A snail that moves like light

"The cause is hidden but the effect is known."
—*Ovid*, Metamorphoses

MOST OF OUR READERS know that light bouncing off a mirror travels along a path that can be adequately described as "the angle of incidence equals the angle of reflection." Light traveling from a point in air to a point in water is certainly more complicated. In this case, the light bends (refracts) at the boundary between the two surfaces. The amount of bending is a property of the water and the color of the light. Light entering other transparent substances, like quartz or diamond, refract by different amounts. Willebrord Snell in 1621 was able to give a mathematical description of the behavior of light, which is now known as Snell's law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$

where n_1 and n_2 are the indices of refraction. We can see that if the light enters water ($n = 1.33$) from air ($n = 1.00$) at an angle of 30°, the angle in water would be 22°:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$
$$1.00 \sin 30° = 1.33 \sin \theta_2,$$
$$\theta_2 = 22°.$$

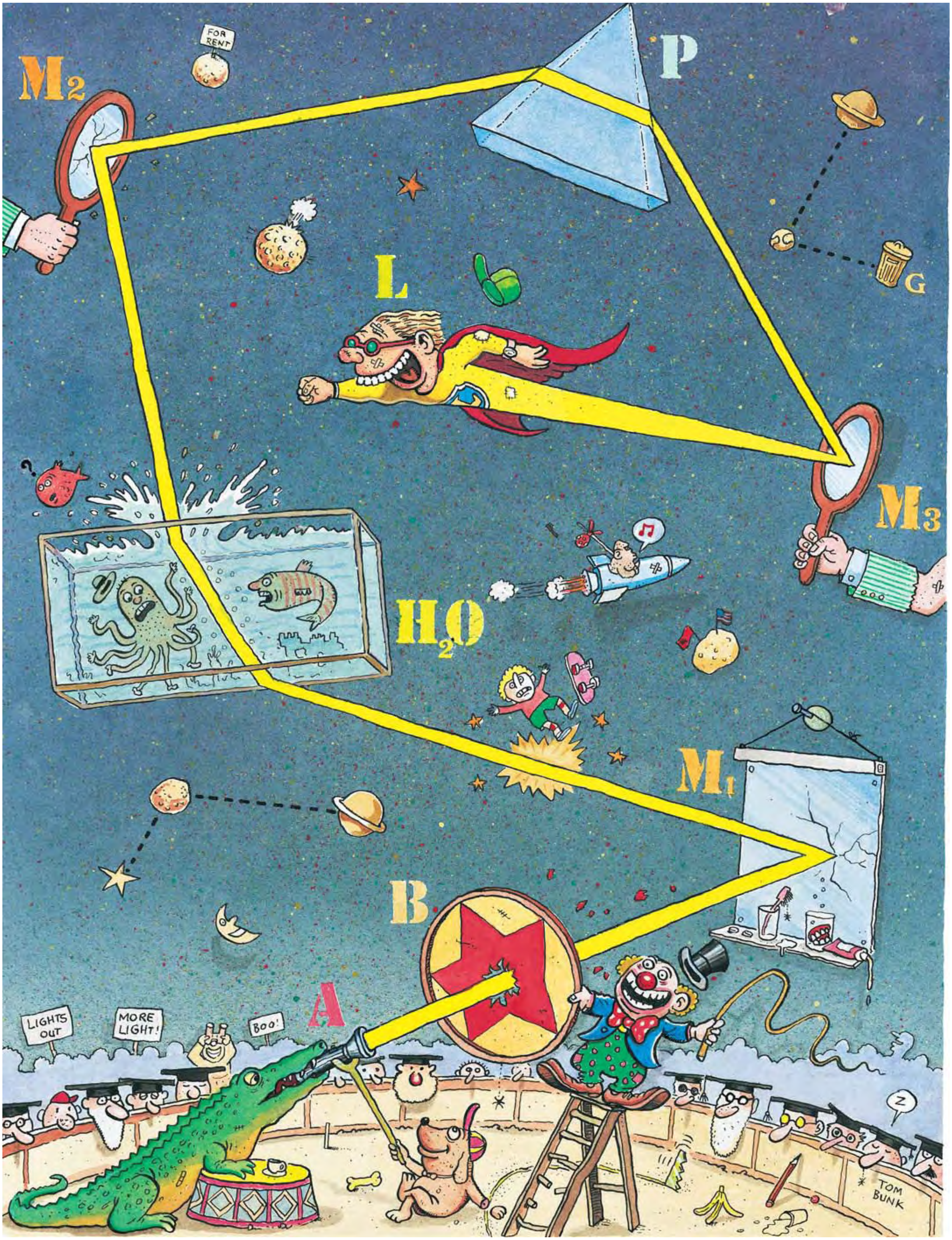
Measuring the angle of refraction is one way to tell whether that's a diamond or a piece of glass in that ring you bought.

