fig. 5). Let G and H be the intersection points of AE and CF, AF and CE, respectively. Then AF and CE are altitudes in the triangle AGC (and also in triangle AHC!), so GH is the third altitude in either triangle, because the altitudes meet at one point.

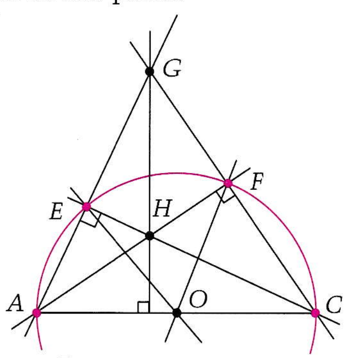


Figure 5

The theory of constructions with a straightedge and standard length is discussed in the famous book *Foundations of Geometry* by David Hilbert in connection with the analysis of geometric axioms. These tools suffice to perform many constructions. In particular, you can draw parallels, construct an angle congruent to a given angle with a given side, and so on. But some constructions that are possible with compass and straightedge are not possible with straightedge and unit length.

M180

Using the given equation, we obtain for any x the relations

$$\begin{aligned} f(x+2) + f(x) &= \sqrt{2} f(x+1) \\ &= \sqrt{2}\,[\sqrt{2} f(x) - f(x-1)] \\ &= 2f(x) - \sqrt{2} f(x-1) \end{aligned}$$

—that is,

$$f(x+2) = f(x) - \sqrt{2} f(x-1).$$

It follows that

$$\begin{aligned} f(x+4) &= f(x+2) - \sqrt{2} f(x+1) \\ &= f(x) - \sqrt{2}\,[f(x+1) + f(x-1)] \\ &= -f(x), \end{aligned}$$

and so

$$f(x+8) = -f(x+4) = f(x).$$

This means that f is a periodic function with period 8. One example of such a function is $f(x) = \sin(\pi x/4)$.

Physics

P176

According to Newton's second law and the universal gravitation law, the acceleration due to gravity at the Martian surface is

$$g_M = G\frac{M}{R_M^2},$$

where G is the gravitational constant, M is the mass of Mars, and R_M is its radius. The radius of Mars can be found directly from the conditions of the problem (fig. 6):

$$R_M = l\tan\frac{\alpha}{2} \cong l\frac{\alpha}{2}$$

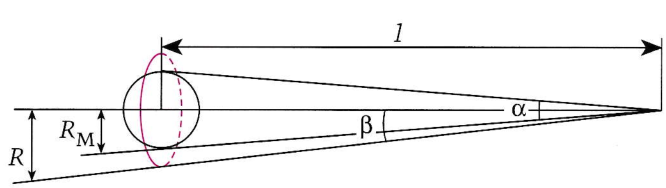


Figure 6

(since α is small, $\tan(\alpha/2) \cong \alpha/2$).

To find the mass of Mars, consider the motion of its moon Phobos. To simplify, assume the satellite's orbit to be circular with radius $R = l\tan\beta \cong l\beta$ (see figure 6). The period of revolution of Phobos is T. The centripetal acceleration $a = \omega^2 R = (2\pi/T)^2 R$ of the moon is provided by the force of gravity $F = G(mM/R^2)$ (m is the mass of Phobos). So $a = F/m$, or $(2\pi/T)^2 R = G(M/R^2)$, from which we get

$$GM = \left(\frac{2\pi}{T}\right)^2 R^3 = \left(\frac{2\pi}{T}\right)^2 (l\beta)^3.$$

Finally,

$$\begin{aligned} g_M = G\frac{M}{R_M^2} &= \left(\frac{2\pi}{T}\right)^2 \frac{(l\beta)^3}{(l\alpha/2)^2} \\ &= \frac{16\pi^2}{T^2}\left(\frac{\beta}{\alpha}\right)^2 l\beta. \end{aligned}$$

Inserting the numerical data into this formula and taking into account that $1'' = 3.14/(180 \cdot 60 \cdot 60)$ rad $\cong 4.9 \cdot 10^{-6}$ rad, we get

$$g_M = 3.64 \text{ m/s}^2.$$

P177

In a small time interval Δt the momentum of a drop of varying mass is

$$\Delta(mv) = mg \cdot \Delta t. \qquad (1)$$

According to the conditions of the problem, the increase in the drop's mass in a time Δt is

$$\Delta m = \alpha\rho v_{av} S \cdot \Delta t, \qquad (2)$$

where ρ is the density of water, v_{av} is the average speed of the drop during the time Δt, $S = 4\pi r^2$ is the drop's surface area, and α is a dimensionless proportionality factor. As for the left-hand side, since $m = {}^4/_3\pi r^3\rho$,

$$\begin{aligned} \Delta m &= 4\pi r^2\rho \cdot \Delta r \\ &= S\rho \cdot \Delta r. \end{aligned}$$

If in time Δt the drop falls the distance Δy, then $\Delta t = \Delta y/v_{av}$. Inserting the expressions for Δm and Δt into equation (2) yields

$$\Delta r = \alpha \cdot \Delta y \sim \Delta y.$$

Because the initial radius is small, the increase in the drop's radius is proportional to the distance traveled—that is, $r \sim y$.

Since the drop falls with a uniform acceleration a, then $y = at^2/2 \sim t^2$. Therefore, $r \sim t^2$ and $m \sim r^3 \sim t^6$. Taking these relationships into account, from equation (1) we get

$$\Delta(t^6 at) = t^6 g \cdot \Delta t.$$

(Because there is a mass on both sides of equation (1), the proportionality constant cancels.) Differentiating the left-hand side results in

$$\Delta(at^7) = 7at^6 \cdot \Delta t = t^6 g \cdot \Delta t,$$

which reduces to

$$a = \frac{g}{7}.$$

P178

The decay of uranium hydride proceeds according to the equation

$$2UH_3 \to 2U + 3H_2$$

—that is, 482 g (2 moles) of uranium hydride yield 476 g (2 moles) of uranium and 6 g (3 moles) of hydrogen. Correspondingly, 1 g of uranium hydride yields $m = 6/482$ g of hydrogen. Assuming hydrogen to be an ideal gas, we calculate its pressure at the given conditions (T = 673 K, $V = 10^{-3}$ m^3) using the ideal gas law:

$$P = \frac{m}{\mu}\frac{RT}{V} \cong 3.5 \cdot 10^4 \text{ N/m}^2.$$

P179

Draw the cross section of the system along its plane of symmetry passing through the given point M (fig. 7—the charged plates go to infinity in both directions from the plane of the page). Draw two lines MA_1A_2 and MB_1B_2 close to one another. Note that the very narrow strip of width A_1B_1 is equivalent to a thread with the linear charge density $\lambda_1 = \sigma \cdot A_1B_1$ located at a distance $r_1 \cong A_1M$ from point M. The electric field $\Delta\mathbf{E}_1$ generated by this thread at point M is directed along A_1M and equals

$$\Delta E_1 = \frac{\lambda_1}{2\pi\varepsilon_0 r_1} = \frac{\sigma \cdot A_1B_1}{2\pi\varepsilon_0 \cdot A_1M},$$

where ε_0 is the permittivity of free space.

The corresponding strip on the lower plate generates a similar field:

$$\Delta E_2 = \frac{\lambda_2}{2\pi\varepsilon_0 r_2} = \frac{-\sigma \cdot A_2B_2}{2\pi\varepsilon_0 \cdot A_2M}.$$

The similarity of triangles MA_1B_1 and MA_2B_2 yields $\Delta E_2 = -\Delta E_1$—that is, the fields produced by the strips A_1B_1 and A_2B_2 balance out exactly. In the long run there will be only one uncompensated source of electric field—the strip P_1Q (fig. 7). As the total width of the plates is large ($P_1N \gg h$), the strength of the electric field formed by the strip P_1Q is directed almost horizontally (to the right). As the triangles MQN and P_1QP_2 are similar, $P_1Q/QN = d/h \ll 1$—that is, the width of the strip P_1Q is small compared to the distance from the strip to point M. So this strip can again be replaced by a thread with linear charge density $\lambda = \sigma \cdot P_1Q$ located at a distance $r \cong QM \cong QN$ from point M.

Thus the desired electric field strength is directed almost horizontally to the right and is approximately equal to

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \cong \frac{\sigma \cdot P_1Q}{2\pi\varepsilon_0 \cdot QN} = \frac{\sigma d}{2\pi\varepsilon_0 h}.$$

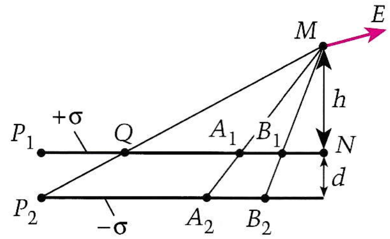


Figure 7

P180

To obtain the brightest strip of light, the screen must be located where the times required for rays traveling along different paths to reach the screen are equal. Assuming that the paths of the rays inside the aquarium are parallel, we can write the following equation for the lowest ray and for the ray traveling at height h (fig. 8):

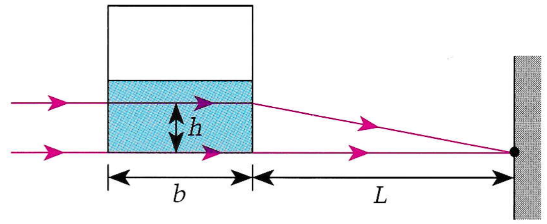


Figure 8

$$\frac{b}{c/n_0} + \frac{L}{c} = \frac{b}{c/\left(n_0 - ah^2\right)} + \frac{\sqrt{L^2 + h^2}}{c},$$

where $b = \sqrt[3]{V}$ is the width of the aquarium, L is the distance to the screen, and c is the speed of light in a vacuum. After some simple transformations (and noting that $h \ll L$), we get

$$L = \frac{1}{2ab} = \frac{1}{2a\sqrt[3]{V}} \cong 2.5 \text{ m}.$$

Note that this result does not contain the parameter h. This means that the conditions for maximum brightness coincide for all the paths of the light rays passing through the water. So this aquarium has the properties of a converging lens.

Brainteasers

B176

See figure 9.

B177

The frictional force is proportional to the normal force. Consider the book just below the book that is

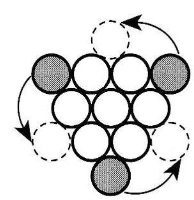


Figure 9

pulled out. The friction between this book and the one beneath it is greater than for the one sliding along it by a value proportional to the weight of one book. This is why the book underneath stays put. The books below this one are kept in place even more firmly.

B178

As a coin rolls a distance equal to its circumference, it makes one full revolution. The perimeter of the rectangle is 12 circumferences. So the outside coin will make 12 revolutions as it rolls along the rectangle's sides. In addition, at every vertex of the rectangle it makes an additional quarter turn (fig. 10). So the total number of revolutions for the outside coin is 13.

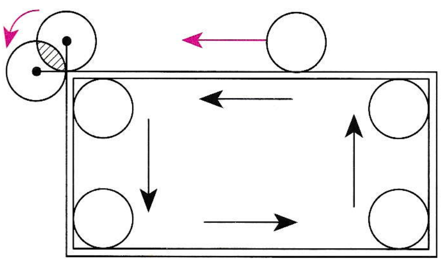


Figure 10

The inside coin travels a distance $12c - 8r$, where c is its circumference and $r = c/2\pi$ is its radius. So it makes $12 - 4/\pi \cong 10.7$ revolutions.

B179

Only the mother and brother or the son and daughter could be twins, so the twins are of opposite genders. Therefore, the winner and the loser are of the same gender. They can't be the mother and daughter, because they must be the same age. So they are the son and the brother. The son couldn't be the winner, because in that case the brother's twin—that is, the mother—would be the same age as her son. This leaves only one possibility free of contradictions: the tournament was won by the brother.

B180

We'll give a solution that may not be the shortest, but it shows that we can turn over any preassigned set of coins. Number the coins 1, 2, 3, ..., 7 in order. Reverse the coins 12345, then 45671, and then 67123. Every coin except coin 1 will have been turned over twice, while coin 1 will have been turned over three times. This amounts to reversing only coin 1. Similarly, we can reverse any other single coin and thus any set of the coins. (V. Dubrovsky)

Kaleidoscope

1. The second time the man will run the distance $5/8 - 3/8 = 1/4$ of the bridge length greater than the first time; the corresponding difference for the car is exactly one bridge length. It follows that the man's speed is 1/4 of the car's—that is, 15 km per hour.