What fascinates many people about the study of physics is the alternative ways of explaining phenomena. The great mathematician Pierre de Fermat recognized (in 1657) that the path of light is the path that requires the least time.[1] If you try all possible paths from the light source *A* to the object *B* after they hit the mirror, you'll find that the shortest path, and so the quickest, is the path through point *D* (fig. 1 on the next page), where the angle of incidence equals the angle of reflection.

[1]The "extremum path."

Light with its human face is having a super blast speeding in a flash through space and objects, breaking mirrors and here and there some rules, shooting through a fishbowl—surprising the wet inhabitants—and through a solid glass prism without getting a headache. This spectacular high-speed performance is taking place in a cosmic circus tent where the alligator and his dog assistant are in charge of the modest flashlight source. In the audience we see the professional science community verifying the correct course and angles of light but not being very impressed by the seriousness of the performance. But that does not bother our lightweight space cadet, who thinks that coming from a flashlight he is Flash Gordon himself.

—T.B.



Light is bending the rules a bit here. (Can you see where?)

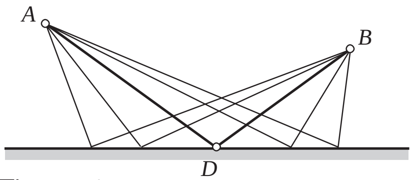


Figure 1

You can demonstrate this for yourself by drawing lots of paths and measuring them. You can also prove it with some simple geometry or by using some calculus.

Fermat's theorem is also valid for refraction: the path light takes when it passes from air to water must be the path requiring the least time. In this case least time is not identical to least distance, since light travels more slowly in water that in air. The speed of light in a substance is equal to the speed of light in a vacuum divided by the substance's index of refraction n.

Proving that the path of the light is the quickest one takes some ingenuity. You can draw lots of paths of light traveling from point A in air to point B in water (fig. 2). You can then measure the lengths of the lines in air and water. But Fermat's theorem states that the path should take the least *time*, not the least distance. We can multiply the lengths in water by 1.33, since the

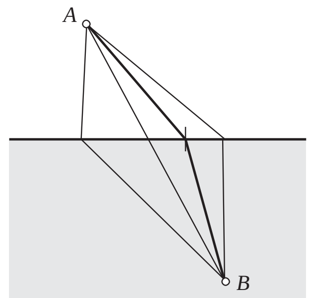
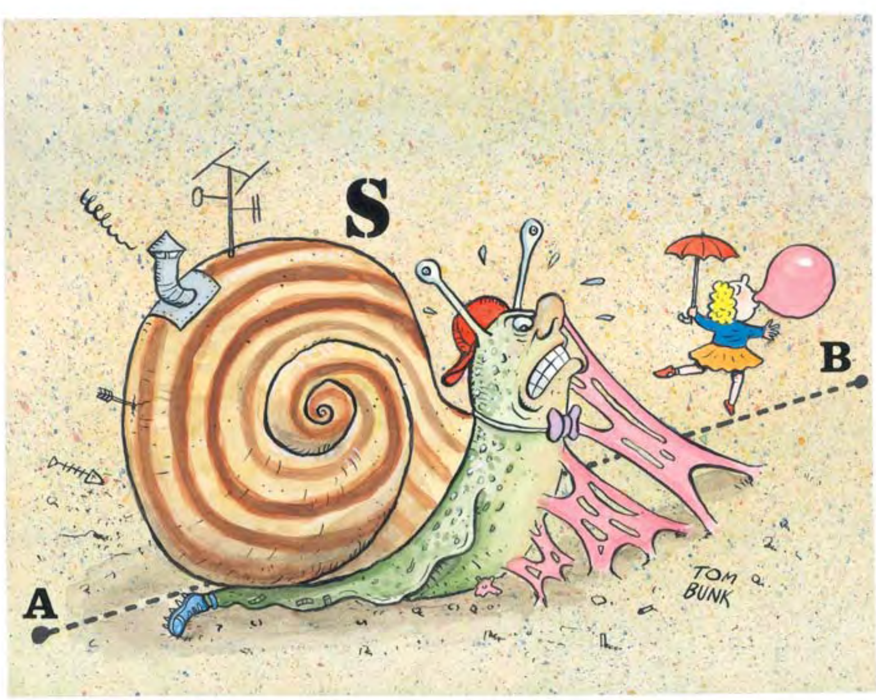