2. The speed of the swimmer with respect to the ball is the same regardless of the direction of swimming. Thus the trip back takes 10 minutes, and the entire trip takes 20 minutes. The ball floated 1 km during these 20 minutes. Therefore, its speed with respect to the shore—that is, the speed of the current—is 1 km per 20 min, or 3 km per hour.

3. Denote by q the percentage of salt in the solution after the entire operation. The initial percentage was $q - p$, and it increased by a factor of $q/(q-p)$. Since the mass of the salt remains the same, this factor is equal to the ratio of the masses of the solution before and after evaporation. A similar consideration shows that the mass of the solution in the test tube decreased by half, so the decrease in the total mass is $1/(2n)$ of its initial value—that is, the total mass changed by a factor of $1 - 1/(2n) = (2n - 1)/(2n)$. This yields the equation

$$\frac{q}{q-p} = \frac{2n}{2n-1},$$

or

$$\frac{1}{1-\frac{p}{q}} = \frac{1}{1-\frac{1}{2n}},$$

from which we immediately get $q = 2np$.

4. Let n be the unknown number of additional machines. They worked for $35 - 6 - 11 = 18$ hours and did the same work as the first 27 machines would have done during the 6 hours that were saved. Thus, we have $18n = 27 \cdot 6$, and so $n = 9$.

5. Let a and b be the masses of copper in the two pieces of the first ingot. Then the corresponding masses for the second ingot are ka and kb, where k is the ratio of concentrations of copper in the two ingots. After alloying the ingots a second time, the mass of copper is $a + kb$ in the first ingot and $b + ka$ in the second ingot. So we have $a + kb = b + ka$, or $(1 - k)(a - b) = 0$. By the condition of the problem, $k \neq 1$; therefore, $a = b$—that is, the ingots were cut in half.

6. The answer is $DF + BE = a$. To prove this equality, rotate the square about A by 90° so that the side AD matches AB (fig. 11). Let G be the image of F under the rotation; then $DF = GB$ and $DF + BE = GB + BE = GE$. So it suffices to prove that $GE = AE$. Since $\triangle ABG$ is congruent to $\triangle ADF$, we have $\angle AGE = \angle AFD = \angle FAB$ (the last equation follows from the fact that $AB \parallel CD$). Further, $\angle FAB = \angle FAE + \angle EAB = \angle FAD + \angle EAB = \angle GAB + \angle EAB = \angle GAE$. Therefore, $\angle AGE = \angle GAE$, the triangle AGE is isosceles, and we're done.

7. (a) The proof is clear from figure 12, in which L is the midpoint of BC, M is the intersection point of the medians, and $LM = MN$. It is a

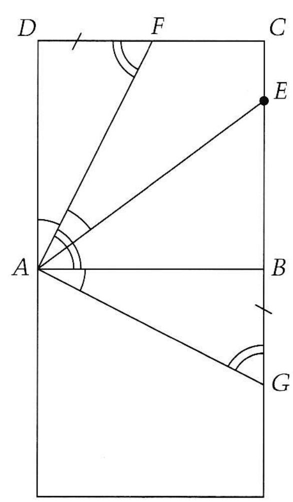


Figure 11

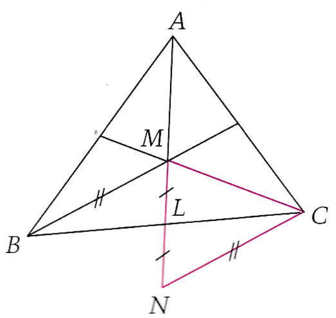


Figure 12

well-known property of medians that point M divides each of them in the ratio 2:1, starting from the triangle's vertices. Thus we have $MN = 2ML = AM$. Also, $CN = BM$ (these segments are symmetric about point L). So each side length of triangle CMN is 2/3 of the corresponding median of triangle ABC, and dilation by a factor of 3/2 takes triangle CMN into the one we have to construct.

(b) In figure 12, the median CL of triangle CMN is exactly one half of the side CB of the initial triangle ABC. So, in view of the dilation above, the length of the corresponding median of triangle $A_1B_1C_1$ is $\frac{3}{4}BC$. Clearly, similar relations will hold for the other two medians, which means that triangle $A_2B_2C_2$ is similar to ABC with the ratio of similarity equal to 3/4.

(c) The last remark immediately yields the answer: the ratio in question is $(3/4)^2 = 9/16$. (Alternately, this result can be obtained from the fact that the ratio of the areas of triangles $A_1B_1C_1$ and ABC is 3/4, which is not hard to see in figure 12.)

8. First, let's find the ratio $CL:LA$. Draw $MN \parallel BL$ (fig. 13). Then $CN = NL$ (since $CM = MB$) and $NL = LA$ (since $MK = KA$). Therefore, $CL = 2LA$. Now we compare the areas of several triangles. Base MB of triangle AMB is half as long as base AB of triangle ABC, and they have the same altitude from A, so $\text{area}(ABM) = \frac{1}{2}\text{area}(ABC) = 1/2$. Similarly, $\text{area}(MKB) = \frac{1}{2}\text{area}(ABM) = 1/4$. And since the base LC of triangle BLC is 2/3 as long as base AC of triangle ABC, and they have the same altitudes from B, $\text{area}(BLC) = \frac{2}{3}\text{area}(ABC) = 2/3$. Finally, $\text{area}(CMKL) = \text{area}(BLC) - \text{area}(BMK) = 2/3 - 1/4 = 5/12$.

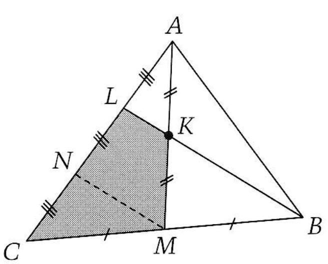


Figure 13

9. Triangles ABC, ACD, and BCD are all similar to one another, and in the notation of figure 14 we have

$$\frac{P_1}{P} = \frac{b}{c}, \quad \frac{P_2}{P} = \frac{a}{c},$$

so

$$\left(\frac{P_1}{P}\right)^2 + \left(\frac{P_2}{P}\right)^2 = \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \frac{a^2 + b^2}{c^2} = 1.$$

It follows that $P = \sqrt{P_1^2 + P_2^2}$.

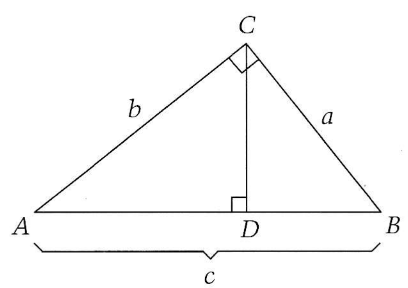


Figure 14

10. Notice that $\sqrt[3]{6} < 2$ and $\sqrt{6} < 3$. Replacing the innermost roots in both terms of the sum in question by 2 and 3, respectively, we will only increase the sum, but now it becomes exactly equal to 5.

11. The function in question, viewed as a function, of the variable x, is a polynomial of degree no greater than 2. If we denote it by $P(x)$, it is not hard to see that $P(a) = P(b) = P(c) = 1$; that is, the polynomial $P(x) - 1$ has at least three roots (the problem stipulates that a, b, and c are all different). But this is possible only if $P(x) - 1 = 0$ for all x, or $P(x) \equiv 1$.

12. Letting $f(x) = \sqrt[3]{2x-1}$, we can rewrite the given equation as $x = \sqrt[3]{2\sqrt[3]{2x-1}-1}$, or $x = f(f(x))$. We'll show that for any increasing function $f(x)$ the equations

$$f(f(x)) = x \qquad (1)$$

and

$$f(x) = x \qquad (2)$$

are equivalent. First we note that any solution of $f(x) = x$ is certainly a solution of $f(f(x)) = x$. Conversely, let x_0 be a solution of equation (1). If $f(x_0) > x_0$, then $f(f(x_0)) > f(x_0)$ (since f increases), so $f(f(x_0)) > x_0$, contrary to our assumption. Similarly, $f(x_0) < x_0$ implies $f(f(x_0)) < x_0$. So x_0 must be a solution of equation (2).

So the given equation reduces to the equation $x = f(x) = \sqrt[3]{2x-1}$, or $x^3 - 2x + 1 = 0$, which can be solved by factoring: $x^3 - 2x + 1 = (x^3 - x) - (x - 1) = (x - 1)(x^2 + x - 1) = 0$. It has three roots: 1 and $(-1 \pm \sqrt{5})/2$.

13. A short solution to this problem involves the scalar product of vectors.

(a) Denote by **a**, **b**, **c** the unit vectors directed with the vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$, respectively. The angle between **a** and **b** is $\pi - \gamma$ (fig. 15). The other angles between these vectors can be expressed in similar form.

We have

$$\begin{aligned} 0 \le (\mathbf{a} + \mathbf{b} + \mathbf{c})^2 &= \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2(\mathbf{ab} \,.\, \mathbf{bc} \,.\, \mathbf{ca}) \\ &= 3 + 2[\cos(\pi - \gamma) + \cos(\pi - \alpha) + \cos(\pi - \beta)] \\ &= 3 - 2(\cos\alpha + \cos\beta + \cos\gamma), \end{aligned}$$

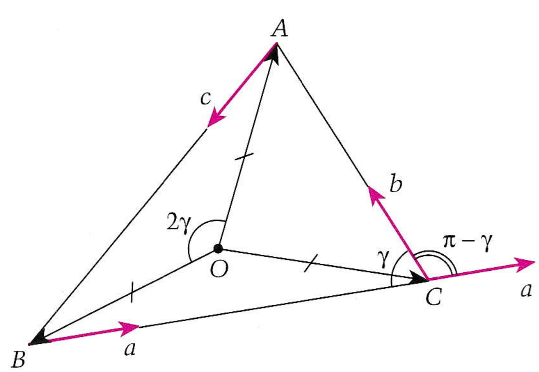


Figure 15

and the desired estimate follows immediately.

(b) The proof of this inequality is almost the same as for the previous one except that the vectors **a**, **b**, **c** must be replaced by $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ (fig. 15), where O is the circumcenter of the given triangle ($|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}|$ is the circumradius, and the angles between these vectors are 2α, 2β, 2γ if the triangle is acute. If, say, the angle α is obtuse, then the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is $2\pi - 2\alpha$, but its cosine still equals $\cos 2\alpha$).

14. (a) Denote the given expression by $f(\alpha)$. Then

$$\begin{aligned} f'(\alpha) &= (3\sin^2\alpha \cdot \cos\alpha \cdot \cos 3\alpha \\ &\quad - 3\sin^3\alpha \sin 3\alpha) \\ &\quad + (3\cos 3\alpha \cdot \cos^3\alpha \\ &\quad - 3\cos^2\alpha \cdot \sin\alpha \cdot \sin 3\alpha) \\ &= 3(\sin^2\alpha + \cos^2\alpha) \\ &\quad \cdot (\cos\alpha \cdot \cos 3\alpha - \sin\alpha \sin 3\alpha) \\ &= 3\cos 4\alpha. \end{aligned}$$

Therefore, $f(\alpha) = {}^3/_4 \sin 4\alpha$ + const. Since $f(0) = 0$, the constant is zero, so $f(\alpha) = {}^3/_4 \sin 4\alpha$.

Infinite descent

(Supplied by the editor)

1. A proof can be made by analogy to the first proof (in the article) of the irrationality of $\sqrt{2}$. Suppose there existed a segment d such that AC and BC are both integral multiples of d. We will construct a smaller triangle, similar to the original, and show that its sides are also integral multiples of d. But this leads to an infinite descent, in which we will eventually get a triangle whose sides are smaller than d itself, which is a contradiction.

To construct the smaller triangle, we bisect angle B and let it intersect AC at C_1 (see figure 16). Then we draw line C_1C_2 parallel to BC. Now $CC_1 = C_1B = AB$ (this depends on the fact that the vertex angle is 36°), so if we divide the perimeter of ABC into segments of length d, then point C will be one of these division points. Also, $AC_1 = C_1C_2 = C_2B$, so C_2 is also one of these division points. Thus the sides of triangle AC_1C_2 can be measured off in seg-

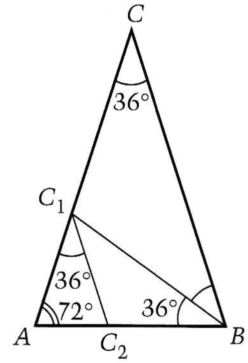


Figure 16

ments of length d, and the infinite descent can begin.