Figure 2

light takes longer to travel in water by a factor of 1.33. Then add this distance to the distance in air. The path that minimizes this sum is the path the light takes. And—guess what? It's the same path described by Snell's law! Those of you who have some calculus background can prove it mathematically.

Leaving light behind, we enter the world of slow-moving mollusks to find our contest problem. A snail must get from one corner of a room (dimensions 5 m × 10 m × 15 m) to the diagonally opposite corner in the least time. The snail can walk on any of the four walls but may not walk on the floor or ceiling. What is the path that the snail should take? In part B of the contest problem, for our more advanced readers, the snail finds that the 15 meter wall that must be traveled is sticky—that is, the snail can only travel at a fraction of its normal speed. If the snail on the sticky wall travels at 1/3 of its normal speed, what is the path that requires the least time for the snail? Finally, in part C, for our most advanced readers, what happens if the snail finds that the stickiness of the first wall is not constant but increases linearly along one dimension of the wall? Specifically, the speed at one end of the wall is the normal speed and the speed at the far end of the wall is 1/3 the normal speed. What will be the path of least time? You may need to use graphical or computer techniques to solve parts B and C. Our best readers are encouraged to see if they can find general proofs for any room (dimensions $l \times w \times h$) and a stickiness factor of s. We are not sure ourselves if such general proofs exist.



Solution

You were asked to help a snail find the quickest path from one corner of a room to a diagonally opposite corner.

In the first case, in which all walls were identical and the dimensions of the room were 5 × 10 × 15, there are at least three ways to solve the problem. The first is to choose different crossover points at the edge between the two walls and calculate the total distance that the snail travels. This numerical method may appear to be tedious, but it will actually converge on the correct solution quickly. A second method is to call the height of the crossover point x, write the total distance traveled in terms of x, and differentiate. By setting the derivative equal to zero, the minimum distance will be revealed as the solution to the equation. The third method is the elegant solution. In this case, the wall is opened up. The room is now a large rectangle of dimensions 25 × 5. The shortest distance will be the diagonal connecting the two corners of the rectangle. If the snail starts at the lower corner of the 15-meter wall, the crossover point can be found by using similar triangles. The crossover point is

$$\frac{x}{15} = \frac{5-x}{10},$$
$$x = 3.$$

In the second case, one of the walls was declared "sticky," meaning that the snail could travel at only 1/3 of its speed on this wall. Unlike the first case, the shortest distance is no longer the shortest time! Since the snail travels at different speeds on the two walls, the quickest path will be the one where the snail travels a greater distance on the faster wall. Once again, the straightforward but tedious solution would be to assign the variable x to the crossover point, write an equation that describes all paths in terms of x, and the minimum time will be revealed.

The more elegant solution in this case is to realize that light always takes the least time to travel, and that this snail traveling on a sticky wall is like light traveling in a slower medium. We then recognize that the solution will be Snell's law (or, if you'll forgive us, "Snail's law"). Even with this knowledge, we are faced with a fourth-order equation, which we choose to solve by numerical techniques:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2.$$

Since the stickiness factor is 3, then $n_1 = 3$ and $n_2 = 1$, and it follows that

$$3\frac{x}{\sqrt{15^2 + x^2}} = \frac{5-x}{\sqrt{(5-x)^2 + 10^2}}.$$

We'll try different values of x and see if the value of the left side of the equation is equal to the value of the right side.

x	*left side*	*right side*
1	0.1996	0.3714
2	0.3965	0.2873
1.5	0.2985	0.3304
1.7	0.3378	0.3134
1.6	0.3182	0.3219
1.63	0.3241	0.3194
1.62	0.3221	0.3202

This method can give us any accuracy we desire. It would certainly be easier to plug the equations into a spreadsheet program and have all values given "instantly."

The third part of the problem, to solve for a wall whose stickiness varies along one dimension, was solved by Jason Jacobs of Harvard University. We will leave this problem as a tease.