2. The existence of the representation in question is equivalent to the solvability of the equation $x^2 + y^2 + z^2 = 7t^2$. An analysis of the remainders modulo 8 shows that all the numbers x, y, z, t are even. So we can divide both sides of the equation by two and repeat the argument, carrying out an infinite descent.

3. Both equations have a single solution, (0, 0, 0). In part (a) use divisibility by 3 to show that any solution (x, y, z) can be represented as $(3x_1, 3y_1, 3z_1)$, where (x_1, y_1, z_1) is another, "smaller" solution. Then "descend to infinity." In part (b) examine the remainders of x, y, z (mod 13). When divided by 13, integer cubes give only the remainders 0, 1, 5, 8, ($\equiv -5$), and 12 ($\equiv -1$). It follows that any three numbers x, y, z satisfying the equation are all divisible by 13, which allows for infinite descent.

4. (a) This equation can be solved as (or even simply reduced to) problem 2 in the article. (b) This problem is, in fact, a particular case of Fermat's Great Theorem—the only case that has an elementary proof, which was found by Fermat himself. This proof uses infinite descent, and we outline it below.

Consider the more general equation

$$x^4 + y^4 = u^2. \quad (1)$$

Suppose it has a solution in nonzero integers. Then it has a solution x, y, u in positive, pairwise relatively prime integers. The numbers x^2, y^2, u form a Pythagorean triple—that is,

$$(x^2)^2 + (y^2)^2 = u^2.$$

One of the numbers x and y is even (consider remainders modulo 4). Let it be x. Then there exist relatively prime positive integers m and n of different parity such that

$$\begin{aligned} x^2 &= 2mn, \\ y^2 &= m^2 - n^2, \\ u &= m^2 + n^2. \end{aligned} \quad (2)$$

From the second of these equation, it follows that n, y, m is also a Pythagorean triple. These three numbers are pairwise relatively prime and, since y is odd, n is even. So we can represent them as

$$\begin{aligned} n &= 2ab, \\ y &+ a^2 - b^2, \\ m &= a^2 + b^2, \end{aligned}$$

with relatively prime a and b. The first of equations (2) gives $x^2 = 4mab$, where m is relatively prime to ab (since $ab = n/2$). Therefore, all three numbers m, a, and b are perfect squares:

$$\begin{aligned} a &= x_1^2, \\ b &= y_1^2, \\ m &= u_1^2. \end{aligned}$$

But this means that x_1, y_1, u_1 is a solution to equation (1) with relatively prime x_1 and y_1:

$$x_1^4 + y_1^4 = a^2 + b^2 = m = u_1^2.$$

It remains to notice that this solution is "smaller" than the initial one in the sense that $u_1 < u$, because

$$u_1 \le u_1^2 = m < m^2 + n^2 = u.$$

5. Prove that the maximum value of the written numbers or the number of these maxima decreases at each step of the process, while all the numbers remain positive. This will mean that the sum of the maxima strictly decreases and must be a fraction after some number of iterations.

6. Drop the perpendicular from the given point P to the plane F_1 of an arbitrary face of the polyhedron. If it falls outside the face, it will intersect another (and only one other) face. Its plane F_2 is closer than F_1 to P. Drop the perpendicular from P to the plane F_2, take the face it intersects, and continue the process. If these perpendiculars never fall

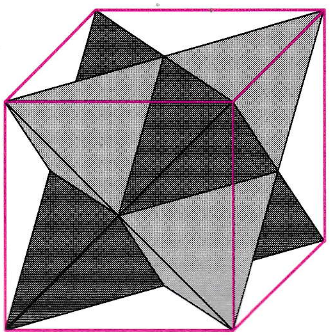


Figure 17

inside a face, we'll get an infinite sequence of faces, all of which are different because the distances from P to their planes F_1, F_2, ... strictly decrease. But there are only finitely many faces.

For nonconvex polyhedrons the statement is not true. A counterexample (Kepler's *stella octangula*) is given in figure 17; the point can be taken at the center of the cube.

Toy Store

Lights in the night. The answer is shown in figure 18. The windows on each floor of the building must be read as the Morse code of a letter! A single lit window is a dot; two lit windows in a row denote a dash.

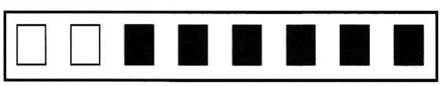


Figure 18

7	1	0	5	1	4	7	1	9
3	9	6	1	5	0	6	1	0
4	0	2	2	1	5	4	6	4
0	2	3	6	0	6	8	0	1
3	8	1	5	2	3	5	3	3

Figure 19

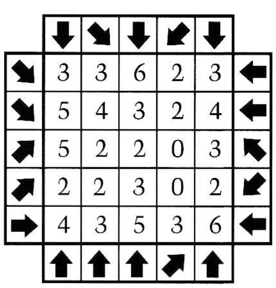


Figure 20

The letters, read from the top down, form the name *Thomas Alva Edison*.

Number pentominoes. See figure 19.

Counting pointers. See figure 20.

Battleships. See figure 21

From X to O. See figure 22.

Compound Latin square. See figure 23. The hidden "surprise" is *Bucharest* (the capital of Romania).

Spinning gold

1. We can prove without an exhaustive search that there are odd numbers greater than 3 not representable in the required form, but our proof is not elementary. On the other hand, an exhaustive search shows that the smallest nonrepresentable number is 149.

2. Since statements (π_1), (σ_1) in both dialogues are the same, the reasoning in the first three sections of the article remains valid. So s_0 belongs to the set

$$C = \{11, 17, 23, 27, 29, 35, 37, 41, 47, 53\},$$

and all these numbers satisfy (σ_1'). Statement (π_2) means that

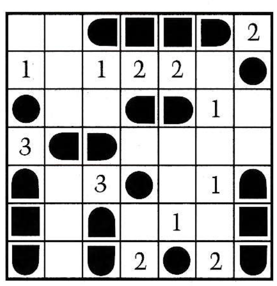


Figure 21

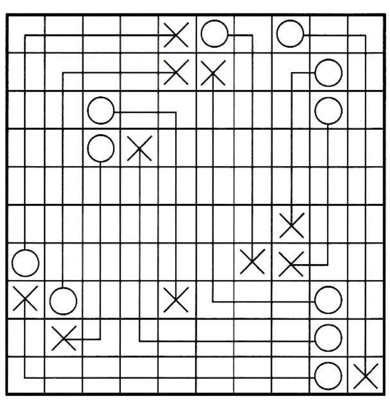


Figure 22

for any factorization of p_0 into the product of two integers satisfying inequalities (1), (2), their sum satisfies [σ_1'). (π_2')

It was proved in the article that any number with property (σ_1') belongs to C. Therefore, we can replace (π_2') with this:

for any factorization of p_0 into the product of integers satisfying inequalities (1) and (2) their sum belongs to C. (π_2'')

Search through all representations of the numbers in C as the sums of two terms: $s = k + l$ (for $2 \le k \le l$) and check whether the products kl satisfy (π_2''). The search can be reduced approximately by half if we prove in advance, using (π_2''), that p_0 is not divisible by 4.

This search shows that only four numbers in C yield a decomposition $s = k + l$ such that kl satisfies (π_2''):

$$23 = 6 + 17,$$
$$35 = 13 + 22,$$
$$37 = 3 + 34 = 11 + 26 = 14 + 23,$$
$$53 = 2 + 51 = 7 + 46.$$

Since S couldn't guess the numbers k_0 and l_0 even after statement (π_2), $s_0 \ne 23$ and $s_0 \ne 35$. So in view of (σ_2), $s_0 = 37$ or $s_0 = 53$. After P had understood this, he managed to determine k_0 and l_0. Therefore, $p_0 = 11 \cdot 26 = 286$, because otherwise P would have had either $p_0 = 102 = 3 \cdot 34 = 2 \cdot 51$ or $p_0 = 322 = 14 \cdot 23 = 7 \cdot 46$, and wouldn't be able to choose between $37 = 3 + 34 = 14 + 23$ and $53 = 2 + 51 = 7 + 46$.

Finally, we get $s_0 = 37$, $p_0 = 286$, $k_0 = 11$, $l_0 = 26$.

A	S	B	C	U	H	R	T	E
U	R	H	E	T	B	S	A	C
E	C	T	A	R	S	B	H	U
T	A	S	B	E	U	C	R	H
B	**U**	**C**	**H**	**A**	**R**	**E**	**S**	**T**
H	E	R	S	C	T	U	B	A
S	B	U	T	H	C	A	E	R
C	H	E	R	S	A	T	U	B
R	T	A	U	B	E	H	C	S

Figure 23

Readers write . . .

Several readers wrote to object to our solution to physics challenge P161 in the January/February 1996 issue. One correspondent, John J. Spokas, professor of physics at Illinois Benedictine College in Lisle, Illinois, sent us a detailed critique of a solution he found "unreasonable and arbitrary." Prof. Spokas writes:

The same notation as found in the published solution will be followed in the discussion below. From the obvious symmetry, it is only necessary to consider placing the added mass in the proximity of leg I. The problem with the published solution has to do with the placement at A of force f_{12} and the placement at B of the equivalent force f_{34}. In selecting points A and B, an unjustified assumption is made regarding the comparative values of f_1 and f_2 and of f_3 and f_4. What argues against the equivalent forces f_{12} and f_{34} being at A' and B' is illustrated in figure 1 below. Point $P(x, y)$ locates the position of the added mass $m/2$. In fact, the most reasonable location of B' is coincident with f_4.

The underlying difficulty with the problem and the solution published is that the problem is indeterminate. Even with the added mass at the center of the table, it is not possible to determine how the load distributes among the four legs. There is one too many legs! Restricting the added mass to the region bounded by the x- and y-axes and the line $x + y = l$, leg III is not required for mechanical stability.

An alternate solution is offered here. Restricting the added mass to the region outlined above and recognizing that leg III is not required, it is eliminated—that is, f_3 is set equal to zero. Incidentally, this corresponds to B' coinciding with f_4. One readily finds that the locus of points for the added mass that makes $f_1 = mg/4$ is the line $x + y = l/2$. Thus the full region of the table where the added mass may be placed without overstressing a leg is outlined in figure 2 by the broken line.

It is a bit strange that by using only three legs it is possible to position the added mass closer to a leg than the published solution, which uses all four legs, allows! Evidently the extra leg results in a greater stress in the leg nearest the added mass. This comes about from the arbitrary and unreasonable manner in which the indeterminacy was handled. Load is being transferred from legs II and IV to leg III, which is farther from leg I, and this requires greater support from leg I.

The difficulties described here would be avoided if the problem pertained to a triangular table in the form of an equilateral triangle with legs at the corners.

We find *Quantum* to be a great magazine, engaging and stimulating. It is an excellent resource for physics problems for students to work up and present at our seminar conducted by the Physics Club on campus.

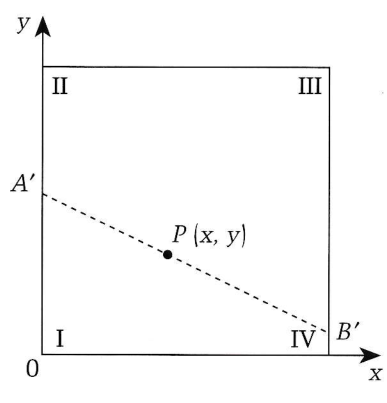


Figure 1

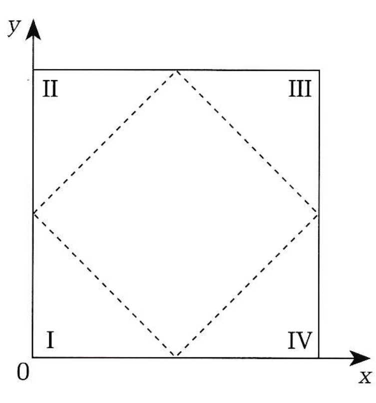


Figure 2

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The World Puzzle Championship

A report on the 1995 convocation in Romania, problems from Brno (1993) and Brasov (1995)

by Vladimir Dubrovsky

WITHOUT MUCH RISK I daresay that the Brainteasers department is one of the most popular in *Quantum*. So our readers might be interested to know that there is an annual international competition that includes solving problems of the brainteaser variety. The history of this competition is not very long. Apparently it dates back to 1984, when puzzle makers from a number of East European countries came to Poland for the first International Crossword Marathon, which grew out of a similar national Polish contest. It was a 24-hour, nonstop team competition to create the longest crossword puzzle (the results were measured in meters). The marathon became an annual event, but as new countries joined, it was understood that this kind of contest is not perfectly fair to all: the Finnish and Dutch teams quite reasonably complained that their native languages are not as suitable for composing long crosswords as, say, English. The marathon could no longer exist on a large international scale—it was held for the last time in 1990 in Croatia.

Thanks to Will Shortz, the American team captain, who then was the editor of *Games* magazine (and now is the puzzle editor of the *New York Times*), the idea of an international puzzle competition was revived in the new format of the World Puzzle Championship (WPC). With the backing of the publisher of *Games* and of Times Books, Shortz organized the first WPC in New York in 1992. It became the pattern for subsequent championships in Brno (Czech Republic) in 1993, Cologne (Germany) in 1994, and Brasov (Romania) in 1995. The American team has won the WPC twice—in 1992 and 1995; the Czechs took the other two contests. In fact, last year the Americans were double winners—an individual victory was gained by Wei-Hwa Huang, a student at MIT.

So far the structure of the Championships has not been firmly established. The schedule of the competition, the checking of solutions and scoring, and the selection (and creation) of the contest puzzles are farmed out to the organizers, which certainly leaves the mark of their taste on the entire event, and not always to the satisfaction of all. For instance, the organizers of the last WPC decided to shift the stress in the selection of puzzles to crosswords, because they consider crosswords the most popular puzzle genre (this is undoubtedly true in western Europe and North America, though I'm not so sure about the rest of the world). So quite a few puzzles required the knowledge of the English (or native) spelling of some personal and geographic names, which was certainly disadvantageous even for participants who use the Roman alphabet but transliterate names phonetically, to say nothing of the Japanese and Russians. These and other problems were discussed at the International Puzzle Congress that was proceeding while the contestants struggled with the puzzles. The general opinion was that such violations of the basic principle of linguistic and cultural neutrality should be avoided in the future. (The next WPC is scheduled for this year in the Netherlands, and the 6th WPC will be hosted by Croatia in 1997.)

So what does one do at the World Puzzle Championships? With regard to the format, this competition resembles mathematical olympiads. All participants solve the same set of problems. In Brasov, for example, there were four 2–3-hour sessions with 16 puzzles to be solved at each. The difference is that the WPC includes special team rounds, where all members of each team can work together on the same puzzle, or they can distribute the given set of puzzles among themselves according to their tastes and abilities—only the *best* individual score for each puzzle counts in the total team score. As to the puzzles themselves, it would be hard even to name the kinds of challenges offered at the four championships held since 1992, although a vast class of puzzles—mechanical or manipulative puzzles—remains unused at the WPC (with very few exceptions). Maybe this is because it's more difficult to supply such puzzles in sufficient quantity. On the other hand, all sorts of printed puzzles have appeared at the competitions. To list but a few, there were general-knowledge quizzes, where you had to guess a country from its flag, say, or its national anthem; various picture puzzles, in which you had to spot the differences between two almost identical pictures, put a scrambled comic strip back together, or find a hidden image; mazes, tilings; and crossword-type puzzles. But first and foremost, one encountered an incredible variety of mathematical and logical brainteasers.

And now let me introduce to you some of the challenges from the World Puzzle Championships. The first two were offered at the second WPC in Brno (both were created by Zdenek Chromy).

Lights in the night. In figure 1 you see a 16-story building with some windows lit and some dark. The pattern of dark and light windows obeys a certain law. Figure out the law and "turn off the lights" in the windows of the top floor according to this law.

Number pentominoes. Cut the grid with numbers shown in figure 2 into pieces made up of five connected squares such that no two of them are of the same shape and the sums of numbers in all the pieces are equal to one another.

It's interesting that originally the score for the first problem was lower than for the second (15 and 20 points, respectively), but it turned out that the first one was the most difficult in the competition—only two participants managed to solve it. This puzzle seems to be of a logical nature, but it isn't! (Consider this a hint.) Personally I would be unable to solve it in principle. (This is another hint.) Such puzzles—they can be called surprise puzzles—are real puzzle gems and naturally, are not encountered very often at the WPC. On the contrary, the second challenge is, in principle, trivial: after all, the total number of possible dissections of the given grid is finite, and if you have enough time you can simply try all possibilities (it might be better to write a computer program that would do this for you). The problem is that you don't have enough time, not at the WPC, at least. However, in this kind of puzzle you can usually find a thread that can be pulled to unravel the entire knot without searching through lots of possibilities.

Such combinatorial puzzles are very popular in Japan. Many of them were invented there (they appear in puzzle magazines published by the Sekai Bunka publishing house in Tokyo), and they are now becoming more popular throughout the world. About a quarter of the puzzles at the fourth WPC were cooked up from Japanese recipes. This was supposed to (but didn't) compensate for the language problems experienced by the Japanese team. Anyway, these puzzles are elegant and attractive indeed, and I want to acquaint *Quantum* readers with some of them. (All the puzzles below are borrowed from the fourth WPC.)

Counting pointers (fig. 3). Draw arrows in the 20 empty squares on the sides of the big square with numbers such that each of them is parallel to a side or diagonal of the big square and each number in the grid equals the

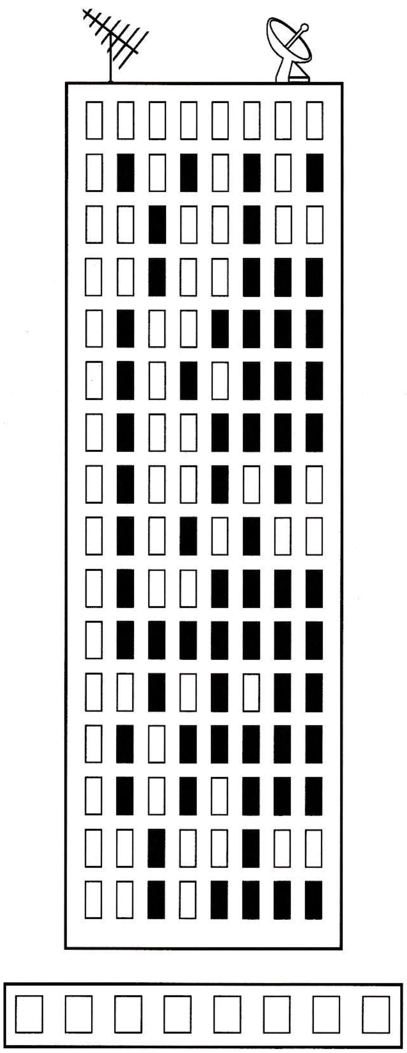


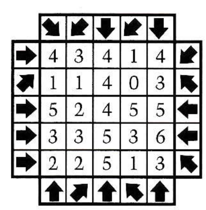
Figure 1

7	1	0	5	1	4	7	1	9
3	9	6	1	5	0	6	1	0
4	0	2	2	1	5	4	6	4
0	2	3	6	0	6	8	0	1
3	8	1	5	2	3	5	3	3

Figure 2

	3	3	6	2	3	
	5	4	3	2	4	
	5	2	2	0	3	
	2	2	3	0	2	
	4	3	5	3	6	

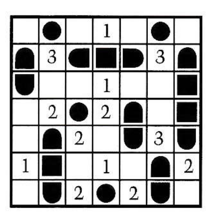
Figure 3



example

						2
1		1	2	2		
					1	
3						
		3			1	
				1		
			2		2	

Figure 4



example

				X	O		O		
				X	X			O	
		O						O	
		O	X						
							X		
O						X	X		
X	O			X				O	
	X							O	
								O	X

Figure 5

A			C		H			E
		H				S		
	C	T		R		B	H	
T			B		U			H
		C				E		
H			S		T			A
	B	U		H		A	E	
		E				T		
R			U		E			S

Figure 6

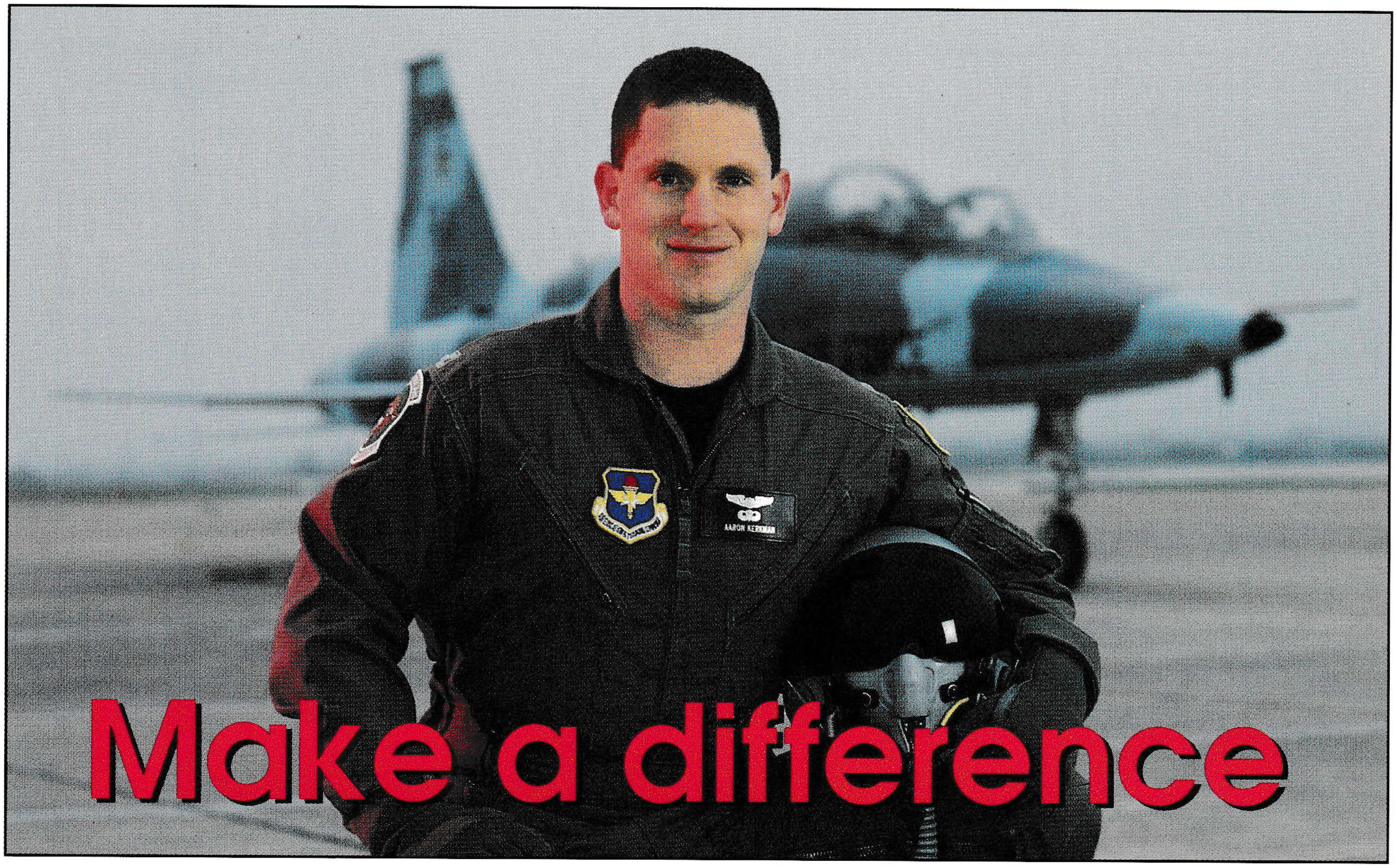
number of arrows pointing at it (as in the example).

Battleships (fig. 4). Insert the standard set of "ships" for the game of battleships (shown in the figure) in the 7 × 7 square grid so that (1) any two squares occupied by the ships never have a common side (though they can have a common vertex); (2) each number given in the grid equals the number of occupied squares adjacent to the square.

From X to O (fig. 5). Draw a path from each X to one of the O's so that the path consists of two perpendicular segments parallel to the grid lines and every grid square belongs to only one path.

Compound Latin square (fig. 6). Write one of the nine letters A, B, C, E, H, R, S, T, U in each empty square of the given grid such that each row, column, and 3 × 3 block marked with bold lines contains each letter once. The resulting array will contain a nine-letter "surprise," which you also have to identify.

ANSWERS, HINTS & SOLUTIONS ON PAGE 53



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